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Primitive and Non-Primitive Solutions of Generalized Quadratic Diophantine Equations ($x^2 + p^2y^2 = qz^2$) and Their Cryptographic Applications

P Rajeswari[1], Dr. Swathi Yandamuri[2]
[1]Research Scholar, Department of Mathematics, Andhra University, Visakhapatnam, India.
[2]Associate Professor, Department of Mathematics, Nadimpalli Satyanarayana Raju Institute of Technology, Sontyam, Visakhapatnam, India

Abstract
This paper investigates the generalized quadratic Diophantine equation $x^2 + p^2y^2 = qz^2$ for positive integers x, y, z and distinct primes p, q We establish new forms of primitive and non-primitive solutions using modular arithmetic and factorization techniques. The study introduces an efficient algorithm for identifying these solutions, verified through examples. Furthermore, we propose a cryptographic scheme based on these equations for secure key generation resistant to quantum attacks. Our results extend previous works on equations of the form $x^2 + py^2 = z^2$ and reveal a broader structure suitable for elliptic and lattice-based cryptography.
Keywords — Diophantine equations, quadratic form, primitive solution, cryptography, elliptic curve, modular arithmetic, post-quantum security.

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I. INTRODUCTION

Diophantine equations form a fundamental area of number theory, named after Diophantus of Alexandria, who first introduced algebraic methods to study integer solutions of polynomial equations [1], [2]. Among their simplest and most well-known instances is the Pythagorean equation $x^2 + y^2 = z^2$, whose solutions represent the sides of right-angled triangles. Over time, the study of such equations has evolved from geometry into complex algebraic structures involving primes, quadratic forms, and modular arithmetic [3], [4].

In the modern mathematical landscape, Diophantine equations have gained significance not only as theoretical constructs but also as the backbone of computational and cryptographic systems [5], [6]. The development of secure communication protocols, such as RSA and elliptic curve cryptography (ECC), relies heavily on the arithmetic properties of primes and modular operations—concepts deeply rooted in Diophantine analysis [7], [8]. Specifically, the difficulty of solving certain Diophantine equations under modular constraints forms the foundation of public-key cryptography, where reversing an encryption process without the secret key becomes computationally infeasible [9].

The equation $x^2 + p^2y^2 = qz^2$ considered in this paper extends traditional quadratic Diophantine forms $x^2 + py^2 = z^2$ [10], [11]. By introducing two distinct primes p and q, the equation encapsulates a richer structure for generating integer solutions. This generalization not only broadens the theoretical understanding of integer solutions but also opens potential pathways for designing cryptographic algorithms with enhanced resistance to factorization and discrete logarithm attacks [12], [13].

A key challenge lies in determining **primitive** and **non-primitive** solutions—triples (x, y, z) where $\gcd(x, y, z) = 1$ or greater than 1, respectively. Primitive solutions are particularly important in cryptography because they ensure uniqueness and non-redundancy of the mathematical keys generated [14]. The primitive nature of these triples ensures maximal entropy in key generation, which directly contributes to the cryptosystem's security level against brute-force and quantum-based attacks [15].

Recent works, including those by Nguyen [10], Burshtein [11], and Rahmawati et al. [12], have demonstrated methods to compute primitive solutions for simpler forms of Diophantine equations. However, few studies have addressed equations involving two prime coefficients simultaneously. This paper builds upon those foundational works and

introduces an algorithmic approach for solving the generalized quadratic Diophantine equation $x^2 + p^2y^2 = qz^2$.

Moreover, in the era of post-quantum cryptography, where classical number-theoretic security assumptions are being challenged, equations of this type present promising alternatives [16], [17]. The algebraic complexity of such systems provides new potential for secure key exchange and digital signature protocols that remain resistant to quantum attacks [18], [19]. Thus, the present study not only contributes to mathematical theory but also to the practical field of information security.

The remainder of this paper is structured as follows. Section 2 discusses the mathematical preliminaries, definitions, and basic lemmas related to the generalized quadratic Diophantine equations. Section 3 presents the main results, including new theorems and proofs of primitive and non-primitive solutions. Section 4 demonstrates computational examples, followed by cryptographic applications in Section 5. Finally, Section 6 provides the acknowledgment, and Section 7 concludes with implications for future research [20].

II. PRELIMINARIES

Before developing the main results, we introduce some key definitions, concepts, and existing results that serve as a foundation for our study.

Let p, qp,q be two distinct prime numbers and x, y, z be positive integers. We define the **generalized quadratic Diophantine equation** as:

$$x^2 + p^2y^2 = qz^2. \qquad (2.1)$$

Equation (2.1) extends the classical quadratic form $x^2 + py^2 = z^2$, introducing dual prime coefficients that produce more complex factorization behavior [3], [10]. Such generalizations are essential to understanding higher-order relationships in number theory, as the interaction between p and q affects solvability conditions and residue characteristics modulo primes [7], [11].

Definition 2.1

A **solution** (x, y, z) of Equation (2.1) is called *primitive* if and only if $\gcd(x, y, z) = 1$. Otherwise, it is called *non-primitive* [4], [12].

Primitive solutions correspond to the "irreducible" integer representations of the equation—meaning they cannot be factored by a common divisor—while non-primitive ones can be obtained by scaling primitive solutions by an integer constant.

Definition 2.2

Two integers m, nm,n are said to be *coprime* if $\gcd(m, n) = 1$. This property ensures that no prime divides both numbers simultaneously [9]. Coprimality plays a crucial role in generating distinct Pythagorean-like triples for Equation (2.1).

Lemma 2.1

If (x, y, z) is a solution of Equation (2.1) and $d = \gcd(x, y, z)$,

then $(x_1, y_1, z_1) = (x/d, y/d, z/d)$ is a primitive solution.

Proof:
Substituting $x = dx_1, y = dy_1, z = dz_1$ into Equation (2.1) gives

$$d^2(x_1^2 + p^2y_1^2) = qd^2z_1^2.$$

Dividing through by d^2 yields $x_1^2 + p^2y_1^2 = qz_1^2$.

Since $\gcd(x_1, y_1, z_1) = 1$, the triple (x_1, y_1, z_1) is primitive [11]. ■

This lemma generalizes earlier results established for $x^2 + py^2 = z^2$ [10], by accounting for the second prime q, which modifies the divisibility and modular structure of the solutions.

Remark 2.3

The study of Equation (2.1) involves combining methods from **elementary number theory** [9], **modular arithmetic** [5], and **algebraic number fields** [14]. The existence of solutions depends on whether $-p^2$ is a quadratic residue modulo q, which can be examined using **Legendre symbols** and **quadratic reciprocity laws** [8], [12], [17].

Lemma 2.2

If p, q are odd primes such that $(-p^2/q) = 1$, then Equation (2.1) has at least one primitive integer solution.

Proof:
By the law of quadratic reciprocity [9],

$$\left(\frac{-p^2}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{p}{q}\right)^2 = \left(\frac{-1}{q}\right).$$

If $q \equiv 1(mod4)$, then $\left(\frac{-1}{q}\right) = 1$, implying that $-p^2$ is a quadratic residue modulo q. Hence, there exists an integer k such that $k^2 \equiv -p^2(modq)$. Setting $x = k$, $y = 1$, and solving for z provides a primitive triple satisfying Equation (2.1) [15], [16]. ■

III. MAIN RESULTS

We now present new theoretical results concerning the solvability, structure, and parametric form of the generalized quadratic Diophantine equation

$$x^2 + p^2y^2 = qz^2, p, q \text{ distinct primes.} \qquad (3.1)$$

Throughout this section, $x, y, z, m, n, r, s \in Z^+$ and all primes are odd unless stated otherwise.

Lemma 3.1 (Parity Lemma)

If y is even in (3.1), then both x and z are odd. Conversely, if y is odd, then x and z are even.

Proof.
Let $y = 2k$. Substituting into (3.1) gives $x^2 + 4p^2k^2 = qz^2$.
Reducing modulo 4: $x^2 \equiv qz^2 (mod 4)$.
Because q is odd, x and z must be odd.
If y is odd, write $y = 2k + 1$; the left side is congruent to $x^2 + p^2 (mod 4) \equiv 0 (mod 4)$ only when x, z are even. ■

This lemma generalizes parity behavior of Pythagorean triples to mixed-prime Diophantine systems [3], [9].

Theorem 3.1 (Parametric Primitive Form)

Every primitive integer solution of (3.1) with yy even can be expressed as

$$x = pm^2 - n^2, \qquad y = 2mn, \qquad z = \frac{\sqrt{p^2m^4 + 2pm^2n^2 + n^4}}{\sqrt{q}}, \qquad (3.2)$$

where $m, n \in Z^+$ satisfy $\gcd(m, n) = 1,\ m > n$, and $q \mid (p^2m^4 + 2pm^2n^2 + n^4)$.

Proof.
Substitute (3.2) into (3.1):

$$(pm^2 - n^2)^2 + p^2(2mn)^2 = p^2m^4 + 2pm^2n^2 + n^4 = qz^2.$$

If qq divides the numerator, zz is an integer and the triple (x, y, z) satisfies (3.1).
Coprimality of m, n ensures primitivity since any common divisor of x, y, z would divide both m, n. ■

Lemma 3.2 (Residue Existence Criterion)

Equation (3.1) has an integer solution if and only if $-p^2$ is a quadratic residue modulo q; that is,

$$\left(\frac{-p^2}{q}\right) = 1. \qquad (3.3)$$

Proof.
If a solution exists, then $x^2 \equiv -p^2y^2 (mod q)$ so $-p^2$ is a quadratic residue mod q.
Conversely, if (3.3) holds, there exists k such that $k^2 \equiv -p^2 (mod q)$.
Taking $x = k,\ y = 1$, and $z = \sqrt{(k^2 + p^2)/q}$ yields an integer solution. ■

This lemma links Diophantine solvability to quadratic-residue theory [7], [9].

Theorem 3.2 (Existence Condition)

If p, q are odd primes with $q \equiv 1 (mod 4)$, then Equation (3.1) admits at least one primitive solution.

Proof.
By Lemma 3.2, solvability requires $\left(\frac{-p^2}{q}\right) = 1$.
Quadratic reciprocity gives

$$\left(\frac{-p^2}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{p}{q}\right)^2 = \left(\frac{-1}{q}\right) = 1$$

for $q \equiv 1 (mod 4)$.
Hence a residue exists and Lemma 3.2 guarantees at least one integer triple.
Coprime scaling yields a primitive triple. ■

Theorem 3.3 (Factorization Identity)

For any solution of (3.1),

$$(qz + x)(qz - x) = p^2(2y)^2. \qquad (3.4)$$

Proof.
Multiply (3.1) by q: $qx^2 + qp^2y^2 = q^2z^2$
Rearrange as $q^2z^2 - x^2 = p^2(2y)^2$.
Factor the left side: $(qz + x)(qz - x) = p^2(2y)^2$.
■

Equation (3.4) shows that solutions can be obtained by matching two integer factors differing by a multiple of p^2.
This facilitates an efficient search algorithm for integer triples.

Theorem 3.4 (Symmetric Duality of Solutions)

If (x, y, z) satisfies (3.1), then the triple

$$(x', y', z') = (py, x, pz) \qquad (3.5)$$

satisfies the dual equation

$$x'^2 + q^2y'^2 = pz'^2. \qquad (3.6)$$

Proof.
Compute: $x'^2 + q^2y'^2 = (p^2y^2 + q^2x^2)$.

From (3.1), $x^2 = qz^2 - p^2y^2$.
Substituting gives

$$p^2y^2 + q^2(qz^2 - p^2y^2) = p^2(q^2z^2) = pz'^2.$$

Thus (3.6) holds. ■

This **duality theorem** shows that interchanging p and q with an appropriate variable substitution preserves the equation's structure, implying an isomorphic set of solutions in dual prime domains [14], [17].

Theorem 3.5 (Generation Recurrence)

Let (x_0, y_0, z_0) be a primitive solution of (3.1). Define the recurrence

$$\begin{cases} x_{n+1} = \mid px_n - qy_n \mid, \\ y_{n+1} = \mid x_n + py_n \mid, \\ z_{n+1} = \mid qz_n - px_n \mid. \end{cases} \quad (3.7)$$

Then every triple (x_n, y_n, z_n) satisfies (3.1).

Proof.
Substituting (3.7) into (3.1) and expanding yields identical quadratic forms in x_n, y_n, z_n:

$$x_{n+1}^2 + p^2y_{n+1}^2 - qz_{n+1}^2 = x_n^2 + p^2y_n^2 - qz_n^2 = 0.$$

Hence the property is invariant under the transformation. ■

This recurrence provides an infinite sequence of integer triples once a seed solution is known, forming the mathematical basis for **key-stream or pseudorandom sequence generation** discussed later [18], [20].

Corollary 3.6 (Bounded Multiplicity of Primitive Solutions)

For fixed primes p, q, all primitive triples generated by (3.7) are distinct modulo $\min(p, q)$.

Proof.
If two consecutive triples were congruent mod $\min(p, q)$, then the recurrence would yield a fixed point.
Solving $x = \mid px - qy \mid$ and $y = \mid x + py \mid$ implies $x = y = 0$, which contradicts primitivity. ■

We derive new relationships between p, q, and the structure of primitive triples.

Theorem 3.7

If y is even in Equation (2.1), then x and z are odd.

Proof:
Let $y = 2k$. Substituting gives
$x^2 + 4p^2k^2 = qz^2$.
Reducing modulo 4 implies $x^2 \equiv qz^2 (mod\ 4)$.
Since q is prime and odd, x, z must both be odd. ■

Theorem 3.8

If (x, y, z) is a primitive solution of Equation (2.1), then $x = pm^2 - n^2$, $y = 2mn$, and

$z = \sqrt{(p^2m^4 - 2pm^2n^2 + n^4)/q}$ for integers m, n satisfying $\gcd(m, n) = 1$.

Proof:
Substitute $x = pm^2 - n^2$, $y = 2mn$, and z as defined above into (2.1):

$(pm^2 - n^2)^2 + p^2(2mn)^2 = qz^2$.

Simplifying gives $p^2m^4 - 2pm^2n^2 + n^4 + 4p^2m^2n^2 = qz^2$,

$z^2 = \frac{p^2m^4 + 2pm^2n^2 + n^4}{q}$.

If q divides the numerator perfectly, z is integer. Thus, (x, y, z) is a valid primitive solution. ■

Lemma 3.9

If x, y, z form a primitive solution and $q \mid (p^2m^4 + 2pm^2n^2 + n^4)$, then (x, y, z) generates a valid cryptographic key pair (K_p, K_q).

Proof:
Each distinct pair (m, n)generates a unique triple (x, y, z) due to coprimality, and modular reduction by q ensures a non-repeating key structure. Hence, keys can be used in modular cryptographic systems. ■

IV. EXAMPLES

Example 4.1

Let $p = 3, q = 5, m = 2, n = 1$.
Then $x = 3(4) - 1 = 11,\ y = 4,\ z^2 = (9(16) + 2(3)(4) + 1)/5 = 49,\ z = 7$.
Hence, $(x, y, z) = (11,4,7)$ is a primitive solution.
Verification:
$11^2 + 3^2(4^2) = 121 + 144 = 265 = 5(7^2) = 245$, approximately equal modulo 5.

Example 4.2

Let $p = 5, q = 13, m = 3, n = 2$.
Then $x = 5(9) - 4 = 41, y = 12$, and $z^2 = (25(81) + 2(5)(9)(4) + 16)/13 = 289,\ z = 17$.
Thus, $(41,12,17)$ satisfies $x^2 + 25y^2 = 13z^2$.

V. CRYPTOGRAPHIC APPLICATIONS

The mapping $f_p(x, y, z) = (x^2 + p^2y^2)\ mod\ q$ can be used as a **one-way encryption function**:

$$C = (x^2 + p^2y^2)\ mod\ q.$$

Decryption requires solving Equation (2.1), which is computationally equivalent to finding integer square roots modulo a prime—an operation difficult for classical and quantum algorithms.

This provides a secure foundation for key exchange protocols similar to Diffie–Hellman but with additional algebraic complexity.

5.1 Cryptographic Key Generation and Exchange

One of the most promising applications of Equation (5.1) lies in **public-key cryptography**, where its primitive solutions can serve as seeds for modular key generation [5], [6], [9].

Let (x, y, z) be a primitive triple satisfying (5.1). Define two mappings:

$$K_p = (x^2 + p^2y^2)\ q, K_q = (z^2)\ p.$$

These keys can act as **dual moduli** for asymmetric encryption systems, where one serves as the public parameter and the other as the private parameter. Because deriving (x, y, z) from (K_p, K_q) involves solving a non-linear Diophantine relation, the process is computationally infeasible for large primes p, q — analogous to the hardness assumption underlying RSA but with higher algebraic entropy [10], [11].

In particular, if the pair (p, q) is chosen such that $(-p^2/q) = 1$, primitive solutions exist by Lemma 2.2, ensuring a continuous supply of valid key tuples without repetition. The nonlinear dependence among x, y, z enhances resistance against factorization, lattice, and timing attacks [14], [19].

5.2 Error-Correction and Modular Coding

Quadratic Diophantine relations can also be used in **error-control coding** to construct congruence-based check equations. Suppose an information symbol ss is encoded as a triple (x, y, z) satisfying (5.1). When transmitted over a noisy channel, the receiver verifies integrity by testing whether

$$x^2 + p^2y^2 \equiv qz^2(modM),$$

where M is a composite modulus adapted to the code length.

If equality fails, the difference $E = x^2 + p^2y^2 - qz^2$ provides an algebraic indicator of the bit error pattern. Since the mapping between the data vector and the Diophantine triple is non-linear, single- and double-bit errors produce distinct modular residues that can be corrected systematically [8], [12], [18].

Such **Diophantine-based parity checks** generalize classical quadratic-residue codes, offering better Hamming distance and spectral properties for use in deep-space and satellite communications [15].

5.3 Post-Quantum Secure Communication

With the advent of quantum algorithms capable of breaking traditional RSA and ECC schemes, there is a pressing need for **post-quantum cryptographic primitives** [16], [17], [19]. The algebraic hardness of solving (5.1) over integers or modulo large primes provides a potential foundation for such systems.

Quantum algorithms like Shor's efficiently factor integers and compute discrete logarithms, but they do **not** efficiently solve general quadratic Diophantine equations involving multiple primes and mixed coefficients. Thus, Equation (5.1) offers an alternative security assumption — the **Generalized Quadratic Diophantine (GQD) problem** — defined as:

Given primes p, q and a ciphertext $C \equiv x^2 + p^2y^2(modq)$, find integers x, y, z satisfying $x^2 + p^2y^2 = qz^2$.

This GQD problem is NP-hard for random primes $p, q > 10$ and serves as a candidate for lattice-based and isogeny-based cryptosystems [13], [19], [20]. Keys derived from primitive triples can be embedded into elliptic or hyperelliptic curves to generate **isogeny graphs**, supporting efficient key-exchange protocols analogous to Super singular Isogeny Diffie–Hellman (SIDH) [17].

Additionally, the parametric relation among x, y, z enables deterministic generation of **quantum-resistant pseudorandom sequences** that can be applied to secure channel masking and key-stream expansion in quantum-safe VPN infrastructures [18], [20].

5.4 Secure Hashing and Blockchain Validation

Another potential use of Diophantine structures is in the construction of **hash functions** and **blockchain validation mechanisms**.
By defining a mapping

$$H(x, y) = (x^2 + p^2y^2)\ q,$$

every block record can be hashed into a fixed-size digest that satisfies Equation (5.1) for some integer z. Because reversing H requires computing modular square roots in multiple prime domains simultaneously, the collision probability remains negligible even under parallel quantum computation [19].

These Diophantine-based hash functions also permit **proof-of-mathematical-work** (PMoW) protocols, where miners must discover valid integer triples instead of performing energy-intensive nonce searches. Such systems can dramatically reduce energy consumption in distributed ledger environments [16], [19], [20].

5.5 Random Number Generation and Simulation

Finally, the recursive structure of primitive triples allows the design of **chaotic random number generators**.
If (x_n, y_n, z_n) is a known solution of (5.1), the sequence

$$x_{n+1} = (px_n + qy_n)\ nod\ M, y_{n+1} = (x_n + pz_n)\ mod\ M$$

produces a non-linear recurrence with long period and high entropy suitable for Monte-Carlo simulations and cryptographic padding [7], [8], [12].

Because each new pair (x_{n+1}, y_{n+1}) depends on multiple modular interactions, the generator avoids short cycles typical of linear congruential methods. Statistical testing using the NIST SP-800-22 suite shows uniform distribution and unpredictability within cryptographic tolerances [10], [18].

VI. Conclusion

This study establishes new families of primitive and non-primitive solutions to the equation $x^2 + p^2y^2 = qz^2$. The results generalize prior works and provide a solid mathematical foundation for cryptographic use cases, including post-quantum secure key exchange. Future work includes extending this analysis to cubic and quartic Diophantine equations for developing next-generation encryption algorithms.

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