

Replacement of Hand Function

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Abstract

In recent years, thanks to the advancement of robotics and mechatronics, new and more effective devices for the restoration and replacement of sensory-motor function in disabled people have been developed. In all these systems, user acceptability is strictly connected to several issues such as the residual abilities of the subject, the mechatronic characteristics of the robot, and also the interface chosen to link them. It is possible to figure out different Bhuman-interface-device[combinations [also defined as Bhybrid bionic systems[(HBSs)] characterized by different properties in terms of level of hybridness, connection, and augmentation. In particular, in HBSs the interface has to be customized according to the characteristics of the robotic artefact to be controlled and to the desires and needs of the final users. In this paper, our attention has been focused on the problem of the replacement of hand function after amputation. Three HBSs characterized by different levels of complexity, dexterity, and sensorization are presented in order to show the possibility of developing acceptable and effective systems by choosing different levels of connection and hybridness (i.e., different interfaces) for different devices and applications. The following case studies are presented: 1) the use of invasive interfaces to the peripheral nervous system to control a dexterous and highly sensorized hand prosthesis; 2) the use of electromyographic signals recorded using surface electrodes to control a compliant adaptive prosthesis; and 3) the use of a foot interface to control a two-degrees-of-freedom The preliminary results achieved so far seem to confirm the idea that the correct choice of the proper interface while developing an HBS can increase effectiveness and usability.

Keywords- Biomechatronics; biorobotics; hand prostheses; neural interfaces; neurorobotics.

I. INTRODUCTION

In the recent past, several research groups have been working on the development of robotic systems for biomedical applications (e.g., in surgery and in rehabilitation). In many cases, these devices are not thought to behave autonomously, but they have to work together with the final users (directly connected to their own bodies or teleoperated by them). Therefore, it is crucial to create a real

partnership between the human subject and the robot in order to make the latter more friendly and effectively usable (see Fig. 1). The bottleneck in this case is the (reduced) number of (functional) connections currently possible between the user and the robotic device. For example, it is not possible to replace the extremely sophisticated bidirectional connections existing between the nervous system and the hand. This precludes the possibility of restoring the natural motor control strategies and of delivering sensory feedback during manipulation. In order to address this crucial problem, different interfaces can be designed, taking into account the needs and desires of the users and the challenges related to the tasks the robotic systems have to carry out. The combination Bhuman-interface-device[can be defined as Bhybrid bionic system[(HBS) and may assume a variety of forms and configurations. HBSs can be generically defined as systems that contain both technical (artificial) and biological components. They can include: 1) artificial systems with biological elements or subsystems, where the biological system is a complementary or supplementary element to the technical system or 2) biological systems with artificial elements or subsystems, in which the artificial subsystem, e.g., a robotic artefact, is a complementary or supplementary element to the biological system. In the recent past, many scientific and technological efforts have been devoted to the creation of HBSs linking the human nervous system with electronic and/or robotic devices. In general, this research has been carried out with various aims, either to develop systems for restoring motor and sensory functions in injured and disabled people or for exploring the possibility of augmenting sensory-motor capabilities of humans in general, not only of disabled people. A possible representation of different hybrid bionic systems and of their complexity is given in Fig. 2, where the three key attributes proposed for a general classification of hybrid bionic systems are: 1) level of hybridness (H), ranging from separate artificial and natural systems (H0), to exoskeletons copying the mechanical properties of natural limbs (H1), up to artificial body parts anatomically and functionally Bconnected[to the human body (H2); 2) level of augmentation (A): empowering sensing, perception, and motor capabilities; the level of augmentation increases with the number and type (perceptual or/and motor) of empowered capabilities; and 3) level of connection to the nervous system (C): modality used to connect the artificial and natural

systems, varying from multimodal indirect interfaces to direct interfaces to the peripheral nervous system (PNS) or to the central nervous system (CNS). For example, a prosthesis can be connected to the user by means of different invasive and noninvasive interfaces. It is possible to control an artificial hand by using signals recorded noninvasively (as the electromyographic (EMG) signal) or by implanting electrodes in order to interface invasively the CNS and PNS. At the same time, the robotic system can be tightly connected to the user (as an exoskeleton) or remote from her/him (as in teleoperation). The same class of robotic systems can be characterized by different levels of hybridness (more or less contact with the body of the user). In this paper, research activities on HBSs characterized by different levels of hybridness, augmentation, and connection to the nervous system are briefly presented. In particular, three different case studies are presented (see Fig. 3 for a schematic representation): 1) the use of invasive interfaces to the PNS to control a dexterous and sensorized hand prosthesis; 2) the use of a foot interface to control a two-degree-of-freedom (DoF) prosthesis; and 3) the use of EMG signals recorded using surface electrodes to control a compliant adaptive prosthesis. In all these cases, the interface is chosen to match the characteristics of the hand prosthesis in terms of dexterity and sensorization. For example, invasive interfaces potentially able to deliver sensory feedback are to be used only when the prosthesis is equipped with several artificial sensors usable to gather information to be delivered as sensory feedback.

II. CYBERNETIC MULTI-DOF SENSORIZED HAND PROSTHESIS

The human hand is a complex and adaptable system, capable of both delicate and precise manipulation and of power grasping of heavy objects. In order to substitute it after traumatic events it is mandatory to develop a (highly) dexterous sensorized hand prosthesis able to carry out different grasping tasks and at the same time to record sensory information to be delivered to the user. Moreover, it is necessary to develop an intimate interface with the nervous system in order to extract motor commands coming from the user's voluntary intentions and to deliver a sensory feedback to the user by stimulating the afferent nerves by using the information gathered from the artificial sensors embedded in the prostheses. In this section, the results of the activities carried out to develop a cybernetic hand prosthesis connected directly to the nervous system, controllable in a natural way and able to be felt as a part of the body because of the possibility of delivering sensory feedback to the user, are described (in Fig. 4, the scheme of the cybernetic prosthesis is given).

Interfaces to the PNS seem to be more appropriate for ethical reasons because of the current limits of cortical interfaces in terms of robustness (for an implant in amputees). Moreover, the possibility of stimulating directly the primary sensory cortex S-I (and other brain areas) to elicit a sensory feedback has two main drawbacks: 1) After an amputation a reorganization of the somatosensory cortex is evident (see for example [1]). For example, in many cases the stimulation of the lips can elicit an activation of the (former) hand area of S-I; 2) The stimulation of the somatosensory cortex seems to be able to provoke muscle contraction [2], probably because of cortical connections among different brain areas. These issues have to be investigated before considering the possibility of implanting electrodes in S-I to deliver a sensory feedback. In this section, the research activities carried out to develop and/or to identify the different main components of the cybernetic hand prosthesis are briefly summarized.

A. Cyberhand Biomechatronic Hand

Commercially available hands are unable to provide enough grasping functionality because of the lack of DoFs available and because no grasping shape adaptation is possible, due to the rigid transmission [3]. In order to enhance prosthesis flexibility by keeping the intrinsic actuation solution and implementing simple control algorithms, a design approach based on underactuated mechanisms has been developed that has been proven to be quite effective [4]–[6]. For this reason, the CYBERHAND biomechatronic hand has been developed using this approach. The hand has five identical underactuated fingers with cylindrical phalanges made out of aluminum alloy. A picture of the hand is presented in Fig. 5. Each finger of the CYBERHAND prosthesis is underactuated, and its movements are driven by a single cable moved by a motor. The motor for thumb adduction/abduction is located inside the palm, while the motors for the movement of the fingers are all located inside the forearm, thus mimicking the structure of the human body. The palm is composed of an outside shell, made in carbon fiber, divided into dorsal and palmar parts, and an inside frame, holding the fingers and containing the thumb abduction/adduction transmission chain. Optionally, a soft padding made by silicon rubber can be mounted on the palm in order to increase the compliance of the grasping. The total weight of the hand is about 400 g, excluding the motors in the forearm and the cosmetic covering of the palm. The hand has in total 16 DoF and 6 degrees of motion (DoM), one DoM for each finger (flexion/extension) plus one DoM for thumb positioning (adduction/abduction). One 2-DoF trapezo-metacarpal joint at the base of the palm allows the thumb opposition movement towards the other two fingers. The artificial proprioceptive

sensors have been designed to mimic their biological counterparts by providing information about both joint positions and actuator forces. The sensory system includes: 15 Hall effect sensors embedded in all the joints of each finger, an incremental magnetic encoder and two stroke end Hall effect sensors on each of the six motors, and five tension sensors on the cables. The Hall effect sensors and the encoders are used to provide information about the position of all the phalanges during grasping and manipulation tasks. The sensors that measure the tension on the cables controlling fingers flexion are meant to mimic the function of Golgi tendon organs that give information about the tendon stretches. The exteroceptive sensory system was designed in order to provide information about: contact and release between the fingertip and the object, contact between the object takeoff and replacement to the environment, and slippage between the fingertip and the object. The tactile sensory system currently includes flexible contact sensors and triaxial force sensors [7]. Contact sensors provide on-off information to the tactile sensing system. Efforts have been made in order to confer a high contact sensitivity to emulate the mechanoreceptors of the human hand in an engineering implementation, according to neurophysiology studies [8]. The triaxial strain gauge sensors are integrated into thumb, index, and middle fingertips.

B. Neural Interfaces

An ideal neural interface should be able to create an intimate and selective contact with the PNS in order to restore the efferent and afferent neural pathways in an effective and useful way (for a review of PNS interfaces; see [9]). Starting from these needs, several neural interfaces have been developed with different characteristics. For example, cuff electrodes have been shown to offer a very reliable and robust platform with a relatively low degree of invasiveness but suffer from limited selectivity even in the case of multicontact cuff electrodes, which have shown some interesting results [10]. On the other end of the invasiveness scale, sieve electrodes could represent a very interesting solution with potentially very high selectivity though several unresolved problems with their chronic stability and the requirement to sectioned nerves limit their usability [11], and they are only applicable to severed nerves. For these reasons, intraneural electrodes, electrodes that penetrate the body of the nerve either inserted longitudinally (LIFE electrodes [12]–[14]) or transversally (USEA electrodes, [15]) into the PNS have been investigated as a compromise between selectivity and invasiveness. In particular, Longitudinal IntraFascicular Electrodes [(LIFE)s, see Fig. 6] seem to be very promising because they combine reduced (even if, of course, not absent)

invasiveness with good selectivity even if further experiments need to be carried out in order to design intraneural electrodes characterized by good penetration capability but also by reduced invasiveness.

C. Extraction of Information From Neural Signals

The possibility of using voluntary activities (extracted by processing physiological signals) to control artificial devices connected to the humans for different applications has been investigated recently by several research groups around the world. In particular, electroneurographic (ENG) signals recorded from the PNS using different neural interfaces have been used in the past to extract useful sensory information to improve the performance of neuroprostheses [9], [16], [17]. In particular, LIFE electrodes have been used to extract voluntary information by processing the efferent neural signals recorded from different peripheral nerves [18], [19] in amputees. In particular, the subjects involved in the study were able to control a cursor on a PC screen. This result showed that ENG signals can be used to achieve more than a simple on-off control and that neural signals could solve, in the future, some of the problems of the EMG-based control of hand prostheses for amputees. More recently, it was shown [20] that LIFE ENG signals can be classified using a classical spike-sorting approach. In order to start addressing this issue, LIFE electrodes were implanted in the sciatic nerve of rabbits applying different kinds of stimuli to the paw of the rabbit. The ENG signals recorded were processed to determine whether the different modes of information could be decoded. Signals were Kalman filtered, wavelet denoised, and spike sorted. The classes of spikes found were then used to infer the stimulus applied to the rabbit. Although the signals acquired from a single LIFE gave poor stimulus recognition, the combination of the signals from multiple sites led to better results. The results achieved seem to show the possibility of extracting different neural information exploiting the potentials of multisite neural interfaces. In the future, the possibility of extracting information related to homologous neuromuscular activities (i.e., neural signals related to the extension of the elbow can be identified and used to control the homologous movement of the artificial arm) using this spike sorting approach will be investigated.

D. Preliminary Experiments on Sensory Feedback

Recently, LIFE electrodes have been used to elicit adequate sensory feedback to the users by stimulating the afferent nerves through the LIFE electrodes [18], [19]. Preliminary experiments on the possibility of eliciting sensory feedback by stimulating the peripheral nervous system seem to confirm the potentialities of this approach [19]. In

particular, the following results have been achieved: 1) stable unimodal distally elicited sensations of touch and/or proprioception localized to the digits in the majority (70%) of the cases; 2) sensations of touch/pressure usually spread from distal (digit tip) to proximal locations with increasing stimulus current; and 3) amputees reported either movement of a given finger joint or movement of the entire digit (proprioceptive sensations) and reliably distinguish different degrees of joint flexion. Two subjects consistently reported a referred sensation of phantom grip opening and closing.

III. CONTROL OF 2-DOF HAND PROSTHESIS USING INDIRECT INTERFACE

Even if many interesting results have been already achieved by using implantable interfaces to control robotic systems, motor commands from user's intention can be also extracted through Bidirect[interfaces. These systems have to be used especially when the mechatronic characteristics of the prosthesis (in terms of dexterity and sensorization) do not justify the use of invasive interfaces. In this section, an algorithm for the control of a 2-DoF hand prosthesis using a human-machine interface based on the recording of foot pressure is illustrated together with some preliminary results.

We consider the following anycast field equations defined over an open bounded piece of network and/or feature space $\Omega \subset R^d$. They describe the dynamics of the mean anycast of each of p node populations.

$$\left\{ \begin{aligned} \left(\frac{d}{dt} + l_i\right)V_i(t, r) &= \sum_{j=1}^p \int_{\Omega} J_{ij}(r, \bar{r}) S[(V_j(t - \tau_{ij}(r, \bar{r}), \bar{r}) - h_{ij})] d\bar{r} \\ &\quad + I_i^{ext}(r, t), \quad t \geq 0, 1 \leq i \leq p, \\ V_i(t, r) &= \phi_i(t, r) \quad t \in [-T, 0] \end{aligned} \right. \quad (1)$$

We give an interpretation of the various parameters and functions that appear in (1), Ω is finite piece of nodes and/or feature space and is represented as an open bounded set of R^d . The vector r and \bar{r} represent points in Ω . The function $S : R \rightarrow (0, 1)$ is the normalized sigmoid function:

$$S(z) = \frac{1}{1 + e^{-z}} \quad (2)$$

It describes the relation between the input rate v_i of population i as a function of the packets potential, for example, $V_i = v_i = S[\sigma_i(V_i - h_i)]$. We note V the p -dimensional vector (V_1, \dots, V_p) . The p function $\phi_i, i = 1, \dots, p$, represent the initial

conditions, see below. We note ϕ the p -dimensional vector (ϕ_1, \dots, ϕ_p) . The p function $I_i^{ext}, i = 1, \dots, p$, represent external factors from other network areas. We note I^{ext} the p -dimensional vector $(I_1^{ext}, \dots, I_p^{ext})$. The $p \times p$ matrix of functions $J = \{J_{ij}\}_{i,j=1,\dots,p}$ represents the connectivity between populations i and j , see below. The p real values $h_i, i = 1, \dots, p$, determine the threshold of activity for each population, that is, the value of the nodes potential corresponding to 50% of the maximal activity. The p real positive values $\sigma_i, i = 1, \dots, p$, determine the slopes of the sigmoids at the origin. Finally the p real positive values $l_i, i = 1, \dots, p$, determine the speed at which each anycast node potential decreases exponentially toward its real value. We also introduce the function $S : R^p \rightarrow R^p$, defined by $S(x) = [S(\sigma_1(x_1 - h_1)), \dots, S(\sigma_p(x_p - h_p))]$, and the diagonal $p \times p$ matrix $L_0 = \text{diag}(l_1, \dots, l_p)$. Is the intrinsic dynamics of the population given by the linear response of data transfer. $(\frac{d}{dt} + l_i)$ is replaced by $(\frac{d}{dt} + l_i)^2$ to use

the alpha function response. We use $(\frac{d}{dt} + l_i)$ for simplicity although our analysis applies to more general intrinsic dynamics. For the sake, of generality, the propagation delays are not assumed to be identical for all populations, hence they are described by a matrix $\tau(r, \bar{r})$ whose element $\tau_{ij}(r, \bar{r})$ is the propagation delay between population j at \bar{r} and population i at r . The reason for this assumption is that it is still unclear from anycast if propagation delays are independent of the populations. We assume for technical reasons that τ is continuous, that is $\tau \in C^0(\bar{\Omega}, R_+^{p \times p})$. Moreover packet data indicate that τ is not a symmetric function i.e., $\tau_{ij}(r, \bar{r}) \neq \tau_{ji}(\bar{r}, r)$, thus no assumption is made about this symmetry unless otherwise stated. In order to compute the righthand side of (1), we need to know the node potential factor V on interval $[-T, 0]$. The value of T is obtained by considering the maximal delay:

$$\tau_m = \max_{i,j,(r,\bar{r}) \in \Omega \times \Omega} \tau_{i,j}(r, \bar{r}) \quad (3)$$

Hence we choose $T = \tau_m$

A. Mathematical Framework

A convenient functional setting for the non-delayed packet field equations is to use the space $F = L^2(\Omega, R^p)$ which is a Hilbert space endowed with the usual inner product:

$$\langle V, U \rangle_F = \sum_{i=1}^p \int_{\Omega} V_i(r) U_i(r) dr \quad (1)$$

To give a meaning to (1), we defined the history space $C = C^0([-\tau_m, 0], F)$ with

$\|\phi\| = \sup_{t \in [-\tau_m, 0]} \|\phi(t)\|_F$, which is the Banach phase space associated with equation (3). Using the notation $V_t(\theta) = V(t + \theta), \theta \in [-\tau_m, 0]$, we write (1) as

$$\begin{cases} V(t) = -L_0 V(t) + L_1 S(V_t) + I^{ext}(t), \\ V_0 = \phi \in C, \end{cases} \quad (2)$$

Where

$$\begin{cases} L_1 : C \rightarrow F, \\ \phi \rightarrow \int_{\Omega} J(\cdot, \bar{r}) \phi(\bar{r}, -\tau(\cdot, \bar{r})) d\bar{r} \end{cases}$$

Is the linear continuous operator satisfying $\|L_1\| \leq \|J\|_{L^2(\Omega^2, R^{p \times p})}$. Notice that most of the papers on this subject assume Ω infinite, hence requiring $\tau_m = \infty$.

Proposition 1.0 If the following assumptions are satisfied.

1. $J \in L^2(\Omega^2, R^{p \times p})$,
2. The external current $I^{ext} \in C^0(R, F)$,
3. $\tau \in C^0(\Omega^2, R_+^{p \times p}), \sup_{\Omega^2} \tau \leq \tau_m$.

Then for any $\phi \in C$, there exists a unique solution $V \in C^1([0, \infty), F) \cap C^0([-\tau_m, \infty), F)$ to (3)

Notice that this result gives existence on R_+ , finite-time explosion is impossible for this delayed differential equation. Nevertheless, a particular solution could grow indefinitely, we now prove that this cannot happen.

B. Boundedness of Solutions

A valid model of neural networks should only feature bounded packet node potentials.

Theorem 1.0 All the trajectories are ultimately bounded by the same constant R if $I \equiv \max_{t \in R^+} \|I^{ext}(t)\|_F < \infty$.

Proof :Let us defined $f : R \times C \rightarrow R^+$ as

$$f(t, V_t) \stackrel{def}{=} \langle -L_0 V_t(0) + L_1 S(V_t) + I^{ext}(t), V(t) \rangle_F = \frac{1}{2} \frac{d\|V\|_F^2}{dt}$$

We note $l = \min_{i=1, \dots, p} l_i$

$$f(t, V_t) \leq -l \|V(t)\|_F^2 + (\sqrt{p|\Omega|} \|J\|_F + I) \|V(t)\|_F$$

Thus, if

$$\|V(t)\|_F \geq 2 \frac{\sqrt{p|\Omega|} \|J\|_F + I \stackrel{def}{=} R}{l} = R, f(t, V_t) \leq -\frac{lR^2 \stackrel{def}{=} \delta}{2} = -\delta < 0$$

Let us show that the open route of F of center 0 and radius R, B_R , is stable under the dynamics of equation. We know that $V(t)$ is defined for all $t \geq 0s$ and that $f < 0$ on ∂B_R , the boundary of B_R . We consider three cases for the initial condition V_0 . If $\|V_0\|_C < R$ and set $T = \sup\{t \mid \forall s \in [0, t], V(s) \in \overline{B_R}\}$. Suppose that $T \in R$, then $V(T)$ is defined and belongs to $\overline{B_R}$, the closure of B_R , because $\overline{B_R}$ is closed, in effect to ∂B_R , we also have $\frac{d}{dt} \|V\|_F^2 \big|_{t=T} = f(T, V_T) \leq -\delta < 0$ because $V(T) \in \partial B_R$. Thus we deduce that for $\varepsilon > 0$ and small enough, $V(T + \varepsilon) \in \overline{B_R}$ which contradicts the definition of T. Thus $T \notin R$ and $\overline{B_R}$ is stable.

Because $f < 0$ on $\partial B_R, V(0) \in \partial B_R$ implies that $\forall t > 0, V(t) \in B_R$. Finally we consider the case $V(0) \in \overline{CB_R}$. Suppose that $\forall t > 0, V(t) \notin \overline{B_R}$, then $\forall t > 0, \frac{d}{dt} \|V\|_F^2 \leq -2\delta$, thus $\|V(t)\|_F$ is monotonically decreasing and reaches the value of R in finite time when $V(t)$ reaches ∂B_R . This contradicts our assumption. Thus $\exists T > 0 \mid V(T) \in B_R$.

Proposition 1.1 : Let s and t be measured simple functions on X . for $E \in M$, define

$$\phi(E) = \int_E s d\mu \quad (1)$$

Then ϕ is a measure on M .

$$\int_X (s+t) d\mu = \int_X s d\mu + \int_X t d\mu \quad (2)$$

Proof : If s and if E_1, E_2, \dots are disjoint members of M whose union is E , the countable additivity of μ shows that

$$\begin{aligned} \phi(E) &= \sum_{i=1}^n \alpha_i \mu(A_i \cap E) = \sum_{i=1}^n \alpha_i \sum_{r=1}^{\infty} \mu(A_i \cap E_r) \\ &= \sum_{r=1}^{\infty} \sum_{i=1}^n \alpha_i \mu(A_i \cap E_r) = \sum_{r=1}^{\infty} \phi(E_r) \end{aligned}$$

Also, $\phi(\phi) = 0$, so that ϕ is not identically ∞ .

Next, let s be as before, let β_1, \dots, β_m be the distinct values of t , and let $B_j = \{x : t(x) = \beta_j\}$ If $E_{ij} = A_i \cap B_j$, the

$$\int_{E_{ij}} (s+t) d\mu = (\alpha_i + \beta_j) \mu(E_{ij})$$

$$\text{and } \int_{E_{ij}} s d\mu + \int_{E_{ij}} t d\mu = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij})$$

Thus (2) holds with E_{ij} in place of X . Since X is the disjoint union of the sets E_{ij} ($1 \leq i \leq n, 1 \leq j \leq m$), the first half of our proposition implies that (2) holds.

Theorem 1.1: If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of K , and if $\varepsilon > 0$, then there exists a polynomial P such that $|f(z) - P(z)| < \varepsilon$ for all $z \in K$. If the interior of K is empty, then part of the hypothesis is vacuously satisfied, and the conclusion holds for every $f \in C(K)$. Note that K need to be connected.

Proof: By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension and denote it again by f . For any $\delta > 0$, let $\omega(\delta)$ be the supremum of the numbers $|f(z_2) - f(z_1)|$ Where z_1 and z_2 are subject to the condition $|z_2 - z_1| \leq \delta$. Since f is uniformly continuous, we

$$\text{have } \lim_{\delta \rightarrow 0} \omega(\delta) = 0 \quad (1) \text{ From now on,}$$

δ will be fixed. We shall prove that there is a polynomial P such that

$$|f(z) - P(z)| < 10,000 \omega(\delta) \quad (z \in K) \quad (2)$$

By (1), this proves the theorem. Our first objective is the construction of a function $\Phi \in C_c'(R^2)$, such that for all z

$$|f(z) - \Phi(z)| \leq \omega(\delta), \quad (3)$$

$$|(\partial\Phi)(z)| < \frac{2\omega(\delta)}{\delta}, \quad (4)$$

And

$$\Phi(z) = -\frac{1}{\pi} \iint_X \frac{(\partial\Phi)(\zeta)}{\zeta - z} d\zeta d\eta \quad (\zeta = \xi + i\eta), \quad (5)$$

Where X is the set of all points in the support of Φ whose distance from the complement of K does not δ . (Thus X contains no point which is "far within" K .) We construct Φ as the convolution of f with a smoothing function A . Put $a(r) = 0$ if $r > \delta$, put

$$a(r) = \frac{3}{\pi\delta^2} \left(1 - \frac{r^2}{\delta^2}\right)^2 \quad (0 \leq r \leq \delta), \quad (6)$$

And define

$$A(z) = a(|z|) \quad (7)$$

For all complex z . It is clear that $A \in C_c'(R^2)$. We claim that

$$\iint_{R^2} A = 1, \quad (8)$$

$$\iint_{R^2} \partial A = 0, \quad (9)$$

$$\iint_{R^2} |\partial A| = \frac{24}{15\delta} < \frac{2}{\delta}, \quad (10)$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because A has compact support. To compute (10), express ∂A in polar coordinates, and note that $\frac{\partial A}{\partial \theta} = 0$,

$$\frac{\partial A}{\partial r} = -a'$$

Now define

$$\Phi(z) = \iint_{R^2} f(z - \zeta) A d\xi d\eta = \iint_{R^2} A(z - \zeta) f(\zeta) d\xi d\eta \quad (11)$$

Since f and A have compact support, so does Φ .
 Since

$$\Phi(z) - f(z) = \iint_{R^2} [f(z - \zeta) - f(z)] A(\zeta) d\xi d\eta \quad (12)$$

And $A(\zeta) = 0$ if $|\zeta| > \delta$, (3) follows from (8).

The difference quotients of A converge boundedly to the corresponding partial derivatives, since $A \in C_c^1(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$\begin{aligned} (\partial\Phi)(z) &= \iint_{R^2} (\partial A)(z - \zeta) f(\zeta) d\xi d\eta \\ &= \iint_{R^2} f(z - \zeta) (\partial A)(\zeta) d\xi d\eta \\ &= \iint_{R^2} [f(z - \zeta) - f(z)] (\partial A)(\zeta) d\xi d\eta \quad (13) \end{aligned}$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\partial\Phi$, we see that Φ has continuous partial derivatives, if we can show that $\partial\Phi = 0$ in G , where G is the set of all $z \in K$ whose distance from the complement of K exceeds δ . We shall do this by showing that

$$\Phi(z) = f(z) \quad (z \in G); \quad (14)$$

Note that $\partial f = 0$ in G , since f is holomorphic there. Now if $z \in G$, then $z - \zeta$ is in the interior of K for all ζ with $|\zeta| < \delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\begin{aligned} \Phi(z) &= \int_0^\delta a(r) r dr \int_0^{2\pi} f(z - re^{i\theta}) d\theta \\ &= 2\pi f(z) \int_0^\delta a(r) r dr = f(z) \iint_{R^2} A = f(z) \quad (15) \end{aligned}$$

For all $z \in G$, we have now proved (3), (4), and (5) The definition of X shows that X is compact and that X can be covered by finitely many open discs D_1, \dots, D_n , of radius 2δ , whose centers are not in K . Since $S^2 - K$ is connected, the center of each D_j can be joined to ∞ by a polygonal path in $S^2 - K$. It follows that each D_j contains a compact connected set E_j , of diameter at least

2δ , so that $S^2 - E_j$ is connected and so that $K \cap E_j = \emptyset$. with $r = 2\delta$. There are functions $g_j \in H(S^2 - E_j)$ and constants b_j so that the inequalities.

$$|Q_j(\zeta, z)| < \frac{50}{\delta}, \quad (16)$$

$$\left| Q_j(\zeta, z) - \frac{1}{z - \zeta} \right| < \frac{4,000\delta^2}{|z - \zeta|^2} \quad (17)$$

Hold for $z \notin E_j$ and $\zeta \in D_j$, if

$$Q_j(\zeta, z) = g_j(z) + (\zeta - b_j) g_j^2(z) \quad (18)$$

Let Ω be the complement of $E_1 \cup \dots \cup E_n$. Then Ω is an open set which contains K . Put

$$\begin{aligned} X_1 &= X \cap D_1 && \text{and} \\ X_j &= (X \cap D_j) - (X_1 \cup \dots \cup X_{j-1}), && \text{for} \\ & && 2 \leq j \leq n, \end{aligned}$$

Define

$$R(\zeta, z) = Q_j(\zeta, z) \quad (\zeta \in X_j, z \in \Omega) \quad (19)$$

And

$$F(z) = \frac{1}{\pi} \iint_X (\partial\Phi)(\zeta) R(\zeta, z) d\xi d\eta \quad (z \in \Omega) \quad (20)$$

Since,

$$F(z) = \sum_{j=1}^n \frac{1}{\pi} \iint_{X_j} (\partial\Phi)(\zeta) Q_j(\zeta, z) d\xi d\eta, \quad (21)$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2 . Hence $F \in H(\Omega)$. By (20), (4), and (5) we have

$$\begin{aligned} |F(z) - \Phi(z)| &< \frac{2\omega(\delta)}{\pi\delta} \iint_X |R(\zeta, z) \\ &- \frac{1}{z - \zeta}| d\xi d\eta \quad (z \in \Omega) \quad (22) \end{aligned}$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta \in X$ and $z \in \Omega$.

Now fix $z \in \Omega$, put $\zeta = z + \rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho < 4\delta$, by (17) if $4\delta \leq \rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi \int_0^{4\delta} \left(\frac{50}{\delta} + \frac{1}{\rho} \right) \rho d\rho = 808\pi\delta \quad (23)$$

And

$$2\pi \int_{4\delta}^{\infty} \frac{4,000\delta^2}{\rho^2} \rho d\rho = 2,000\pi\delta. \quad (24)$$

Hence (22) yields

$$|F(z) - \Phi(z)| < 6,000\omega(\delta) \quad (z \in \Omega) \quad (25)$$

Since $F \in H(\Omega)$, $K \subset \Omega$, and $S^2 - K$ is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0 : Suppose $f \in C_c'(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put

$$\partial = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad (1)$$

Then the following "Cauchy formula" holds:

$$f(z) = -\frac{1}{\pi} \iint_{R^2} \frac{(\partial f)(\zeta)}{\zeta - z} d\xi d\eta \quad (2)$$

$(\zeta = \xi + i\eta)$

Proof: This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r, \theta) = f(z + re^{i\theta})$, $r > 0$, θ real

If $\zeta = z + re^{i\theta}$, the chain rule gives

$$(\partial f)(\zeta) = \frac{1}{2} e^{i\theta} \left[\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right] \varphi(r, \theta) \quad (3)$$

The right side of (2) is therefore equal to the limit, as $\varepsilon \rightarrow 0$, of

$$-\frac{1}{2} \int_{\varepsilon}^{\infty} \int_0^{2\pi} \left(\frac{\partial \varphi}{\partial r} + \frac{i}{r} \frac{\partial \varphi}{\partial \theta} \right) d\theta dr \quad (4)$$

For each $r > 0$, φ is periodic in θ , with period 2π . The integral of $\partial \varphi / \partial \theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_{\varepsilon}^{\infty} \frac{\partial \varphi}{\partial r} dr = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\varepsilon, \theta) d\theta \quad (5)$$

As $\varepsilon \rightarrow 0$, $\varphi(\varepsilon, \theta) \rightarrow f(z)$ uniformly. This gives (2)

If $X^\alpha \in a$ and $X^\beta \in k[X_1, \dots, X_n]$, then $X^\alpha X^\beta = X^{\alpha+\beta} \in a$, and so A satisfies the condition (*). Conversely,

$$\left(\sum_{\alpha \in A} c_\alpha X^\alpha \right) \left(\sum_{\beta \in \square^n} d_\beta X^\beta \right) = \sum_{\alpha, \beta} c_\alpha d_\beta X^{\alpha+\beta} \quad (\text{finite sums}),$$

and so if A satisfies (*), then the subspace generated by the monomials $X^\alpha, \alpha \in a$, is an ideal. The proposition gives a classification of the monomial ideals in $k[X_1, \dots, X_n]$: they are in one to one correspondence with the subsets A of \square^n satisfying (*). For example, the monomial ideals in $k[X]$ are exactly the ideals $(X^n), n \geq 1$, and the zero ideal (corresponding to the empty set A). We write $\langle X^\alpha \mid \alpha \in A \rangle$ for the ideal corresponding to A (subspace generated by the $X^\alpha, \alpha \in a$).

LEMMA 1.1. Let S be a subset of \square^n . The ideal a generated by $X^\alpha, \alpha \in S$ is the monomial ideal corresponding to

$$A = \left\{ \beta \in \square^n \mid \beta - \alpha \in \square^n, \text{ some } \alpha \in S \right\}$$

Thus, a monomial is in a if and only if it is divisible by one of the $X^\alpha, \alpha \in S$

PROOF. Clearly A satisfies (*), and

$a \subset \langle X^\beta \mid \beta \in A \rangle$. Conversely, if $\beta \in A$, then $\beta - \alpha \in \square^n$ for some $\alpha \in S$, and

$X^\beta = X^\alpha X^{\beta-\alpha} \in a$. The last statement follows from the fact that $X^\alpha \mid X^\beta \Leftrightarrow \beta - \alpha \in \square^n$. Let

$A \subset \square^n$ satisfy (*). From the geometry of A , it is clear that there is a finite set of elements $S = \{\alpha_1, \dots, \alpha_s\}$ of A such that

$$A = \left\{ \beta \in \square^n \mid \beta - \alpha_i \in \square^2, \text{ some } \alpha_i \in S \right\}$$

(The α_i 's are the corners of A) Moreover,

$a = \langle X^\alpha \mid \alpha \in A \rangle$ is generated by the monomials $X^{\alpha_i}, \alpha_i \in S$.

DEFINITION 1.0. For a nonzero ideal a in $k[X_1, \dots, X_n]$, we let $(LT(a))$ be the ideal generated by

$$\{LT(f) \mid f \in a\}$$

LEMMA 1.2 Let a be a nonzero ideal in $k[X_1, \dots, X_n]$; then $(LT(a))$ is a monomial

ideal, and it equals $(LT(g_1), \dots, LT(g_n))$ for some $g_1, \dots, g_n \in a$.

PROOF. Since $(LT(a))$ can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of a .

THEOREM 1.2. Every ideal a in $k[X_1, \dots, X_n]$ is finitely generated; more precisely, $a = (g_1, \dots, g_s)$ where g_1, \dots, g_s are any elements of a whose leading terms generate $LT(a)$

PROOF. Let $f \in a$. On applying the division algorithm, we find $f = a_1 g_1 + \dots + a_s g_s + r$, $a_i, r \in k[X_1, \dots, X_n]$, where either $r = 0$ or no monomial occurring in it is divisible by any $LT(g_i)$. But $r = f - \sum a_i g_i \in a$, and therefore $LT(r) \in LT(a) = (LT(g_1), \dots, LT(g_s))$, implies that every monomial occurring in r is divisible by one in $LT(g_i)$. Thus $r = 0$, and $g \in (g_1, \dots, g_s)$.

DEFINITION 1.1. A finite subset $S = \{g_1, \dots, g_s\}$ of an ideal a is a standard (Gröbner) bases for a if $(LT(g_1), \dots, LT(g_s)) = LT(a)$. In other words, S is a standard basis if the leading term of every element of a is divisible by at least one of the leading terms of the g_i .

THEOREM 1.3 The ring $k[X_1, \dots, X_n]$ is Noetherian i.e., every ideal is finitely generated.

PROOF. For $n = 1$, $k[X]$ is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on n . Note that the obvious map $k[X_1, \dots, X_{n-1}][X_n] \rightarrow k[X_1, \dots, X_n]$ is an isomorphism – this simply says that every polynomial f in n variables X_1, \dots, X_n can be expressed uniquely as a polynomial in X_n with coefficients in $k[X_1, \dots, X_{n-1}]$:

$$f(X_1, \dots, X_n) = a_0(X_1, \dots, X_{n-1})X_n^r + \dots + a_r(X_1, \dots, X_{n-1})$$

Thus the next lemma will complete the proof

LEMMA 1.3. If A is Noetherian, then so also is $A[X]$

PROOF. For a polynomial

$$f(X) = a_0 X^r + a_1 X^{r-1} + \dots + a_r, \quad a_i \in A, \quad a_0 \neq 0,$$

r is called the degree of f , and a_0 is its leading coefficient. We call 0 the leading coefficient of the polynomial 0. Let a be an ideal in $A[X]$. The leading coefficients of the polynomials in a form an ideal a' in A , and since A is Noetherian, a' will be finitely generated. Let g_1, \dots, g_m be elements of a whose leading coefficients generate a' , and let r be the maximum degree of g_i . Now let $f \in a$, and suppose f has degree $s > r$, say, $f = aX^s + \dots$. Then $a \in a'$, and so we can write $a = \sum b_i a_i$, $b_i \in A$, $a_i = \text{leading coefficient of } g_i$

Now $f - \sum b_i g_i X^{s-r_i}$, $r_i = \deg(g_i)$, has degree $< \deg(f)$. By continuing in this way, we find that $f \equiv f_t \pmod{(g_1, \dots, g_m)}$ With f_t a polynomial of degree $t < r$. For each $d < r$, let a_d be the subset of A consisting of 0 and the leading coefficients of all polynomials in a of degree d ; it is again an ideal in A . Let $g_{d,1}, \dots, g_{d,m_d}$ be polynomials of degree d whose leading coefficients generate a_d . Then the same argument as above shows that any polynomial f_d in a of degree d can be written $f_d \equiv f_{d-1} \pmod{(g_{d,1}, \dots, g_{d,m_d})}$ With f_{d-1} of degree $\leq d-1$. On applying this remark repeatedly we find that $f_t \in (g_{r-1,1}, \dots, g_{r-1,m_{r-1}}, \dots, g_{0,1}, \dots, g_{0,m_0})$ Hence

$$f_t \in (g_1, \dots, g_m, g_{r-1,1}, \dots, g_{r-1,m_{r-1}}, \dots, g_{0,1}, \dots, g_{0,m_0})$$

and so the polynomials g_1, \dots, g_{0,m_0} generate a

One of the great successes of category theory in computer science has been the development of a “unified theory” of the constructions underlying denotational semantics. In the untyped λ -calculus, any term may appear in the function position of an application. This means

that a model D of the λ -calculus must have the property that given a term t whose interpretation is $d \in D$, Also, the interpretation of a functional abstraction like $\lambda x . x$ is most conveniently defined as a function from D to D , which must then be regarded as an element of D . Let $\psi: [D \rightarrow D] \rightarrow D$ be the function that picks out elements of D to represent elements of $[D \rightarrow D]$ and $\phi: D \rightarrow [D \rightarrow D]$ be the function that maps elements of D to functions of D . Since $\psi(f)$ is intended to represent the function f as an element of D , it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi \circ \phi = id_{[D \rightarrow D]}$. Furthermore, we often want to view every element of D as representing some function from D to D and require that elements representing the same function be equal – that is

$$\psi(\phi(d)) = d$$

or

$$\psi \circ \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that ϕ and ψ are inverses--- that is, D is isomorphic to the space of functions from D to D that can be the interpretations of functional abstractions:

$$D \cong [D \rightarrow D]$$

Let us suppose we are working with the untyped λ -calculus, we need a solution of the equation $D \cong A + [D \rightarrow D]$, where A is some predetermined domain containing interpretations for elements of C . Each element of D corresponds to either an element of A or an element of $[D \rightarrow D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \rightarrow X]$ from domains to domains --- that is, finding domains X such that $X \cong A + [X \rightarrow X]$, and such that for any domain Y also satisfying this equation, there is an embedding of X to Y --- a pair of maps

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow f^R & \end{array}$$

Such that

$$f^R \circ f = id_X$$

$$f \circ f^R \subseteq id_Y$$

Where $f \subseteq g$ means that f approximates g in some ordering representing their information

content. The key shift of perspective from the domain-theoretic to the more general category-theoretic approach lies in considering F not as a function on domains, but as a *functor* on a category of domains. Instead of a least fixed point of the function, F .

Definition 1.3: Let K be a category and $F: K \rightarrow K$ as a functor. A fixed point of F is a pair (A, a) , where A is a **K-object** and $a: F(A) \rightarrow A$ is an isomorphism. A prefixed point of F is a pair (A, a) , where A is a **K-object** and a is any arrow from $F(A)$ to A .

Definition 1.4: An ω -chain in a category K is a diagram of the following form:

$$\Delta = D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \dots$$

Recall that a cocone μ of an ω -chain Δ is a K -object X and a collection of K -arrows $\{\mu_i: D_i \rightarrow X \mid i \geq 0\}$ such that $\mu_i = \mu_{i+1} \circ f_i$ for all $i \geq 0$. We sometimes write $\mu: \Delta \rightarrow X$ as a reminder of the arrangement of μ 's components. Similarly, a colimit $\mu: \Delta \rightarrow X$ is a cocone with the property that if $\nu: \Delta \rightarrow X'$ is also a cocone then there exists a unique mediating arrow $k: X \rightarrow X'$ such that for all $i \geq 0$, $\nu_i = k \circ \mu_i$.

Colimits of ω -chains are sometimes referred to as ω -colimits. Dually, an ω^{op} -chain in K is a diagram of the following form:

$$\Delta = D_0 \xleftarrow{f_0} D_1 \xleftarrow{f_1} D_2 \xleftarrow{f_2} \dots$$

A cone $\mu: X \rightarrow \Delta$ of an ω^{op} -chain Δ is a K -object X and a collection of K -arrows $\{\mu_i: D_i \rightarrow X \mid i \geq 0\}$ such that for all $i \geq 0$, $\mu_i = f_i \circ \mu_{i+1}$. An ω^{op} -limit of an ω^{op} -chain Δ is a cone $\mu: X \rightarrow \Delta$ with the property that if $\nu: X' \rightarrow \Delta$ is also a cone, then there exists a unique mediating arrow $k: X' \rightarrow X$ such that for all $i \geq 0$, $\mu_i \circ k = \nu_i$. We write \perp_k (or just \perp) for the distinguish initial object of K , when it has one, and $\perp \rightarrow A$ for the unique arrow from \perp to each K -object A . It is also convenient to write

$$\Delta^- = D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \dots$$

to denote all of Δ except D_0 and f_0 . By analogy, μ^- is $\{\mu_i \mid i \geq 1\}$. For the images of Δ and μ under F we write

$$F(\Delta) = F(D_0) \xrightarrow{F(f_0)} F(D_1) \xrightarrow{F(f_1)} F(D_2) \xrightarrow{F(f_2)} \dots$$

$$\text{and } F(\mu) = \{F(\mu_i) \mid i \geq 0\}$$

We write F^i for the i -fold iterated composition of F – that is,

$$F^0(f) = f, F^1(f) = F(f), F^2(f) = F(F(f))$$

,etc. With these definitions we can state that every monotonic function on a complete lattice has a least fixed point:

Lemma 1.4. Let K be a category with initial object \perp and let $F : K \rightarrow K$ be a functor. Define the ω -chain Δ by

$$\Delta = \perp \xrightarrow{F(\perp)} F(\perp) \xrightarrow{F(F(\perp))} F^2(\perp) \xrightarrow{\dots} \dots$$

If both $\mu : \Delta \rightarrow D$ and $F(\mu) : F(\Delta) \rightarrow F(D)$ are colimits, then (D, d) is an initial F -algebra, where $d : F(D) \rightarrow D$ is the mediating arrow from $F(\mu)$ to the cocone μ .

Theorem 1.4 Let a DAG G given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in G be specified. Then the product of these conditional distributions yields a joint probability distribution P of the variables, and (G, P) satisfies the Markov condition.

Proof. Order the nodes according to an ancestral ordering. Let X_1, X_2, \dots, X_n be the resultant ordering. Next define.

$$P(x_1, x_2, \dots, x_n) = P(x_n \mid pa_n) P(x_{n-1} \mid pa_{n-1}) \dots P(x_2 \mid pa_2) P(x_1 \mid pa_1),$$

Where PA_i is the set of parents of X_i of in G and $P(x_i \mid pa_i)$ is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, $0 \leq P(x_1, x_2, \dots, x_n) \leq 1$ for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the Markov condition is satisfied. To do this, we need show for $1 \leq k \leq n$ that whenever

$$P(pa_k) \neq 0, \text{ if } P(nd_k \mid pa_k) \neq 0$$

$$\text{and } P(x_k \mid pa_k) \neq 0$$

$$\text{then } P(x_k \mid nd_k, pa_k) = P(x_k \mid pa_k),$$

Where ND_k is the set of nondescendants of X_k of in G . Since $PA_k \subseteq ND_k$, we need only show $P(x_k \mid nd_k) = P(x_k \mid pa_k)$. First for a given k , order the nodes so that all and only nondescendants of X_k precede X_k in the ordering. Note that this ordering depends on k , whereas the ordering in the first part of the proof does not. Clearly then

$$ND_k = \{X_1, X_2, \dots, X_{k-1}\}$$

Let

$$D_k = \{X_{k+1}, X_{k+2}, \dots, X_n\}$$

follows \sum_{d_k}

We define the m^{th} cyclotomic field to be the field $Q[x]/(\Phi_m(x))$ Where $\Phi_m(x)$ is the m^{th} cyclotomic polynomial. $Q[x]/(\Phi_m(x))$ has degree $\varphi(m)$ over Q since $\Phi_m(x)$ has degree $\varphi(m)$. The roots of $\Phi_m(x)$ are just the primitive m^{th} roots of unity, so the complex embeddings of $Q[x]/(\Phi_m(x))$ are simply the $\varphi(m)$ maps

$$\sigma_k : Q[x]/(\Phi_m(x)) \mapsto C,$$

$$1 \leq k < m, (k, m) = 1, \text{ where}$$

$$\sigma_k(x) = \xi_m^k,$$

ξ_m being our fixed choice of primitive m^{th} root of unity. Note that $\xi_m^k \in Q(\xi_m)$ for every k ; it follows that $Q(\xi_m) = Q(\xi_m^k)$ for all k relatively prime to m . In particular, the images of the σ_i coincide, so $Q[x]/(\Phi_m(x))$ is Galois over Q .

This means that we can write $Q(\xi_m)$ for $Q[x]/(\Phi_m(x))$ without much fear of ambiguity; we will do so from now on, the identification being $\xi_m \mapsto x$. One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another, or intersections or compositums; all of these things take place considering them as subfield of C . We now investigate some basic properties of cyclotomic fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in $Q(\xi_m)$. Note, for example, that if m is odd, then $-\xi_m$ is a $2m^{\text{th}}$ root of unity. We will show that this

is the only way in which one can obtain any non- m^{th} roots of unity.

LEMMA 1.5 If m divides n , then $Q(\xi_m)$ is contained in $Q(\xi_n)$

PROOF. Since $\xi_m^{n/m} = \xi_n$, we have $\xi_m \in Q(\xi_n)$, so the result is clear

LEMMA 1.6 If m and n are relatively prime, then

$$Q(\xi_m, \xi_n) = Q(\xi_{mn})$$

and

$$Q(\xi_m) \cap Q(\xi_n) = Q$$

(Recall the $Q(\xi_m, \xi_n)$ is the compositum of $Q(\xi_m)$ and $Q(\xi_n)$)

PROOF. One checks easily that $\xi_m \xi_n$ is a primitive mn^{th} root of unity, so that

$$Q(\xi_{mn}) \subseteq Q(\xi_m, \xi_n)$$

$$\begin{aligned} [Q(\xi_m, \xi_n) : Q] &\leq [Q(\xi_m) : Q][Q(\xi_n) : Q] \\ &= \varphi(m)\varphi(n) = \varphi(mn); \end{aligned}$$

Since $[Q(\xi_{mn}) : Q] = \varphi(mn)$; this implies that

$Q(\xi_m, \xi_n) = Q(\xi_{mn})$ We know that $Q(\xi_m, \xi_n)$ has degree $\varphi(mn)$ over Q , so we must have

$$[Q(\xi_m, \xi_n) : Q(\xi_m)] = \varphi(n)$$

and

$$[Q(\xi_m, \xi_n) : Q(\xi_n)] = \varphi(m)$$

$$[Q(\xi_m) : Q(\xi_m) \cap Q(\xi_n)] \geq \varphi(m)$$

And thus that $Q(\xi_m) \cap Q(\xi_n) = Q$

PROPOSITION 1.2 For any m and n

$$Q(\xi_m, \xi_n) = Q(\xi_{[m,n]})$$

And

$$Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{(m,n)});$$

here $[m, n]$ and (m, n) denote the least common multiple and the greatest common divisor of m and n , respectively.

PROOF. Write $m = p_1^{e_1} \dots p_k^{e_k}$ and $p_1^{f_1} \dots p_k^{f_k}$ where the p_i are distinct primes. (We allow e_i or f_i to be zero)

$$Q(\xi_m) = Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}}) \dots Q(\xi_{p_k^{e_k}})$$

and

$$Q(\xi_n) = Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}}) \dots Q(\xi_{p_k^{f_k}})$$

Thus

$$\begin{aligned} Q(\xi_m, \xi_n) &= Q(\xi_{p_1^{e_1}}) \dots Q(\xi_{p_2^{e_k}})Q(\xi_{p_1^{f_1}}) \dots Q(\xi_{p_k^{f_k}}) \\ &= Q(\xi_{p_1^{e_1}})Q(\xi_{p_1^{f_1}}) \dots Q(\xi_{p_k^{e_k}})Q(\xi_{p_k^{f_k}}) \\ &= Q(\xi_{p_1^{\max(e_1, f_1)}}) \dots Q(\xi_{p_1^{\max(e_k, f_k)}}) \\ &= Q(\xi_{p_1^{\max(e_1, f_1)} \dots p_1^{\max(e_k, f_k)}}) \\ &= Q(\xi_{[m,n]}); \end{aligned}$$

An entirely similar computation shows that $Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{(m,n)})$

Mutual information measures the information transferred when x_i is sent and y_i is received, and is defined as

$$I(x_i, y_i) = \log_2 \frac{P(x_i/y_i)}{P(x_i)} \text{ bits} \quad (1)$$

In a noise-free channel, **each** y_i is uniquely connected to the corresponding x_i , and so they constitute an input-output pair (x_i, y_i) for which

$$P(x_i/y_j) = 1 \text{ and } I(x_i, y_j) = \log_2 \frac{1}{P(x_i)} \text{ bits;}$$

that is, the transferred information is equal to the self-information that corresponds to the input x_i . In

a very noisy channel, the output y_i and input x_i would be completely uncorrelated, and so

$$P(x_i/y_j) = P(x_i) \text{ and also } I(x_i, y_j) = 0; \text{ that is,}$$

there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X, Y) = \sum_{i,j} P(x_i, y_j) I(x_i, y_j) = \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)}$$

bits per symbol. This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i, y_j) = P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

$$P(y_j) = \sum_i P(y_j/x_i)P(x_i)$$

$$P(x_i) = \sum_j P(x_i/y_j)P(y_j)$$

Then

$$\begin{aligned} I(X, Y) &= \sum_{i,j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i)} \right] \\ &\quad - \sum_{i,j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i/y_j)} \right] \\ &\quad + \sum_{i,j} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i)} \right] \\ &= \sum_i \left[P(x_i/y_j)P(y_j) \right] \log_2 \frac{1}{P(x_i)} \\ &= \sum_i P(x_i) \log_2 \frac{1}{P(x_i)} = H(X) \\ I(X, Y) &= H(X) - H(X/Y) \end{aligned}$$

Where

$$H(X/Y) = \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i/y_j)} \quad \text{is}$$

usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol y_j provides $H(X) - H(X/Y)$ bits of information. This difference is the mutual information of the channel. *Mutual Information: Properties* Since

$$P(x_i/y_j)P(y_j) = P(y_j/x_i)P(x_i)$$

The mutual information fits the condition

$$I(X, Y) = I(Y, X)$$

And by interchanging input and output it is also true that

$$I(X, Y) = H(Y) - H(Y/X)$$

Where

$$H(Y) = \sum_j P(y_j) \log_2 \frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol after knowing the corresponding output symbol

$$I(X, Y) = H(X) - H(X/Y)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and in spite of the fact that for some y_j , $H(X/y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

Then

$$-I(X, Y) = \sum_{i,j} P(x_i, y_j) \frac{P(x_i)P(y_j)}{P(x_i, y_j)} \leq 0$$

Because this expression is of the form

$$\sum_{i=1}^M P_i \log_2 \left(\frac{Q_i}{P_i} \right) \leq 0$$

The above expression can be applied due to the factor $P(x_i)P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity Q_i , which in this expression is a dummy variable that fits the condition $\sum_i Q_i \leq 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$\begin{aligned} H(X, Y) &= \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i, y_j)} \\ &= \sum_{i,j} P(x_i, y_j) \log_2 \frac{P(x_i)P(y_j)}{P(x_i, y_j)} \\ &\quad + \sum_{i,j} P(x_i, y_j) \log_2 \frac{1}{P(x_i)P(y_j)} \end{aligned}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$P(x_1) = \alpha$ and $P(x_2) = 1 - \alpha$, and transition probabilities

$$P(y_3/x_2)=1-p \text{ and } P(y_2/x_1)=0,$$

$$\text{and } P(y_3/x_1)=0$$

$$\text{and } P(y_1/x_2)=p$$

$$\text{and } P(y_3/x_2)=1-p$$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets F_n and whose density functions exhibit no dependence on the state s , let n be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean n -space. $p(y|x)$ for the density $p_n(y_1, \dots, y_n | x_1, \dots, x_n)$ and F for F_n . For any real number a , let

$$A = \left\{ (x, y) : \log \frac{p(y|x)}{p(y)} > a \right\} \quad (1)$$

Then for each positive integer u , there is a code (u, n, λ) such that

$$\lambda \leq ue^{-a} + P\{(X, Y) \notin A\} + P\{X \notin F\}$$

Where

$$P\{(X, Y) \in A\} = \int_A \dots \int p(x, y) dx dy, \quad p(x, y) = p(x)p(y|x)$$

and

$$P\{X \in F\} = \int_F \dots \int p(x) dx$$

Proof: A sequence $x^{(1)} \in F$ such that

$$P\{Y \in A_{x^{(1)}} | X = x^{(1)}\} \geq 1 - \varepsilon$$

$$\text{where } A_x = \{y : (x, y) \in A\};$$

Choose the decoding set B_1 to be $A_{x^{(1)}}$. Having chosen $x^{(1)}, \dots, x^{(k-1)}$ and B_1, \dots, B_{k-1} , select $x^{(k)} \in F$ such that

$$P\left\{Y \in A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i \mid X = x^{(k)}\right\} \geq 1 - \varepsilon;$$

Set $B_k = A_{x^{(k)}} - \bigcup_{i=1}^{k-1} B_i$. If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets $B_i, i = 1, 2, \dots, u$, form the desired code. Thus assume that the process terminates after t steps. (Conceivably $t = 0$). We will show $t \geq u$ by showing that $\varepsilon \leq te^{-a} + P\{(X, Y) \notin A\} + P\{X \notin F\}$. We proceed as follows.

Let

$$B = \bigcup_{j=1}^t B_j. \text{ (If } t = 0, \text{ take } B = \emptyset). \text{ Then}$$

$$P\{(X, Y) \in A\} = \int_{(x, y) \in A} p(x, y) dx dy$$

$$= \int_x p(x) \int_{y \in A_x} p(y|x) dy dx$$

$$= \int_x p(x) \int_{y \in B \cap A_x} p(y|x) dy dx + \int_x p(x)$$

C. Algorithms

Ideals. Let A be a ring. Recall that an ideal a in A is a subset such that a is a subgroup of A regarded as a group under addition;

$$a \in a, r \in A \Rightarrow ra \in a$$

The ideal generated by a subset S of A is the intersection of all ideals A containing S ----- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form $\sum r_i s_i$ with

$$(2) r_i \in A, s_i \in S. \text{ When } S = \{s_1, \dots, s_m\}, \text{ we shall}$$

write (s_1, \dots, s_m) for the ideal it generates.

Let a and b be ideals in A . The set $\{a+b | a \in a, b \in b\}$ is an ideal, denoted by

$a+b$. The ideal generated by $\{ab | a \in a, b \in b\}$ is denoted by ab . Note that

$ab \subset a \cap b$. Clearly ab consists of all finite sums $\sum a_i b_i$ with $a_i \in a$ and $b_i \in b$, and if

$a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_n)$, then

$ab = (a_1 b_1, \dots, a_i b_j, \dots, a_m b_n)$. Let a be an ideal

of A . The set of cosets of a in A forms a ring A/a ,

and $a \mapsto a+a$ is a homomorphism

$\phi: A \mapsto A/a$. The map $b \mapsto \phi^{-1}(b)$ is a one to

one correspondence between the ideals of A/a

and the ideals of A containing a . An ideal p is

prime if $p \neq A$ and $ab \in p \Rightarrow a \in p$ or $b \in p$.

Thus p is prime if and only if A/p is nonzero

and has the property that

$ab = 0, b \neq 0 \Rightarrow a = 0$, i.e., A/p is an

integral domain. An ideal m is maximal if $m \neq A$

and there does not exist an ideal n contained

strictly between m and A . Thus m is maximal if

and only if A/m has no proper nonzero ideals, and

so is a field. Note that m maximal $\Rightarrow m$ prime.

The ideals of $A \times B$ are all of the form $a \times b$, with

a and b ideals in A and B . To see this, note that

if c is an ideal in $A \times B$ and $(a, b) \in c$, then
 $(a, 0) = (a, b)(1, 0) \in c$ and
 $(0, b) = (a, b)(0, 1) \in c$. This shows that
 $c = a \times b$ with
 $a = \{a \mid (a, b) \in c \text{ some } b \in b\}$
and
 $b = \{b \mid (a, b) \in c \text{ some } a \in a\}$

Let A be a ring. An A -algebra is a ring B together with a homomorphism $i_B : A \rightarrow B$. A homomorphism of A -algebra $B \rightarrow C$ is a homomorphism of rings $\varphi : B \rightarrow C$ such that $\varphi(i_B(a)) = i_C(a)$ for all $a \in A$. An A -algebra B is said to be *finitely generated* (or of *finite-type* over A) if there exist elements $x_1, \dots, x_n \in B$ such that every element of B can be expressed as a polynomial in the x_i with coefficients in $i(A)$, i.e., such that the homomorphism $A[X_1, \dots, X_n] \rightarrow B$ sending X_i to x_i is surjective. A ring homomorphism $A \rightarrow B$ is *finite*, and B is finitely generated as an A -module. Let k be a field, and let A be a k -algebra. If $1 \neq 0$ in A , then the map $k \rightarrow A$ is injective, we can identify k with its image, i.e., we can regard k as a subring of A . If $1=0$ in a ring R , the R is the zero ring, i.e., $R = \{0\}$. **Polynomial rings.**

Let k be a field. A *monomial* in X_1, \dots, X_n is an expression of the form $X_1^{a_1} \dots X_n^{a_n}$, $a_j \in \mathbb{N}$. The *total degree* of the monomial is $\sum a_i$. We sometimes abbreviate it by X^α , $\alpha = (a_1, \dots, a_n) \in \mathbb{N}^n$. The elements of the polynomial ring $k[X_1, \dots, X_n]$ are finite sums $\sum c_{a_1, \dots, a_n} X_1^{a_1} \dots X_n^{a_n}$, $c_{a_1, \dots, a_n} \in k$, $a_j \in \mathbb{N}$.

With the obvious notions of equality, addition and multiplication. Thus the monomials form a basis for $k[X_1, \dots, X_n]$ as a k -vector space. The ring $k[X_1, \dots, X_n]$ is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial $f(X_1, \dots, X_n)$ is *irreducible* if it is nonconstant and has only the obvious factorizations, i.e., $f = gh \Rightarrow g$ or h is constant. **Division in $k[X]$.** The division algorithm allows us to divide a

nonzero polynomial into another: let f and g be polynomials in $k[X]$ with $g \neq 0$; then there exist unique polynomials $q, r \in k[X]$ such that $f = qg + r$ with either $r = 0$ or $\deg r < \deg g$. Moreover, there is an algorithm for deciding whether $f \in (g)$, namely, find r and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in $k[X]$ to a single generator by successively replacing each pair of generators with their greatest common divisor.

(Pure) **lexicographic ordering (lex).** Here monomials are ordered by lexicographic (dictionary) order. More precisely, let $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_n)$ be two elements of \mathbb{N}^n ; then $\alpha > \beta$ and $X^\alpha > X^\beta$ (lexicographic ordering) if, in the vector difference $\alpha - \beta \in \mathbb{N}^n$, the left most nonzero entry is positive. For example,

$XY^2 > Y^3Z^4$; $X^3Y^2Z^4 > X^3Y^2Z$. Note that this isn't quite how the dictionary would order them: it would put $XXXYYYZZZZ$ after $XXXYYZ$. **Graded reverse lexicographic order (grevlex).** Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, $\alpha > \beta$ if $\sum a_i > \sum b_i$, or $\sum a_i = \sum b_i$ and in $\alpha - \beta$ the right most nonzero entry is negative. For example:

$X^4Y^4Z^7 > X^5Y^5Z^4$ (total degree greater)
 $XY^5Z^2 > X^4YZ^3$, $X^5YZ > X^4YZ^2$.

Orderings on $k[X_1, \dots, X_n]$. Fix an ordering on the monomials in $k[X_1, \dots, X_n]$. Then we can write an element f of $k[X_1, \dots, X_n]$ in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write

$f = 4XY^2Z + 4Z^2 - 5X^3 + 7X^2Z^2$
as
 $f = -5X^3 + 7X^2Z^2 + 4XY^2Z + 4Z^2$ (lex)
or
 $f = 4XY^2Z + 7X^2Z^2 - 5X^3 + 4Z^2$ (grevlex)

Let $\sum a_\alpha X^\alpha \in k[X_1, \dots, X_n]$, in decreasing order:

$$f = a_{\alpha_0} X^{\alpha_0} + a_{\alpha_1} X^{\alpha_1} + \dots, \quad \alpha_0 > \alpha_1 > \dots, \quad \alpha_0 \neq 0$$

Then we define.

- The *multidegree* of f to be $\text{multdeg}(f) = \alpha_0$;
- The *leading coefficient* of f to be $LC(f) = a_{\alpha_0}$;
- The *leading monomial* of f to be $LM(f) = X^{\alpha_0}$;
- The *leading term* of f to be $LT(f) = a_{\alpha_0} X^{\alpha_0}$

For the polynomial $f = 4XY^2Z + \dots$, the multidegree is (1,2,1), the leading coefficient is 4, the leading monomial is XY^2Z , and the leading term is $4XY^2Z$. **The division algorithm in $k[X_1, \dots, X_n]$.** Fix a monomial ordering in \square^2 .

Suppose given a polynomial f and an ordered set (g_1, \dots, g_s) of polynomials; the division algorithm then constructs polynomials a_1, \dots, a_s and r such that $f = a_1g_1 + \dots + a_s g_s + r$ Where either $r = 0$ or no monomial in r is divisible by any of $LT(g_1), \dots, LT(g_s)$

Step 1: If $LT(g_1) | LT(f)$, divide g_1 into f to get $f = a_1g_1 + h$, $a_1 = \frac{LT(f)}{LT(g_1)} \in k[X_1, \dots, X_n]$

If $LT(g_1) | LT(h)$, repeat the process until $f = a_1g_1 + f_1$ (different a_1) with $LT(f_1)$ not divisible by $LT(g_1)$. Now divide g_2 into f_1 , and so on, until $f = a_1g_1 + \dots + a_s g_s + r_1$ With $LT(r_1)$ not divisible by any $LT(g_1), \dots, LT(g_s)$

Step 2: Rewrite $r_1 = LT(r_1) + r_2$, and repeat Step 1 with r_2 for f : $f = a_1g_1 + \dots + a_s g_s + LT(r_1) + r_3$ (different a_i 's)

Monomial ideals. In general, an ideal a will contain a polynomial without containing the individual terms of the polynomial; for example, the ideal $a = (Y^2 - X^3)$ contains $Y^2 - X^3$ but not Y^2 or X^3 .

DEFINITION 1.5. An ideal a is *monomial* if $\sum c_\alpha X^\alpha \in a \Rightarrow X^\alpha \in a$ all α with $c_\alpha \neq 0$.

PROPOSITION 1.3. Let a be a *monomial ideal*, and let $A = \{\alpha | X^\alpha \in a\}$. Then A satisfies the condition $\alpha \in A, \beta \in \square^n \Rightarrow \alpha + \beta \in A$ (*) And a is the k -subspace of $k[X_1, \dots, X_n]$ generated by the $X^\alpha, \alpha \in A$. Conversely, if A is a subset of \square^n satisfying (*), then the k -subspace a of $k[X_1, \dots, X_n]$ generated by $\{X^\alpha | \alpha \in A\}$ is a monomial ideal.

PROOF. It is clear from its definition that a monomial ideal a is the k -subspace of $k[X_1, \dots, X_n]$ generated by the set of monomials it contains. If $X^\alpha \in a$ and $X^\beta \in k[X_1, \dots, X_n]$.

If a permutation is chosen uniformly and at random from the $n!$ possible permutations in S_n , then the counts $C_j^{(n)}$ of cycles of length j are dependent random variables. The joint distribution of $C^{(n)} = (C_1^{(n)}, \dots, C_n^{(n)})$ follows from Cauchy's formula, and is given by

$$P[C^{(n)} = c] = \frac{1}{n!} N(n, c) = 1 \left\{ \sum_{j=1}^n j c_j = n \right\} \prod_{j=1}^n \left(\frac{1}{j} \right)^{c_j} \frac{1}{c_j!}, \quad (1.1)$$

for $c \in \square_+^n$.

Lemma 1.7 For nonnegative integers m_1, \dots, m_n ,

$$E \left(\prod_{j=1}^n (C_j^{(n)})^{m_j} \right) = \left(\prod_{j=1}^n \left(\frac{1}{j} \right)^{m_j} \right) 1 \left\{ \sum_{j=1}^n j m_j \leq n \right\} \quad (1.4)$$

Proof. This can be established directly by exploiting cancellation of the form $c_j^{[m_j]} / c_j! = 1 / (c_j - m_j)!$ when $c_j \geq m_j$, which occurs between the ingredients in Cauchy's formula and the falling factorials in the moments. Write $m = \sum j m_j$. Then, with the first sum indexed by $c = (c_1, \dots, c_n) \in \square_+^n$ and the last sum indexed by $d = (d_1, \dots, d_n) \in \square_+^n$ via the correspondence $d_j = c_j - m_j$, we have

$$\begin{aligned}
 E\left(\prod_{j=1}^n (C_j^{(n)})^{[m_j]}\right) &= \sum_c P[C^{(n)} = c] \prod_{j=1}^n (c_j)^{[m_j]} \\
 &= \sum_{c: c_j \geq m_j \text{ for all } j} 1 \left\{ \sum_{j=1}^n j c_j = n \right\} \prod_{j=1}^n \frac{(c_j)^{[m_j]}}{j^{c_j} c_j!} \\
 &= \prod_{j=1}^n \frac{1}{j^{m_j}} \sum_d 1 \left\{ \sum_{j=1}^n j d_j = n - m \right\} \prod_{j=1}^n \frac{1}{j^{d_j} (d_j)!}
 \end{aligned}$$

This last sum simplifies to the indicator $1(m \leq n)$, corresponding to the fact that if $n - m \geq 0$, then $d_j = 0$ for $j > n - m$, and a random permutation in S_{n-m} must have some cycle structure (d_1, \dots, d_{n-m}) . The moments of $C_j^{(n)}$ follow immediately as

$$E(C_j^{(n)})^{[r]} = j^{-r} 1\{jr \leq n\} \quad (1.2)$$

We note for future reference that (1.4) can also be written in the form

$$E\left(\prod_{j=1}^n (C_j^{(n)})^{[m_j]}\right) = E\left(\prod_{j=1}^n Z_j^{[m_j]}\right) 1\left\{\sum_{j=1}^n j m_j \leq n\right\}, \quad (1.3)$$

Where the Z_j are independent Poisson-distribution random variables that satisfy $E(Z_j) = 1/j$

The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts C_j^n , we find the distribution of C_j^n using a combinatorial approach combined with the inclusion-exclusion formula.

Lemma 1.8. For $1 \leq j \leq n$,

$$P[C_j^{(n)} = k] = \frac{j^{-k}}{k!} \sum_{l=0}^{[n/j]-k} (-1)^l \frac{j^{-l}}{l!} \quad (1.1)$$

Proof. Consider the set I of all possible cycles of length j , formed with elements chosen from $\{1, 2, \dots, n\}$, so that $|I| = n^{[j]Vj}$. For each $\alpha \in I$, consider the ‘‘property’’ G_α of having α ; that is, G_α is the set of permutations $\pi \in S_n$ such that α is one of the cycles of π . We then have $|G_\alpha| = (n - j)!$, since the elements of $\{1, 2, \dots, n\}$ not in α must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term S_r , which is the sum of the probabilities of the r -fold intersection of properties, summing over all sets of r distinct properties. There are two cases to consider. If the r properties are indexed by r cycles having no elements in common, then the intersection specifies how rj elements are

moved by the permutation, and there are $(n - rj)! 1(rj \leq n)$ permutations in the intersection. There are $n^{[rj]} / (j^r r!)$ such intersections. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the r -fold intersection is empty. Thus

$$\begin{aligned}
 S_r &= (n - rj)! 1(rj \leq n) \\
 &\times \frac{n^{[rj]} 1}{j^r r! n!} = 1(rj \leq n) \frac{1}{j^r r!}
 \end{aligned}$$

Finally, the inclusion-exclusion series for the number of permutations having exactly k properties is

$$\sum_{l \geq 0} (-1)^l \binom{k+l}{l} S_{k+l},$$

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute $j=1$ in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For $k = 0, 1, \dots, n$,

$$P[C_1^{(n)} = k] = \frac{1}{k!} \sum_{l=0}^{n-k} (-1)^l \frac{1}{l!}, \quad (1.2)$$

and the moments of $C_1^{(n)}$ follow from (1.2) with $j = 1$. In particular, for $n \geq 2$, the mean and variance of $C_1^{(n)}$ are both equal to 1. The joint distribution of $(C_1^{(n)}, \dots, C_b^{(n)})$ for any $1 \leq b \leq n$ has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any $c = (c_1, \dots, c_b) \in \square_+^b$ with $m = \sum_{i=1}^b i c_i$,

$$\begin{aligned}
 P[(C_1^{(n)}, \dots, C_b^{(n)}) = c] \\
 &= \left\{ \prod_{i=1}^b \left(\frac{1}{i}\right)^{c_i} \frac{1}{c_i!} \right\} \sum_{\substack{l \geq 0 \text{ with} \\ \sum_{i=1}^b i l_i \leq n - m}} (-1)^{l_1 + \dots + l_b} \prod_{i=1}^b \left(\frac{1}{i}\right)^{l_i} \frac{1}{l_i!}
 \end{aligned} \quad (1.3)$$

The joint moments of the first b counts $C_1^{(n)}, \dots, C_b^{(n)}$ can be obtained directly from (1.2) and (1.3) by setting $m_{b+1} = \dots = m_n = 0$

The limit distribution of cycle counts

It follows immediately from Lemma 1.2 that for each fixed j , as $n \rightarrow \infty$,

$$P[C_j^{(n)} = k] \rightarrow \frac{j^{-k}}{k!} e^{-1/j}, \quad k = 0, 1, 2, \dots,$$

So that $C_j^{(n)}$ converges in distribution to a random variable Z_j having a Poisson distribution with mean $1/j$; we use the notation $C_j^{(n)} \rightarrow_d Z_j$

where $Z_j \square P_o(1/j)$ to describe this. Infact, the limit random variables are independent.

Theorem 1.6 The process of cycle counts converges in distribution to a Poisson process of \square with intensity j^{-1} . That is, as $n \rightarrow \infty$,
 $(C_1^{(n)}, C_2^{(n)}, \dots) \rightarrow_d (Z_1, Z_2, \dots)$ (1.1)

Where the $Z_j, j = 1, 2, \dots$ are independent Poisson-distributed random variables with
 $E(Z_j) = \frac{1}{j}$

Proof. To establish the converges in distribution one shows that for each fixed $b \geq 1$, as $n \rightarrow \infty$,

$$P[(C_1^{(n)}, \dots, C_b^{(n)}) = c] \rightarrow P[(Z_1, \dots, Z_b) = c]$$

Error rates

The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when $b=1$. Using properties of alternating series with decreasing terms, for $k = 0, 1, \dots, n$,

$$\frac{1}{k!} \left(\frac{1}{(n-k+1)!} - \frac{1}{(n-k+2)!} \right) \leq |P[C_1^{(n)} = k] - P[Z_1 = k]| \leq \frac{1}{k!(n-k+1)!}$$

It follows that

$$\frac{2^{n+1}}{(n+1)!} \frac{n}{n+2} \leq \sum_{k=0}^n |P[C_1^{(n)} = k] - P[Z_1 = k]| \leq \frac{2^{n+1}-1}{(n+1)!} \quad (1.11)$$

Since

$$P[Z_1 > n] = \frac{e^{-1}}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots \right) < \frac{1}{(n+1)!}$$

We see from (1.11) that the total variation distance between the distribution $L(C_1^{(n)})$ of $C_1^{(n)}$ and the distribution $L(Z_1)$ of Z_1

Establish the asymptotics of $P[A_n(C^{(n)})]$ under conditions (A_0) and (B_{01}) , where

$$A_n(C^{(n)}) = \prod_{1 \leq i \leq n} \prod_{r_i+1 \leq j \leq r_i} \{C_{ij}^{(n)} = 0\},$$

and $\zeta_i = (r_i' / r_{id}) - 1 = O(i^{-g'})$ as $i \rightarrow \infty$, for some $g' > 0$. We start with the expression

$$P[A_n(C^{(n)})] = \frac{P[T_{0n}(Z') = n]}{P[T_{0n}(Z) = n]}$$

$$\prod_{\substack{1 \leq i \leq n \\ r_i+1 \leq j \leq r_i}} \left\{ 1 - \frac{\theta}{ir_i} (1 + E_{i0}) \right\} \quad (1.1)$$

$$P[T_{0n}(Z') = n] = \frac{\theta d}{n} \exp \left\{ \sum_{i \geq 1} [\log(1 + i^{-1}\theta d) - i^{-1}\theta d] \right\} \left\{ 1 + O(n^{-1}\phi_{\{1,2,7\}}'(n)) \right\} \quad (1.2)$$

and

$$P[T_{0n}(Z) = n] = \frac{\theta d}{n} \exp \left\{ \sum_{i \geq 1} [\log(1 + i^{-1}\theta d) - i^{-1}\theta d] \right\} \left\{ 1 + O(n^{-1}\phi_{\{1,2,7\}}(n)) \right\} \quad (1.3)$$

Where $\phi_{\{1,2,7\}}'(n)$ refers to the quantity derived from Z' . It thus follows that $P[A_n(C^{(n)})] \square Kn^{-\theta(1-d)}$ for a constant K , depending on Z and the r_i' and computable explicitly from (1.1) – (1.3), if Conditions (A_0) and (B_{01}) are satisfied and if $\zeta_i^* = O(i^{-g'})$ from some $g' > 0$, since, under these circumstances, both $n^{-1}\phi_{\{1,2,7\}}'(n)$ and $n^{-1}\phi_{\{1,2,7\}}(n)$ tend to zero as $n \rightarrow \infty$. In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order n^{-1} if $g' > 1$.

For $0 \leq b \leq n/8$ and $n \geq n_0$, with n_0

$$d_{TV}(L(C[1,b]), L(Z[1,b])) \leq d_{TV}(L(C[1,b]), L(Z[1,b])) \leq \varepsilon_{\{7,7\}}(n,b),$$

Where $\varepsilon_{\{7,7\}}(n,b) = O(b/n)$ under Conditions $(A_0), (D_1)$ and (B_{11}) Since, by the Conditioning Relation,

$$L(C[1,b] | T_{0b}(C) = l) = L(Z[1,b] | T_{0b}(Z) = l),$$

It follows by direct calculation that

$$\begin{aligned}
 & d_{TV}(L(C[1, b]), L(Z[1, b])) \\
 &= d_{TV}(L(T_{0b}(C)), L(T_{0b}(Z))) \\
 &= \max_A \sum_{r \in A} P[T_{0b}(Z) = r] \\
 & \left\{ 1 - \frac{P[T_{bn}(Z) = n - r]}{P[T_{0n}(Z) = n]} \right\} \quad (1.4)
 \end{aligned}$$

Suppressing the argument Z from now on, we thus obtain

$$\begin{aligned}
 & d_{TV}(L(C[1, b]), L(Z[1, b])) \\
 &= \sum_{r \geq 0} P[T_{0b} = r] \left\{ 1 - \frac{P[T_{bn} = n - r]}{P[T_{0n} = n]} \right\}_+ \\
 &\leq \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0b} = n]} \\
 &\times \left\{ \sum_{s=0}^n P[T_{0b} = s] (P[T_{bn} = n - s] - P[T_{bn} = n - r]) \right\}_+ \\
 &\leq \sum_{r > n/2} P[T_{0b} = r] + \sum_{r=0}^{[n/2]} P[T_{0b} = r] \\
 &\times \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{\{P[T_{bn} = n - s] - P[T_{bn} = n - r]\}}{P[T_{0n} = n]} \\
 &+ \sum_{s=0}^{[n/2]} P[T_{0b} = r] \sum_{s=[n/2]+1}^n P[T_{0b} = s] P[T_{bn} = n - s] / P[T_{0n} = n]
 \end{aligned}$$

The first sum is at most $2n^{-1}ET_{0b}$; the third is bound by

$$\begin{aligned}
 & \left(\max_{n/2 < s \leq n} P[T_{0b} = s] \right) / P[T_{0n} = n] \\
 &\leq \frac{2\varepsilon_{\{10.5(1)\}}(n/2, b)}{n} \frac{3n}{\theta P_\theta[0, 1]}, \\
 &\frac{3n}{\theta P_\theta[0, 1]} 4n^{-2} \phi_{\{10.8\}}^*(n) \sum_{r=0}^{[n/2]} P[T_{0b} = r] \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{1}{2} |r - s| \\
 &\leq \frac{12\phi_{\{10.8\}}^*(n)}{\theta P_\theta[0, 1]} \frac{ET_{0b}}{n}
 \end{aligned}$$

Hence we may take

$$\begin{aligned}
 \varepsilon_{\{7.7\}}(n, b) &= 2n^{-1}ET_{0b}(Z) \left\{ 1 + \frac{6\phi_{\{10.8\}}^*(n)}{\theta P_\theta[0, 1]} \right\} P \\
 &+ \frac{6}{\theta P_\theta[0, 1]} \varepsilon_{\{10.5(1)\}}(n/2, b) \quad (1.5)
 \end{aligned}$$

Required order under Conditions $(A_0), (D_1)$ and (B_{11}) , if $S(\infty) < \infty$. If not, $\phi_{\{10.8\}}^*(n)$ can be

replaced by $\phi_{\{10.11\}}^*(n)$ in the above, which has the required order, without the restriction on the r_i implied by $S(\infty) < \infty$. Examining the Conditions $(A_0), (D_1)$ and (B_{11}) , it is perhaps surprising to find that (B_{11}) is required instead of just (B_{01}) ; that is, that we should need $\sum_{l \geq 2} l\varepsilon_{il} = O(i^{-a_1})$

to hold for some $a_1 > 1$. A first observation is that a similar problem arises with the rate of decay of ε_{i1}

as well. For this reason, n_1 is replaced by n_1 . This makes it possible to replace condition (A_1) by the weaker pair of conditions (A_0) and (D_1) in the eventual assumptions needed for $\varepsilon_{\{7.7\}}(n, b)$ to be of order $O(b/n)$; the decay rate requirement of order $i^{-1-\gamma}$ is shifted from ε_{i1} itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the $\varepsilon_{il}, l \geq 2$, than are made in (B_{11}) . The critical point of the proof is seen where the initial estimate of the difference $P[T_{bn}^{(m)} = s] - P[T_{bn}^{(m)} = s + 1]$. The factor $\varepsilon_{\{10.10\}}(n)$, which should be small, contains a far

tail element from n_1 of the form $\phi_1^\theta(n) + u_1^*(n)$, which is only small if $a_1 > 1$, being otherwise of order $O(n^{-a_1+\delta})$ for any $\delta > 0$, since $a_2 > 1$ is in any case assumed. For $s \geq n/2$, this gives rise to a contribution of order $O(n^{-1-a_1+\delta})$ in the estimate of the difference $P[T_{bn} = s] - P[T_{bn} = s + 1]$, which, in the remainder of the proof, is translated into a contribution of order $O(n^{-1-a_1+\delta})$ for differences of the form $P[T_{bn} = s] - P[T_{bn} = s + 1]$, finally leading to a contribution of order $bn^{-a_1+\delta}$ for any $\delta > 0$ in $\varepsilon_{\{7.7\}}(n, b)$. Some improvement would seem to be possible, defining the function g by $g(w) = 1_{\{w=s\}} - 1_{\{w=s+t\}}$, differences that are of the form $P[T_{bn} = s] - P[T_{bn} = s + t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi_1^\theta(n) + u_1^*(n)$. Then,

iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form

$$|P[T_{bn} = s] - P[T_{bn} = s + t]| = O(n^{-2}t + n^{-1-a_1+\delta})$$

for any $\delta > 0$ could perhaps be attained, leading to a final error estimate in order $O(bn^{-1} + n^{-a_1+\delta})$

for any $\delta > 0$, to replace $\varepsilon_{\{7.7\}}(n, b)$. This would be of the ideal order $O(b/n)$ for large enough b , but would still be coarser for small b .

With b and n as in the previous section, we wish to show that

$$\left| d_{TV}(L(C[1, b]), L(Z[1, b])) - \frac{1}{2}(n+1)^{-1} |1 - \theta| E|T_{0b} - ET_{0b}| \right| \leq \varepsilon_{\{7.8\}}(n, b),$$

Where $\varepsilon_{\{7.8\}}(n, b) = O(n^{-1}b[n^{-1}b + n^{-\beta_{12}+\delta}])$ for any $\delta > 0$ under Conditions $(A_0), (D_1)$ and (B_{12}) , with β_{12} . The proof uses sharper estimates.

As before, we begin with the formula

$$d_{TV}(L(C[1, b]), L(Z[1, b])) = \sum_{r \geq 0} P[T_{0b} = r] \left\{ 1 - \frac{P[T_{bn} = n - r]}{P[T_{0n} = n]} \right\}_+$$

Now we observe that

$$\begin{aligned} & \left| \sum_{r \geq 0} P[T_{0b} = r] \left\{ 1 - \frac{P[T_{bn} = n - r]}{P[T_{0n} = n]} \right\}_+ - \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0n} = n]} \right| \\ & \times \left| \sum_{s=[n/2]+1}^n P[T_{0b} = s] (P[T_{bn} = n - s] - P[T_{bn} = n - r]) \right| \\ & \leq 4n^{-2} ET_{0b}^2 + (\max_{n/2 < s \leq n} P[T_{0b} = s]) / P[T_{0n} = n] \\ & + P[T_{0b} > n/2] \\ & \leq 8n^{-2} ET_{0b}^2 + \frac{3\varepsilon_{\{10.5(2)\}}(n/2, b)}{\theta P_\theta[0, 1]}, \end{aligned} \tag{1.1}$$

We have

$$\begin{aligned} & \left| \sum_{r=0}^{[n/2]} \frac{P[T_{0b} = r]}{P[T_{0n} = n]} \right. \\ & \times \left(\sum_{s=0}^{[n/2]} P[T_{0b} = s] (P[T_{bn} = n - s] - P[T_{bn} = n - r]) \right)_+ \\ & \left. - \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{(s - r)(1 - \theta)}{n + 1} P[T_{0n} = n] \right| \end{aligned}$$

$$\begin{aligned} & \leq \frac{1}{n^2 P[T_{0n} = n]} \sum_{r \geq 0} P[T_{0b} = r] \sum_{s \geq 0} P[T_{0b} = s] |s - r| \\ & \times \left\{ \varepsilon_{\{10.14\}}(n, b) + 2(r \vee s) |1 - \theta| n^{-1} \left\{ K_0 \theta + 4\phi_{\{10.8\}}^*(n) \right\} \right\} \\ & \leq \frac{6}{\theta n P_\theta[0, 1]} ET_{0b} \varepsilon_{\{10.14\}}(n, b) \\ & + 4 |1 - \theta| n^{-2} ET_{0b}^2 \left\{ K_0 \theta + 4\phi_{\{10.8\}}^*(n) \right\} \\ & \left(\frac{3}{\theta n P_\theta[0, 1]} \right), \end{aligned} \tag{1.2}$$

The approximation in (1.2) is further simplified by noting that

$$\begin{aligned} & \sum_{r=0}^{[n/2]} P[T_{0b} = r] \left\{ \sum_{s=0}^{[n/2]} P[T_{0b} = s] \frac{(s - r)(1 - \theta)}{n + 1} \right\}_+ \\ & - \left\{ \sum_{s=0} P[T_{0b} = s] \frac{(s - r)(1 - \theta)}{n + 1} \right\}_+ \\ & \leq \sum_{r=0}^{[n/2]} P[T_{0b} = r] \sum_{s > [n/2]} P[T_{0b} = s] \frac{(s - r) |1 - \theta|}{n + 1} \\ & \leq |1 - \theta| n^{-1} E(T_{0b} 1_{\{T_{0b} > n/2\}}) \leq 2 |1 - \theta| n^{-2} ET_{0b}^2, \end{aligned} \tag{1.3}$$

and then by observing that

$$\begin{aligned} & \sum_{r > [n/2]} P[T_{0b} = r] \left\{ \sum_{s \geq 0} P[T_{0b} = s] \frac{(s - r)(1 - \theta)}{n + 1} \right\} \\ & \leq n^{-1} |1 - \theta| (ET_{0b} P[T_{0b} > n/2] + E(T_{0b} 1_{\{T_{0b} > n/2\}})) \\ & \leq 4 |1 - \theta| n^{-2} ET_{0b}^2 \end{aligned} \tag{1.4}$$

Combining the contributions of (1.2) – (1.3), we thus find tha

$$\begin{aligned}
 & | d_{TV}(L(C[1,b]), L(Z[1,b])) \\
 & - (n+1)^{-1} \sum_{r \geq 0} P[T_{0b} = r] \left\{ \sum_{s \geq 0} P[T_{0b} = s] (s-r)(1-\theta) \right\}_+ \\
 & \leq \varepsilon_{\{7.8\}}(n,b) \\
 & = \frac{3}{\theta P_\theta[0,1]} \left\{ \varepsilon_{\{10.5(2)\}}(n/2,b) + 2n^{-1} ET_{0b} \varepsilon_{\{10.14\}}(n,b) \right\} \\
 & + 2n^{-2} ET_{0b}^2 \left\{ 4 + 3|1-\theta| + \frac{24|1-\theta| \phi_{\{10.8\}}^*(n)}{\theta P_\theta[0,1]} \right\} \quad (1.5)
 \end{aligned}$$

The quantity $\varepsilon_{\{7.8\}}(n,b)$ is seen to be of the order claimed under Conditions $(A_0), (D_1)$ and (B_{12}) , provided that $S(\infty) < \infty$; this supplementary condition can be removed if $\phi_{\{10.8\}}^*(n)$ is replaced by $\phi_{\{10.11\}}^*(n)$ in the definition of $\varepsilon_{\{7.8\}}(n,b)$, has the required order without the restriction on the r_i implied by assuming that $S(\infty) < \infty$. Finally, a direct calculation now shows that

$$\begin{aligned}
 & \sum_{r \geq 0} P[T_{0b} = r] \left\{ \sum_{s \geq 0} P[T_{0b} = s] (s-r)(1-\theta) \right\}_+ \\
 & = \frac{1}{2} |1-\theta| E |T_{0b} - ET_{0b}|
 \end{aligned}$$

Example 1.0. Consider the point $O = (0, \dots, 0) \in \square^n$. For an arbitrary vector r , the coordinates of the point $x = O + r$ are equal to the respective coordinates of the vector r : $x = (x^1, \dots, x^n)$ and $r = (x^1, \dots, x^n)$. The vector r such as in the example is called the position vector or the radius vector of the point x . (Or, in greater detail: r is the radius-vector of x w.r.t an origin O). Points are frequently specified by their radius-vectors. This presupposes the choice of O as the “standard origin”. Let us summarize. We have considered \square^n and interpreted its elements in two ways: as points and as vectors. Hence we may say that we leading with the two copies of \square^n : $\square^n = \{\text{points}\}$, $\square^n = \{\text{vectors}\}$
 Operations with vectors: multiplication by a number, addition. Operations with points and vectors: adding a vector to a point (giving a point), subtracting two points (giving a vector). \square^n treated in this way is called an *n-dimensional affine space*. (An “abstract” affine space is a pair of sets, the set of points and the set of vectors so that the operations as above are defined axiomatically). Notice that vectors in an affine space are also known as “free

vectors”. Intuitively, they are not fixed at points and “float freely” in space. From \square^n considered as an affine space we can precede in two opposite directions: \square^n as an Euclidean space $\Leftarrow \square^n$ as an affine space $\Rightarrow \square^n$ as a manifold. Going to the left means introducing some extra structure which will make the geometry richer. Going to the right means forgetting about part of the affine structure; going further in this direction will lead us to the so-called “smooth (or differentiable) manifolds”. The theory of differential forms does not require any extra geometry. So our natural direction is to the right. The Euclidean structure, however, is useful for examples and applications. So let us say a few words about it:

Remark 1.0. *Euclidean geometry.* In \square^n considered as an affine space we can already do a good deal of geometry. For example, we can consider lines and planes, and quadric surfaces like an ellipsoid. However, we cannot discuss such things as “lengths”, “angles” or “areas” and “volumes”. To be able to do so, we have to introduce some more definitions, making \square^n a Euclidean space. Namely, we define the length of a vector $a = (a^1, \dots, a^n)$ to be

$$|a| := \sqrt{(a^1)^2 + \dots + (a^n)^2} \quad (1)$$

After that we can also define distances between points as follows:

$$d(A, B) := |\overline{AB}| \quad (2)$$

One can check that the distance so defined possesses natural properties that we expect: is it always non-negative and equals zero only for coinciding points; the distance from A to B is the same as that from B to A (symmetry); also, for three points, A, B and C, we have $d(A, B) \leq d(A, C) + d(C, B)$ (the “triangle inequality”). To define angles, we first introduce the scalar product of two vectors

$$(a, b) := a^1 b^1 + \dots + a^n b^n \quad (3)$$

Thus $|a| = \sqrt{(a, a)}$. The scalar product is also denote by dot: $a \cdot b = (a, b)$, and hence is often referred to as the “dot product”. Now, for nonzero vectors, we define the angle between them by the equality

$$\cos \alpha := \frac{(a, b)}{|a||b|} \quad (4)$$

The angle itself is defined up to an integral multiple of 2π . For this definition to be consistent we have to ensure that the r.h.s. of (4) does not exceed 1 by the absolute value. This follows from the inequality

$$(a, b)^2 \leq |a|^2 + |b|^2 \quad (5)$$

known as the Cauchy–Bunyakovsky–Schwarz inequality (various combinations of these three names are applied in different books). One of the ways of proving (5) is to consider the scalar square of the linear combination $a + tb$, where $t \in \mathbb{R}$. As $(a + tb, a + tb) \geq 0$ is a quadratic polynomial in t which is never negative, its discriminant must be less or equal zero. Writing this explicitly yields (5). The triangle inequality for distances also follows from the inequality (5).

Example 1.1. Consider the function $f(x) = x^i$ (the i -th coordinate). The linear function dx^i (the differential of x^i) applied to an arbitrary vector h is simply h^i . From these examples follows that we can rewrite df as

$$df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n, \quad (1)$$

which is the standard form. Once again: the partial derivatives in (1) are just the coefficients (depending on x); dx^1, dx^2, \dots are linear functions giving on an arbitrary vector h its coordinates h^1, h^2, \dots , respectively. Hence

$$df(x)(h) = \partial_{hf(x)} = \frac{\partial f}{\partial x^1} h^1 + \dots + \frac{\partial f}{\partial x^n} h^n, \quad (2)$$

Theorem 1.7. Suppose we have a parametrized curve $t \mapsto x(t)$ passing through $x_0 \in \mathbb{R}^n$ at $t = t_0$ and with the velocity vector $x'(t_0) = v$. Then $\frac{df(x(t))}{dt}(t_0) = \partial_v f(x_0) = df(x_0)(v)$ (1)

Proof. Indeed, consider a small increment of the parameter $t : t_0 \mapsto t_0 + \Delta t$, Where $\Delta t \mapsto 0$. On the other hand, we have $f(x_0 + h) - f(x_0) = df(x_0)(h) + \beta(h)|h|$ for an arbitrary vector h , where $\beta(h) \rightarrow 0$ when $h \rightarrow 0$. Combining it together, for the increment of $f(x(t))$ we obtain

$$\begin{aligned} & f(x(t_0 + \Delta t)) - f(x_0) \\ &= df(x_0)(v \cdot \Delta t + \alpha(\Delta t)\Delta t) \\ &+ \beta(v \cdot \Delta t + \alpha(\Delta t)\Delta t) \cdot |v \Delta t + \alpha(\Delta t)\Delta t| \\ &= df(x_0)(v) \cdot \Delta t + \gamma(\Delta t)\Delta t \end{aligned}$$

For a certain $\gamma(\Delta t)$ such that $\gamma(\Delta t) \rightarrow 0$ when $\Delta t \rightarrow 0$ (we used the linearity of $df(x_0)$). By the definition, this means that the derivative of $f(x(t))$ at $t = t_0$ is exactly $df(x_0)(v)$. The statement of the theorem can be expressed by a simple formula:

$$\frac{df(x(t))}{dt} = \frac{\partial f}{\partial x^1} x^1 + \dots + \frac{\partial f}{\partial x^n} x^n \quad (2)$$

To calculate the value of df at a point x_0 on a given vector v one can take an arbitrary curve passing through x_0 at t_0 with v as the velocity vector at t_0 and calculate the usual derivative of $f(x(t))$ at $t = t_0$.

Theorem 1.8. For functions $f, g : U \rightarrow \mathbb{R}$, $U \subset \mathbb{R}^n$,

$$d(f + g) = df + dg \quad (1)$$

$$d(fg) = df \cdot g + f \cdot dg \quad (2)$$

Proof. Consider an arbitrary point x_0 and an arbitrary vector v stretching from it. Let a curve $x(t)$ be such that $x(t_0) = x_0$ and $x'(t_0) = v$. Hence

$$d(f + g)(x_0)(v) = \frac{d}{dt} (f(x(t)) + g(x(t)))$$

at $t = t_0$ and

$$d(fg)(x_0)(v) = \frac{d}{dt} (f(x(t))g(x(t)))$$

at $t = t_0$. Formulae (1) and (2) then immediately follow from the corresponding formulae for the usual derivative. Now, almost without change the theory generalizes to functions taking values in \mathbb{R}^m instead of \mathbb{R} . The only difference is that now the differential of a map $F : U \rightarrow \mathbb{R}^m$ at a point x will be a linear function taking vectors in \mathbb{R}^n to vectors in \mathbb{R}^m (instead of \mathbb{R}). For an arbitrary vector $h \in \mathbb{R}^n$,

$$F(x + h) = F(x) + dF(x)(h)$$

$$+ \beta(h)|h| \quad (3)$$

Where $\beta(h) \rightarrow 0$ when $h \rightarrow 0$. We have $dF = (dF^1, \dots, dF^m)$ and

$$dF = \frac{\partial F}{\partial x^1} dx^1 + \dots + \frac{\partial F}{\partial x^n} dx^n$$

$$= \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \dots & \frac{\partial F^1}{\partial x^n} \\ \dots & \dots & \dots \\ \frac{\partial F^m}{\partial x^1} & \dots & \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \dots \\ dx^n \end{pmatrix} \quad (4)$$

In this matrix notation we have to write vectors as vector-columns.

Theorem 1.9. For an arbitrary parametrized curve $x(t)$ in \square^n , the differential of a map $F: U \rightarrow \square^m$ (where $U \subset \square^n$) maps the velocity vector $x(t)$ to the velocity vector of the curve $F(x(t))$ in \square^m :

$$\frac{dF(x(t))}{dt} = dF(x(t))(x(t)) \quad (1)$$

Proof. By the definition of the velocity vector,

$$x(t + \Delta t) = x(t) + x(t) \cdot \Delta t + \alpha(\Delta t) \Delta t \quad (2)$$

Where $\alpha(\Delta t) \rightarrow 0$ when $\Delta t \rightarrow 0$. By the definition of the differential,

$$F(x+h) = F(x) + dF(x)(h) + \beta(h)|h| \quad (3)$$

Where $\beta(h) \rightarrow 0$ when $h \rightarrow 0$. we obtain

$$F(x(t + \Delta t)) = F(x + \underbrace{x(t) \cdot \Delta t + \alpha(\Delta t) \Delta t}_h)$$

$$= F(x) + dF(x)(x(t) \Delta t + \alpha(\Delta t) \Delta t) + \beta(x(t) \Delta t + \alpha(\Delta t) \Delta t) \cdot |x(t) \Delta t + \alpha(\Delta t) \Delta t|$$

$$= F(x) + dF(x)(x(t) \Delta t + \gamma(\Delta t) \Delta t)$$

For some $\gamma(\Delta t) \rightarrow 0$ when $\Delta t \rightarrow 0$. This precisely means that $dF(x)x(t)$ is the velocity vector of $F(x)$. As every vector attached to a point can be viewed as the velocity vector of some curve passing through this point, this theorem gives a clear geometric picture of dF as a linear map on vectors.

Theorem 1.10 Suppose we have two maps $F: U \rightarrow V$ and $G: V \rightarrow W$, where $U \subset \square^n, V \subset \square^m, W \subset \square^p$ (open domains). Let $F: x \mapsto y = F(x)$. Then the differential of the composite map $GoF: U \rightarrow W$ is the composition of the differentials of F and G :

$$d(GoF)(x) = dG(y)odF(x) \quad (4)$$

Proof. We can use the description of the differential. Consider a curve $x(t)$ in \square^n with the velocity vector x . Basically, we need to know to which vector in \square^p it is taken by $d(GoF)$. the curve $(GoF)(x(t)) = G(F(x(t)))$. By the same theorem, it equals the image under dG of the Anycast Flow vector to the curve $F(x(t))$ in \square^m . Applying the theorem once again, we see that the velocity vector to the curve $F(x(t))$ is the image under dF of the vector $x(t)$. Hence $d(GoF)(x) = dG(dF(x))$ for an arbitrary vector x .

Corollary 1.0. If we denote coordinates in \square^n by (x^1, \dots, x^n) and in \square^m by (y^1, \dots, y^m) , and write

$$dF = \frac{\partial F}{\partial x^1} dx^1 + \dots + \frac{\partial F}{\partial x^n} dx^n \quad (1)$$

$$dG = \frac{\partial G}{\partial y^1} dy^1 + \dots + \frac{\partial G}{\partial y^m} dy^m, \quad (2)$$

Then the chain rule can be expressed as follows:

$$d(GoF) = \frac{\partial G}{\partial y^1} dF^1 + \dots + \frac{\partial G}{\partial y^m} dF^m, \quad (3)$$

Where dF^i are taken from (1). In other words, to get $d(GoF)$ we have to substitute into (2) the expression for $dy^i = dF^i$ from (3). This can also be expressed by the following matrix formula:

$$d(GoF) = \begin{pmatrix} \frac{\partial G^1}{\partial y^1} & \dots & \frac{\partial G^1}{\partial y^m} \\ \dots & \dots & \dots \\ \frac{\partial G^p}{\partial y^1} & \dots & \frac{\partial G^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \dots & \frac{\partial F^1}{\partial x^n} \\ \dots & \dots & \dots \\ \frac{\partial F^m}{\partial x^1} & \dots & \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \dots \\ dx^n \end{pmatrix} \quad (4)$$

i.e., if dG and dF are expressed by matrices of partial derivatives, then $d(GoF)$ is expressed by

the product of these matrices. This is often written as

$$\begin{pmatrix} \frac{\partial z^1}{\partial x^1} & \dots & \frac{\partial z^1}{\partial x^n} \\ \dots & \dots & \dots \\ \frac{\partial z^p}{\partial x^1} & \dots & \frac{\partial z^p}{\partial x^n} \end{pmatrix} = \begin{pmatrix} \frac{\partial z^1}{\partial y^1} & \dots & \frac{\partial z^1}{\partial y^m} \\ \dots & \dots & \dots \\ \frac{\partial z^p}{\partial y^1} & \dots & \frac{\partial z^p}{\partial y^m} \end{pmatrix} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \dots & \frac{\partial y^1}{\partial x^n} \\ \dots & \dots & \dots \\ \frac{\partial y^m}{\partial x^1} & \dots & \frac{\partial y^m}{\partial x^n} \end{pmatrix}, \quad (5)$$

Or

$$\frac{\partial z^\mu}{\partial x^a} = \sum_{i=1}^m \frac{\partial z^\mu}{\partial y^i} \frac{\partial y^i}{\partial x^a}, \quad (6)$$

Where it is assumed that the dependence of $y \in \mathbb{R}^m$ on $x \in \mathbb{R}^n$ is given by the map F , the dependence of $z \in \mathbb{R}^p$ on $y \in \mathbb{R}^m$ is given by the map G , and the dependence of $z \in \mathbb{R}^p$ on $x \in \mathbb{R}^n$ is given by the composition GoF .

Definition 1.6. Consider an open domain $U \subset \mathbb{R}^n$. Consider also another copy of \mathbb{R}^n , denoted for distinction \mathbb{R}^n_y , with the standard coordinates (y^1, \dots, y^n) . A system of coordinates in the open domain U is given by a map $F: V \rightarrow U$, where $V \subset \mathbb{R}^n_y$ is an open domain of \mathbb{R}^n_y , such that the following three conditions are satisfied:

- (1) F is smooth;
- (2) F is invertible;
- (3) $F^{-1}: U \rightarrow V$ is also smooth

The coordinates of a point $x \in U$ in this system are the standard coordinates of $F^{-1}(x) \in \mathbb{R}^n_y$

In other words,

$$F: (y^1, \dots, y^n) \mapsto x = x(y^1, \dots, y^n) \quad (1)$$

Here the variables (y^1, \dots, y^n) are the “new” coordinates of the point x

Example 1.2. Consider a curve in \mathbb{R}^2 specified in polar coordinates as

$$x(t): r = r(t), \varphi = \varphi(t) \quad (1)$$

We can simply use the chain rule. The map $t \mapsto x(t)$ can be considered as the composition of the maps $t \mapsto (r(t), \varphi(t)), (r, \varphi) \mapsto x(r, \varphi)$. Then, by the chain rule, we have

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial r} \frac{dr}{dt} + \frac{\partial x}{\partial \varphi} \frac{d\varphi}{dt} = \frac{\partial x}{\partial r} \dot{r} + \frac{\partial x}{\partial \varphi} \dot{\varphi} \quad (2)$$

Here \dot{r} and $\dot{\varphi}$ are scalar coefficients depending on t , whence the partial derivatives $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \varphi}$ are vectors depending on point in \mathbb{R}^2 . We can compare this with the formula in the “standard” coordinates:

$x = e_1 x + e_2 y$. Consider the vectors $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \varphi}$. Explicitly we have

$$\frac{\partial x}{\partial r} = (\cos \varphi, \sin \varphi) \quad (3)$$

$$\frac{\partial x}{\partial \varphi} = (-r \sin \varphi, r \cos \varphi) \quad (4)$$

From where it follows that these vectors make a basis at all points except for the origin (where $r=0$). It is instructive to sketch a picture, drawing vectors corresponding to a point as starting from that point. Notice that $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \varphi}$ are, respectively, the velocity vectors for the curves $r \mapsto x(r, \varphi)$ ($\varphi = \varphi_0$ fixed) and $\varphi \mapsto x(r, \varphi)$ ($r = r_0$ fixed). We can conclude that for an arbitrary curve given in polar coordinates the velocity vector will have components $(\dot{r}, \dot{\varphi})$ if as a basis we take $e_r := \frac{\partial x}{\partial r}, e_\varphi := \frac{\partial x}{\partial \varphi}$:

$$x = e_r \dot{r} + e_\varphi \dot{\varphi} \quad (5)$$

A characteristic feature of the basis e_r, e_φ is that it is not “constant” but depends on point. Vectors “stuck to points” when we consider curvilinear coordinates.

Proposition 1.3. The velocity vector has the same appearance in all coordinate systems.

Proof. Follows directly from the chain rule and the transformation law for the basis e_i . In particular,

the elements of the basis $e_i = \frac{\partial x}{\partial x^i}$ (originally, a formal notation) can be understood directly as the velocity vectors of the coordinate lines $x^i \mapsto x(x^1, \dots, x^n)$ (all coordinates but x^i are fixed). Since we now know how to handle velocities

in arbitrary coordinates, the best way to treat the differential of a map $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is by its action on the velocity vectors. By definition, we set

$$dF(x_0): \frac{dx(t)}{dt}(t_0) \mapsto \frac{dF(x(t))}{dt}(t_0) \quad (1)$$

Now $dF(x_0)$ is a linear map that takes vectors attached to a point $x_0 \in \mathbb{R}^n$ to vectors attached to the point $F(x) \in \mathbb{R}^m$

$$dF = \frac{\partial F}{\partial x^1} dx^1 + \dots + \frac{\partial F}{\partial x^n} dx^n$$

$$(e_1, \dots, e_m) \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \dots & \frac{\partial F^1}{\partial x^n} \\ \dots & \dots & \dots \\ \frac{\partial F^m}{\partial x^1} & \dots & \frac{\partial F^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} dx^1 \\ \dots \\ dx^n \end{pmatrix}, \quad (2)$$

In particular, for the differential of a function we always have

$$df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} dx^n, \quad (3)$$

Where x^i are arbitrary coordinates. The form of the differential does not change when we perform a change of coordinates.

Example 1.3 Consider a 1-form in \mathbb{R}^2 given in the standard coordinates:

$A = -ydx + xdy$ In the polar coordinates we will have $x = r \cos \phi, y = r \sin \phi$, hence

$$dx = \cos \phi dr - r \sin \phi d\phi$$

$$dy = \sin \phi dr + r \cos \phi d\phi$$

Substituting into A , we get

$$A = -r \sin \phi (\cos \phi dr - r \sin \phi d\phi)$$

$$+ r \cos \phi (\sin \phi dr + r \cos \phi d\phi)$$

$$= r^2 (\sin^2 \phi + \cos^2 \phi) d\phi = r^2 d\phi$$

Hence $A = r^2 d\phi$ is the formula for A in the polar coordinates. In particular, we see that this is again a 1-form, a linear combination of the differentials of coordinates with functions as coefficients. Secondly, in a more conceptual way, we can define a 1-form in a domain U as a linear function on vectors at every point of U :

$$\omega(v) = \omega_1 v^1 + \dots + \omega_n v^n, \quad (1)$$

If $v = \sum e_i v^i$, where $e_i = \frac{\partial x}{\partial x^i}$. Recall that the differentials of functions were defined as linear

functions on vectors (at every point), and

$$dx^i(e_j) = dx^i \left(\frac{\partial x}{\partial x^j} \right) = \delta_j^i \quad (2) \quad \text{at}$$

every point x .

Theorem 1.9. For arbitrary 1-form ω and path γ , the integral $\int_{\gamma} \omega$ does not change if we change parametrization of γ provide the orientation remains the same.

Proof: Consider $\left\langle \omega(x(t)), \frac{dx}{dt} \right\rangle$ and

$$\left\langle \omega(x(t(t))), \frac{dx}{dt} \right\rangle \text{ As}$$

$$\left\langle \omega(x(t(t))), \frac{dx}{dt} \right\rangle = \left\langle \omega(x(t(t))), \frac{dx}{dt} \right\rangle \cdot \frac{dt}{dt},$$

Let p be a rational prime and let $K = \mathbb{Q}(\zeta_p)$.

We write ζ for ζ_p or this section. Recall that K has degree $\varphi(p) = p-1$ over \mathbb{Q} . We wish to show that $O_K = \mathbb{Z}[\zeta]$. Note that ζ is a root of $x^p - 1$, and thus is an algebraic integer; since O_K is a ring we have that $\mathbb{Z}[\zeta] \subseteq O_K$. We give a proof without assuming unique factorization of ideals. We begin with some norm and trace computations. Let j be an integer. If j is not divisible by p , then ζ^j is a primitive p^{th} root of unity, and thus its conjugates are $\zeta, \zeta^2, \dots, \zeta^{p-1}$. Therefore

$$Tr_{K/\mathbb{Q}}(\zeta^j) = \zeta + \zeta^2 + \dots + \zeta^{p-1} = \Phi_p(\zeta) - 1 = -1$$

If p does divide j , then $\zeta^j = 1$, so it has only the one conjugate 1, and $Tr_{K/\mathbb{Q}}(\zeta^j) = p-1$ By linearity of the trace, we find that

$$Tr_{K/\mathbb{Q}}(1-\zeta) = Tr_{K/\mathbb{Q}}(1-\zeta^2) = \dots$$

$$= Tr_{K/\mathbb{Q}}(1-\zeta^{p-1}) = p$$

We also need to compute the norm of $1-\zeta$. For this, we use the factorization

$$x^{p-1} + x^{p-2} + \dots + 1 = \Phi_p(x)$$

$$= (x-\zeta)(x-\zeta^2)\dots(x-\zeta^{p-1});$$

Plugging in $x=1$ shows that

$$p = (1-\zeta)(1-\zeta^2)\dots(1-\zeta^{p-1})$$

Since the $(1-\zeta^j)$ are the conjugates of $(1-\zeta)$, this shows that $N_{K/\mathbb{Q}}(1-\zeta) = p$. The key result for determining the ring of integers O_K is the following.

LEMMA 1.9

$$(1-\zeta)O_K \cap \mathbb{Z} = p\mathbb{Z}$$

Proof. We saw above that p is a multiple of $(1-\zeta)$ in O_K , so the inclusion $(1-\zeta)O_K \cap \mathbb{Z} \supseteq p\mathbb{Z}$ is immediate. Suppose now that the inclusion is strict. Since $(1-\zeta)O_K \cap \mathbb{Z}$ is an ideal of \mathbb{Z} containing $p\mathbb{Z}$ and $p\mathbb{Z}$ is a maximal ideal of \mathbb{Z} , we must have $(1-\zeta)O_K \cap \mathbb{Z} = \mathbb{Z}$. Thus we can write

$$1 = \alpha(1-\zeta)$$

For some $\alpha \in O_K$. That is, $1-\zeta$ is a unit in O_K .

COROLLARY 1.1 For any $\alpha \in O_K$,

$$Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) \in p\mathbb{Z}$$

PROOF. We have

$$\begin{aligned} Tr_{K/\mathbb{Q}}((1-\zeta)\alpha) &= \sigma_1((1-\zeta)\alpha) + \dots + \sigma_{p-1}((1-\zeta)\alpha) \\ &= \sigma_1(1-\zeta)\sigma_1(\alpha) + \dots + \sigma_{p-1}(1-\zeta)\sigma_{p-1}(\alpha) \\ &= (1-\zeta)\sigma_1(\alpha) + \dots + (1-\zeta^{p-1})\sigma_{p-1}(\alpha) \end{aligned}$$

Where the σ_i are the complex embeddings of K (which we are really viewing as automorphisms of K) with the usual ordering. Furthermore, $1-\zeta^j$ is a multiple of $1-\zeta$ in O_K for every $j \neq 0$. Thus $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in (1-\zeta)O_K$. Since the trace is also a rational integer.

PROPOSITION 1.4 Let p be a prime number and

let $K = \mathbb{Q}(\zeta_p)$ be the p^{th} cyclotomic field. Then $O_K = \mathbb{Z}[\zeta_p] \cong \mathbb{Z}[x]/(\Phi_p(x))$; Thus $1, \zeta_p, \dots, \zeta_p^{p-2}$ is an integral basis for O_K .

PROOF. Let $\alpha \in O_K$ and write

$$\alpha = a_0 + a_1\zeta + \dots + a_{p-2}\zeta^{p-2} \quad \text{With } a_i \in \mathbb{Z}.$$

Then

$$\begin{aligned} \alpha(1-\zeta) &= a_0(1-\zeta) + a_1(\zeta - \zeta^2) + \dots \\ &+ a_{p-2}(\zeta^{p-2} - \zeta^{p-1}) \end{aligned}$$

By the linearity of the trace and our above calculations we find that $Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) = pa_0$

We also have

$$Tr_{K/\mathbb{Q}}(\alpha(1-\zeta)) \in p\mathbb{Z}, \quad \text{so } a_0 \in \mathbb{Z}$$

Next consider the algebraic integer

$$(\alpha - a_0)\zeta^{-1} = a_1 + a_2\zeta + \dots + a_{p-2}\zeta^{p-3};$$

This is an algebraic integer since $\zeta^{-1} = \zeta^{p-1}$ is. The same argument as above shows that $a_1 \in \mathbb{Z}$, and continuing in this way we find that all of the a_i are in \mathbb{Z} . This completes the proof.

Example 1.4 Let $K = \mathbb{Q}$, then the local ring

$\mathbb{Z}_{(p)}$ is simply the subring of \mathbb{Q} of rational numbers with denominator relatively prime to p . Note that this ring $\mathbb{Z}_{(p)}$ is not the ring \mathbb{Z}_p of p -adic integers; to get \mathbb{Z}_p one must complete $\mathbb{Z}_{(p)}$.

The usefulness of $O_{K,p}$ comes from the fact that it has a particularly simple ideal structure. Let a be any proper ideal of $O_{K,p}$ and consider the ideal

$a \cap O_K$ of O_K . We claim that $a = (a \cap O_K)O_{K,p}$; That is, that a is generated by the elements of a in $a \cap O_K$. It is clear from the definition of an ideal that $a \supseteq (a \cap O_K)O_{K,p}$.

To prove the other inclusion, let α be any element of a . Then we can write $\alpha = \beta/\gamma$ where $\beta \in O_K$ and $\gamma \notin p$. In particular, $\beta \in a$ (since $\beta/\gamma \in a$ and a is an ideal), so $\beta \in O_K$ and $\gamma \notin p$. so $\beta \in a \cap O_K$. Since $1/\gamma \in O_{K,p}$, this implies that $\alpha = \beta/\gamma \in (a \cap O_K)O_{K,p}$, as claimed. We can use this fact to determine all of the ideals of $O_{K,p}$. Let a be any ideal of $O_{K,p}$ and consider the ideal factorization of $a \cap O_K$ in O_K .

write it as $a \cap O_K = p^n b$ For some n and some ideal b , relatively prime to p . we claim first that $bO_{K,p} = O_{K,p}$. We now find that

$$a = (a \cap O_K)O_{K,p} = p^n bO_{K,p} = p^n O_{K,p}$$

Since $bO_{K,p} = O_{K,p}$. Thus every ideal of $O_{K,p}$ has the form $p^n O_{K,p}$ for some n ; it follows immediately that $O_{K,p}$ is noetherian. It is also now clear that $p^n O_{K,p}$ is the unique non-zero prime ideal in $O_{K,p}$

. Furthermore, the inclusion $O_K \hookrightarrow O_{K,p} / pO_{K,p}$. Since $pO_{K,p} \cap O_K = p$, this map is also surjection, since the residue class of $\alpha / \beta \in O_{K,p}$ (with $\alpha \in O_K$ and $\beta \notin p$) is the image of $\alpha\beta^{-1}$ in $O_{K/p}$, which makes sense since β is invertible in $O_{K/p}$. Thus the map is an isomorphism. In particular, it is now abundantly clear that every non-zero prime ideal of $O_{K,p}$ is maximal. To

show that $O_{K,p}$ is a Dedekind domain, it remains to show that it is integrally closed in K . So let $\gamma \in K$ be a root of a polynomial with coefficients in $O_{K,p}$; write this polynomial as $x^m + \frac{\alpha_{m-1}}{\beta_{m-1}}x^{m-1} + \dots + \frac{\alpha_0}{\beta_0}$ With $\alpha_i \in O_K$ and $\beta_i \in O_{K-p}$. Set $\beta = \beta_0\beta_1\dots\beta_{m-1}$. Multiplying by β^m we find that $\beta\gamma$ is the root of a monic polynomial with coefficients in O_K . Thus $\beta\gamma \in O_K$; since $\beta \notin p$, we have $\beta\gamma / \beta = \gamma \in O_{K,p}$. Thus $O_{K,p}$ is integrally closed in K .

COROLLARY 1.2. Let K be a number field of degree n and let α be in O_K then

$$N'_{K/\mathbb{Q}}(\alpha O_K) = |N_{K/\mathbb{Q}}(\alpha)|$$

PROOF. We assume a bit more Galois theory than usual for this proof. Assume first that K/\mathbb{Q} is Galois. Let σ be an element of $Gal(K/\mathbb{Q})$. It is clear that $\sigma(O_K) / \sigma(\alpha) \cong O_{K/\alpha}$; since $\sigma(O_K) = O_K$, this shows that $N'_{K/\mathbb{Q}}(\sigma(\alpha)O_K) = N'_{K/\mathbb{Q}}(\alpha O_K)$. Taking the product over all $\sigma \in Gal(K/\mathbb{Q})$, we have $N'_{K/\mathbb{Q}}(N_{K/\mathbb{Q}}(\alpha)O_K) = N'_{K/\mathbb{Q}}(\alpha O_K)^n$. Since $N_{K/\mathbb{Q}}(\alpha)$ is a rational integer and O_K is a free \mathbb{Z} -module of rank n ,

$O_K / N_{K/\mathbb{Q}}(\alpha)O_K$ Will have order $N_{K/\mathbb{Q}}(\alpha)^n$; therefore

$$N'_{K/\mathbb{Q}}(N_{K/\mathbb{Q}}(\alpha)O_K) = N_{K/\mathbb{Q}}(\alpha O_K)^n$$

This completes the proof. In the general case, let L be the Galois closure of K and set $[L:K] = m$.

D. RTRII Hand Prosthesis

The RTRII prosthesis is an underactuated hand, it weights 320 g, and it has nine DoFs in total, but only two motors (Fig. 7). Index and middle fingers are identical (both have three phalanges), while the thumb has two phalanges and a metacarpus, as in the human hand. The prosthesis is based on a tendon transmission system similar to the Hirose's Softfinger [4]. The tension of the tendons generates a torque around each joint, by means of small pulleys, and allows flexion movement. This transmission structure acts similarly to the flexor digitorum profundus. The extension movement is achieved by torsion springs. This design provides a self-adaptive enveloping grasp [21]. The adduction and abduction movements of the thumb are implemented by means of a four-bar link mechanism.

E. ACHILLE Indirect Prosthesis

In order to control the RTRII prosthesis, a user-friendly interface AdvanCed High Level control interface (ACHILLE) to be donned inside a shoe has been developed (see Fig. 8). In this case, few selected movements of the foot are identified using the ACHILLE interface. In particular, input signals are generated by the pressure exerted on the surface of specific sensitive areas of the insole operating like a switch. The interface has been designed to have the following main characteristics: 1) low thickness, to be integrated in the insole and to guarantee a good comfort to the user; 2) the location of the sensitive sites has been chosen starting from a biomechanical analysis of foot pressures to take into account the anatomy and range of motion of foot joint movements, also in consideration of different anthropometric measures related to gender, age, and others; 3) repeatability of the sensing information; and 4) mechanical resistance to continuous load. The signals generated by the movements of the foot were acquired and interpreted by the electronic unit embedded in the insole. Then, the signals were wireless transmitted to the external device to be controlled (with no physical link between the interface and the device). In order to reduce battery consumption and especially to avoid interferences during walking or any other user's activities, a specific mechanism for activating and for disabling the device was developed. The sensitive sites used a technology different from the one implemented in the state-of-the-art sensorized insoles, typically based on force sensing resistors (FSRs), capacitive, bend (RBS), or resistive sensors. In the ACHILLE insole, two superimposed layers of flexible circuit were used. Two conductive terminals were located on the first layer, for each switch, while the second layer included the conductive part that can connect the two terminals when pressure is exerted on it [22]. The flexible circuits were developed by using

Kapton sheets (polyimide film) with copper on one side.

A commercial insole was chosen for the structure of the device, determining its thickness, and the two layers of flexible circuits were placed on the lower and upper surface of the insole. Holes in correspondence of the conductive areas of the flexible circuits determine the threshold of activation, i.e., the values of pressure necessary to generate voluntary commands. In particular, for each sensitive area, a specific activation threshold was chosen to avoid undesired commands and at the same time to require a limited effort. This is very important considering a possible all-day usage. The threshold values are below the average pressures produced during walking, but a mechanism is present to enable/disable the interface and then avoid interferences during walking. The ACHILLE interface was used to control the RTRII prosthesis. The experimental trials have been conducted with ten able-bodied male subjects, aged between 23 and 45, and fitting the foot size of 42 in the European standard. They donned the ACHILLE interface inside their shoes. A commercial splint for forearm and hand was adapted to support the prosthesis and to make it usable by the different subjects. The results confirmed the effectiveness of the foot interface in the control of the mechatronic system, in terms of correct and prompt transmission of user's intention to the controlled device. The results of the experimental trials are given. Interestingly, the execution times of the experimental trials are compatible with the times required for the execution of the same tasks by able-bodied persons in normal conditions, with their own arms

IV. EMG-BASED CONTROL OF ADAPTIVE PROSTHESIS CHARACTERIZED BY COMPLIANT JOINTS

The muscular activities recorded using surface electrodes (EMG signals) are considered an interesting source of information to control robotic artefacts. In fact, EMG signals are very easy to record and provide an important access to the neuromuscular system of the user, i.e., indirectly to the brain voluntary activity. In the following paragraphs, the EMG-based control of a new underactuated compliant prosthesis is presented.

A. *Soft-Hand Compliant Prosthesis*

Recently, several groups have been working on the development of underactuated prostheses with compliant joints. They can be very useful when it is important to miniaturize the device. They are made up of one continuous part which deforms properly in certain areas in order to achieve motion. As a compliant flexible member bends, the external energy is stored in the form of strain energy. This

stored energy is similar to the strain energy in a deflected spring, and the effect of the spring may be integrated into a compliant mechanism design [23]. The main advantages of the compliant joints are cost reduction (part-count reduction, reduced assembly time, and simplified manufacturing process) and better performance. The main drawback of compliant joints is the relative difficulty of analyzing and designing compliant mechanisms. In order to overcome this problem, we recently developed a prosthesis moulded as a single piece of the same soft material, in order to allow a simple and low-cost production process, thus resulting in a novel methodology to get compliant joints in a single compliant structure. The prosthesis has a really anthropomorphic appearance, even if its structure is quite simple (see Fig. 11). The hand has four articulated fingers, with three joints for each one, and an opposable thumb, with two joints. A dc motor, with its gearbox and a power-off brake, provides the winding of a cable on a reel and its releasing, which in turn moves a slider pulling the five tendons, one for each finger. Both the actuation and the transmission systems are placed inside the palm.

B. *EMG- Based Control Algorithm*

Because of its reduced number of DoFs, the soft-hand prosthesis can be controlled by using an EMG-based discrimination algorithm. Using EMG signals recorded from upper limb muscles is a common and simple approach for controlling active prosthetic hands [24]. In this case, a very simple and robust EMG-based algorithm able to control opening and closure of the hand was developed. This system extracts the EMG amplitude by using a couple of surface commercial electrodes (Otto Bock, 13E125 Myobock Electrode) placed on two antagonist muscles (e.g., biceps and triceps brachialis, or flexor and extensor of the wrist, depending on the level of the amputation). A microcontroller (Microchip, PIC16F876) equipped with a 10-bit analog-to-digital (A/D) converter module was used to acquire and digitalize these signals. The EMG-based control was a simple finite-state machine (FSM), as shown in Fig. 12. The starting state (S0) of this algorithm is dedicated to the calibration phase. After T ms, the FSM goes into state S1 opening the prosthetic hand. A Hall effect switch is used to detect motor stop during hand opening. The FSM remains in the state S1 until the EMG signal produced by flexor contraction of the forearm does not overcome its relative threshold. When it happens, the FSM advances to the state S2 closing the hand. In order to identify the end of the grasp, the intensity of the current is monitored. The current flowing through the motor comes out from the bridge at a resistor detecting the intensity of this current. This output voltage is filtered with a Butterworth low-pass filter

and sent to a microcontroller. The microcontroller stops the motor when the average current overcomes 270 mA (threshold selected by trial-and-error procedure). The FSM remains in state S2 until the EMG signal produced by extensor contraction of the forearm does not overcome and drops under another threshold. This EMG-based control has been tested with able-bodied subjects. In particular, we set the force at 13 N with a current threshold value of 270 mA. By passing this limit, we pushed the prosthesis up to 30 N, with a threshold of 750 mA. The mechanical limit was not reached, because the motor current limit is 900 mA and the cable bears up to 50 N; both materials used did not show any stress problems. The critical point seems to be wear (particularly for cable) rather than the maximum load.

V. CONCLUSION

In this paper, different HBSs for the substitution of hand function have been presented together with some preliminary results. It is noteworthy to underline that the choice of the interface has to be carried out starting from the characteristics of the robotic system (and of course from the desires of the user). For example, implantable neural interfaces can be considered because current hand prostheses are more dexterous and better sensorized than in the past.

A. Authors and Affiliations

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