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Bijective Additive Singularity or Invertibility Preservers on Hermitian Matrices

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ABSTRACT
Let M_n be the set of all $n\times n$ complex matrices, and H_n be all Hermitian matrices in M_n .Suppose $n\geq 2$, ϕ is a bijective additive map on H_n which maps the set of singular or invertible matrices into itself. Then there exists an invertible matrix $P\in M_n$ and a constant $c\in\{-1,1\}$ such that $\phi(X)=cPXP^*$ or $cPX^TP^*, \forall X\in H_n$.

Keywords - Additive preservers; Hermitian; Singularity; Invertibility

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I. INTRODUCTION

Let M_n be the set of all $n\times n$ complex matrices, and H_n be all Hermitian matrices in M_n. Linear Preserver Problems is an active research area in matrix and operator theory. These problems involve linear maps on matrices and operators or other algebraic objects that keep some properties, sets or relations invariant. The earliest paper on such a problem dates back to 1897 when Frobenius [12] proved that a linear map $L: M_n\to M_n$ satisfies $\det(L(A))=\det(A)$ for all $A\in M_n$ if and only if there exist $M,N\in M_n$ with $\det(MN)=1$ such that L has the form $A\mapsto MAN$ or $A\mapsto MA^TN$,where A^T denotes the transpose of A. The above linear map is called determinant preserver. More general, researchers study linear map which only preserve singularity (i.e. maps singular matrices to singular matrices) and invertibility (i.e. maps invertible matrices to invertible matrices). Dieudonné[13] determined bijective linear singularity preservers on M_n, while Marcus and Moyls[14] described linear invertibility preservers on M_n. Chooi and Lim[15] described linear map which preserver both singularity and invertibility on T_n (the subset of M_n containing all up triangular matrices). Fošner and Šemrl [16] characterized almost surjective additive singularity and invertibility preservers on M_n. Recently, in quantum physics, quantum states of a system with n physical states are represented as $n\times n$ density matrices in H_n, i.e., positive semi-definite with trace one. See [1-11] for linear preserver problem and linear preserver problems arising in quantum information science. However, the maps on Hermitian matrices preserving singularity or invertibility have not been investigated. The propose of this paper is to characteristic bijective additive singularity or invertibility preservers on H_n.

Throughout this paper, the symbol E_{ij}, $1\leq i,j\leq n$, will be used for a $n\times n$ matrix with 1 in (i,j)-th position and 0 elsewhere. We denote by I_n and 0 the $n\times n$ identity matrix, and zero matrix of appropriate size, respectively. We denote by X^* the conjugate transpose of a matrix X.
The main result can be stated as:
Theorem 1. Suppose $n\geq 2$, ϕ is a bijective additive map on H_n which maps the set of singular matrices into itself. Then there exist an

invertible matrix $P \in M_n$ and a constant $c \in \{-1,1\}$ such that $\phi(X) = cPXP^*$ or $cPX^T P^*, \forall X \in H_n$.

Remark 2. The form of general singularity preserver on H_n may be complexity. For example

$$A = \begin{bmatrix} a & b \\ \bar{b} & d \end{bmatrix} \mapsto \begin{bmatrix} f(A) & 0 \\ 0 & 0 \end{bmatrix}$$

where f is any additive map from H_2 to real field.

Theorem 3. Suppose $n \geq 2$, ϕ is a bijective additive map on H_n which maps the set of invertible matrices into itself. Then there exist an invertible matrix $P \in M_n$ and a constant $c \in \{-1,1\}$ such that $\phi(X) = cPXP^*$ or $cPX^T P^*, \forall X \in H_n$.

II. PRELIMINARY RESULTS

Let us prove now two lemmas which would make the proof more concise.

Lemma 4. Let $r < n$ be a positive integer. Suppose $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ with $A_1 \in H_r$ invertible and $X = \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_4 \end{bmatrix} \in H_n$ with $X_1 \in H_r$. If for all integer λ, $\lambda A + X$ is singular matrix, Then X_4 is singular.

Proof. We assume that X_4 is invertible and we are going to find a contradiction.

Since $\begin{bmatrix} \lambda A_1 + X_1 & X_2 \\ X_2^* & X_4 \end{bmatrix}$ is singular, we have

$$0 = \det \begin{bmatrix} \lambda A_1 + X_1 & X_2 \\ X_2^* & X_4 \end{bmatrix} = \det \begin{bmatrix} \lambda A_1 - X_2 X_4^{-1} X_2^* & 0 \\ 0 & X_4 \end{bmatrix}.$$

Thus, $\det(\lambda A_1 - X_2 X_4^{-1} X_2^*) = 0$ for all integer λ. On the other hand, $\det(\lambda A_1 - X_2 X_4^{-1} X_2^*)$ is a polynomial of degree r, hence, there is no more than r different integer such that $\det(\lambda A_1 - X_2 X_4^{-1} X_2^*) = 0$, which is a contradiction.

Lemma 5. Let $G \subset H_n$ is an additive group containing only singular matrices. Then the linear span of G is a proper subset of H_n.

Proof. Choose $A \in G$ with the maximum rank, i.e., for any $B \in G$, $rankB \leq rankA$. Denote $rankA = r$. Because A is singular, $r < n$.

Set $A = P \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & a_r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P^*$, where P is an invertible matrix. For any $B \in G$, we can write

$B = P \begin{bmatrix} B_1 & B_2^* \\ B_2^* & B_4 \end{bmatrix} P^*$, where $B_1 \in H_r$. We only need to prove $B_4 = 0$, then the lemma follows. We assume there exists a $B \in G$ such that $b_{p,q}$, the $(p,q)-$th element of B_4, is nonzero. For any integer λ, we consider $(\lambda A + B)[1, \cdots, r, p; 1, \cdots, r, q]$, the submatrix of $\lambda A + B \in G$ obtained by rows $1, \cdots, r, p$ and column $1, \cdots, r, q$. By $\det(\lambda A + B)[1, \cdots, r, p; 1, \cdots, r, q] = a_1 \cdots a_r b_{pq} \lambda^r + g(\lambda)$ where, g is some polynomial of λ with degree less than r and $a_1 \cdots a_r b_{pq} \neq 0$, we obtain that there exists integer λ_0 such that $\det(\lambda A + B)[1, \cdots, r, p; 1, \cdots, q] \neq 0.$

Hence $\lambda_0 A + B$ has a rank bigger than A, which is a contradiction.

Lemma 6. Let $m > n$ be a positive integer. Suppose that $X_1, \cdots, X_m \in H_n$ satisfying for any proper subset $J \subset \{1, \cdots, m\}$, $\det\left(\Sigma_{j \in J} X_j\right) = 0$. Then $\det\left(\Sigma_{j=1}^m X_j\right) = 0$.

Proof. The proof is similar to [16].

Lemma 7. [17]Let L is a bijective additive map on H_n. Then $rankL(A) = 1$ whenever $rank(A) = 1$ if and only if there exist an invertible matrix $P \in M_n$ and a constant such that

$L(X)=cPXP^*$ or $cPX^TP^*, \forall X\in H_n.$

III. THE PROOF OF THEOREM 1

Inspired by Lemma 7, we intend to prove ϕ satisfying the assumption of Lemma 7.

If $n=2$, $\phi(X)\neq 0$ for any $X\neq 0$ since ϕ is a bijective map. Because ϕ preserves singularity, then $rank\phi(A)=1$ whenever $rank(A)=1$.

We next assume $n\geq 3$.

If there exist a rank-1 matrix A such that $1<r=rank\phi(A)<n$, we want to find a contradiction.

Because $rankA=1$, we may find invertible matrix Q such that

$$A=Q\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}Q^*$$

and

$$\phi(A)=T\begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & b_r & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}T^*.$$

After composing ϕ with the linear transformation $A\mapsto T^{-1}QAQ^*T^{*-1}$, we may assume that $\phi(aE_{11})=b_1E_{11}+\cdots+b_rE_{rr}$.

For any $X\in H_n$ with $rankX\leq n-2$.

Becaus $rank(X+\lambda aE_{11})\leq rankX+1\leq n-1$, for any integer $\lambda>0$, we have $rank\phi(X+\lambda aE_{11})\leq n-1$, that is, for all integer λ, $\lambda A+X$

$$\phi(X)+\lambda\begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & b_r & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

is singular matrix. By Lemma 4, we have $\phi(X)=\begin{bmatrix} X_1 & X_2 \\ X_2^* & X_4 \end{bmatrix}$ with $X_4\in H_{n-r}$ is singular.

For any fixed matrix $Y\in H_n$ with $rankY=n-1$. We can write Y as $Y=Y_1+\cdots+Y_{n-1}$, with $rankY_k=1$. Let $\phi(Y_k)[r+1,\cdots,n]$ be the principal ($n-r$)-square submatrix of $\phi(Y_k)\in G$ obtained by deleting rows $1,\cdots,r$, and columns $1,\cdots,r$. We have $\det\phi(Y_k)[r+1,\cdots,n]=0.$

Because $r>1$, we have $n-r<n-1$. Then, for any proper subset $J\subset\{1,\cdots,n-1\}$, it is obvious that $rank(\Sigma_{j\in J}Y_j)\leq n-2$. Apply Lemma 6 to $\phi(Y_k)[r+1,\cdots,n]$, we have

$$\det\phi(Y)[r+1,\cdots,n]=\det\left(\Sigma_{k=1}^{n-1}\phi(Y_k)[r+1,\cdots,n]\right)=0.$$

Again, for any fixed invertible matrix $Z\in H_n$, we can write Z as $Z=Z_1+\cdots+Z_n$, with $rankZ_k=1$. for any proper subset $J\subset\{1,\cdots,n\}$, $rank\left(\Sigma_{j\in J}Z_j\right)\leq n-1$, apply Lemma 6 to $\phi(Z_k)[r+1,\cdots,n]$, we have

$$\det\phi(Z)[r+1,\cdots,n]=\det\left(\Sigma_{k=1}^{n}\phi(Z_k)[r+1,\cdots,n]\right)=0.$$

Counting the above, we have $\det\phi(A)[r+1,\cdots,n]=0$, for all $A\in H_n$. Thus, we obtain by using Lemma 5 that ϕ is not a bijective map. So we have ϕ preserve rank-1. Using lemma 7 we complete the proof of Theorem.

IV. PROOF OF THEOREM 3

Lemma 8. Let ϕ is an additive map on real space V, p/q a rational number and $x\in V$, then $\phi\left(\frac{p}{q}x\right)=\frac{p}{q}\phi(x)$.

Proof. Because ϕ is additive, we have $\phi(0)=0$, and $\phi(-x)=-\phi(x)$, moreover, $\phi(px)=\phi(x+\cdots+x)=\phi(x)+\cdots+\phi(x)=p\phi(x)$ and

$$\phi(x)=\phi\left(q\frac{1}{q}x\right)=q\phi\left(\frac{1}{q}x\right)$$

then

$\phi\left(\frac{1}{q}x\right)=\frac{1}{q}\phi(x)$.Hence $\phi\left(\frac{p}{q}x\right)=\frac{p}{q}\phi(x)$.

Lemma 9. Let ϕ is bijective additive invertibility preverser on H_n, then there is a positive or negative definite matrix $A \in H_n$ such that $\phi(A) = I_n$. Moreover, ϕ is both singurlar and invertibility preverser.

Proof. Since ϕ is bijective there exists $A \in H_n$ such that $\phi(A) = I_n$.

We prove that A is positive or negative semi-definite. Otherwise, if A is indefinite, then there exists an invertible $T \in M_n$ such that $A = Tdiag(1,-1,\varepsilon_3,\cdots,\varepsilon_n)T^*$, with $\varepsilon_k \in \{-1,0,1\}$. Let

$$B_x = T\left(\begin{bmatrix} 0 & x \\ \overline{x} & 0 \end{bmatrix} \oplus 0\right)T^*, x \in C$$

then $A + \lambda B_x$ is invertible for any rational number λ, so is $I + \lambda\phi(B_x)$. Hence $\phi(B_x) = 0$, $\forall x \in C$, which is contradicting that ϕ is bijective. Without of loss generality, we assume A is positive semi-definite.

We next prove A must be invertible. Otherwise, there exists an invertible $T \in M_n$ such that $A = T(I_r \oplus 0_t)T^*$ with $t \geq 1$. Let $B = T(0_r \oplus I_t)T^*$, then for any negative rational number λ, we have $A + \lambda B$ is invertible, and $\phi(A + \lambda B) = I + \lambda\phi(B)$. Set $|\lambda|$ small enough, then $\phi(A + \lambda B)$ is positive definite. Replacing ϕ by $X \mapsto [I + \lambda\phi(B)]^{-1/2}\phi(X)[I + \lambda\phi(B)]^{-1/2}$, then $\phi(A + \lambda B) = I$ and $A + \lambda B$ is indefinite, which is a contradiction.

Define $\psi : H_n \to H_n$, $\psi(X) = \phi(A^{1/2}XA^{1/2})$, then ψ is a bijective map on H_n preversing invertible, and $\psi(I) = \phi(I) = I$. For any $\psi(X) \in H_n$ is singurlar, then X must be singurlar, i.e. ψ^{-1} maps the singurlar set of H_n into singurlar matrices set. By Theorem 1, ψ^{-1} maps the invertible matrices set of H_n into invertible matrices set, so is ψ.

Proof of Theorem 3 By Lemma 9, we have ϕ is satifying Theorem 1, and we drive the result directly.

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