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B_4- Metric spaces and Contractions

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Abstract—As a generalisation of metric spaces, the notion of B_4-metric spaces are introduced. In this article, we consider a relation between a metric space a B_4-metric space, S-metric space and a metric space. We show that a B_4-metric space can be generated by S-metric space. We also show that a S-metric space, gives rise to a B_4-metric space and give some examples. We also study the relationship of contractions of self-maps on B_4-metric space and
S-metric spaces.

Keywords—B_4-metric space, S-metric space, metric space, S-metric space with index λ.

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I. INTRODUCTION

"Recently, Sedghi, Shobe, and Aliouche [2] have defined the concept of a S-metric space as a generalization of a metric space as follows:"

***Definition* 01**: [1] "Let$Y \neq \phi$, and $S: Y^3 \to [0, \infty)$be a function meeting the criteria below:
for all$f_0, g_0, h_0, a \in Y$
1. $S(f_0,g_0,h_0) = 0$ if and only if $f_0 = g_0 = h_0$
2. $S(f_0,g_0,h_0) \leq S(f_0,f_0,a) + S(g_0,g_0,a) + S(h_0,h_0,a)$.
Then, S is called an S-metric on Y and the pair (Y,S) is called an S-metric space."

***Example*01**: Let $Y \neq \phi$ and define the function $S: Y^3 \to [0, \infty)$ as

$$S(f_0, g_0, h_0) = \begin{cases} 0, & \text{if} f_0 = g_0 = h_0 \\ 1, & \text{otherwise} \end{cases}$$

Then it's simple to verified that$S$ is an S-metric and $(Y,S)$is an S-metric space on Y.
This S-metric is called the trivial S-metricon Y.

Notation:
The terms "R" and "N" refer, respectively, to the set of real numbers and the set of positive integers.
Now we introduce the notion of B_4-metricspaces follows:

***Definition* 02**: Let $Y \neq \phi$ and $B_4: Y^4 \to R$ be a function meeting the criteria below:
for all $f_0,g_0,h_0,t_0,a \in Y$
(i) $B_4(f_0,g_0,h_0,t_0) = 0$ if and only if $f_0 = g_0 = h_0 = t_0$,
(ii) $B_4(f_0,g_0,h_0,t_0) \leq B_4(f_0,f_0,f_0,a) + B_4(g_0,g_0,g_0,a) + B_4(h_0,h_0,h_0,a) + B_4(t_0,t_0,t_0,a)$.
Then, B_4 is called B_4-metric on Y and the pair (Y, B_4) is called a B_4-metric space.
The following two examples give an insight into the concept of B_4-metric spaces.

***Example*02**: Let $R = Y$ and define the function $B_4: Y^4 \to R$ by
$B_4(f_0,g_0,h_0,t_0) = |f_0 - h_0| + |g_0 - h_0| + |f_0 + g_0 + h_0 - 3t_0|$, for all$f_0,g_0,h_0,t_0 \in R$.
Then, B_4 is a B_4 -metric on Y.

***Example*03**: Let $Y \neq \phi$ and define the function $B_4: Y^4 \to R$ as

$B_4\ (f_0, g_0, h_0, t_0\ ) = \begin{cases} 0, & \text{if } f_0 = g_0 = h_0 = t_0 \\ 1, & \text{otherwise} \end{cases}$

Then, B_4 is a B_4-metric on Y.

Now we introduce the notions oflimits, convergence, Cauchy sequence and completeness in a B_4-metric space.

***Definition*03**: Let(Y, B_4) be a B_4-metric space and $A \subset Y$.

1.***Convergence*:** A sequence $\{f_{0\,n}\}$ in Y **Converges** to f_0 if $B_4(f_{0\,n}, f_{0\,n}, f_{0\,n}, f_0) \rightarrow 0$ as $n \rightarrow \infty$. That is given $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$, $B_4(f_n, f_n, f_n, f_0) < \varepsilon$.

We denote this by$\lim_{n\rightarrow\infty} f_{0\,n} = f_0$.

2.***CauchySequence***: A sequence $\{f_{0\,n}\}$ in Y is called a **Cauchy Sequence** if $B_4(f_{0\,n}, f_{0\,n}, f_{0\,n}, f_{0\,m}) \rightarrow 0\ as\ n, m \rightarrow \infty$. That is, given $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $B_4(f_n, f_n, f_n, f_m) < \varepsilon$.

3.***Completeness*:** AB_4-metric space $(Y, B_4)$is called **Complete** if every Cauchy Sequence in Y is convergent to a point $f_0 \in Y$.

We now prove a few lemmas, which we use in our further development.

Lemma 01: *Let (Y, B_4)be a B_4-metricspace. Then$B_4(f_0, f_0, f_0, g_0) = B_4(g_0, g_0, g_0, f_0)$, for all $f_0, g_0 \in Y$.*

Proof: Suppose(Y, B_4) is a B_4-metric space.

By definition 2, replacing a by f_0 we get,

$B_4(f_0, f_0, f_0, g_0) \leq B_4(f_0, f_0, f_0, a) + B_4(f_0, f_0, f_0, a) + B_4(f_0, f_0, f_0, a) + B_4(g_0, g_0, g_0, a)$

$B_4(f_0, f_0, f_0, g_0) \leq B_4(f_0, f_0, f_0, f_0) + B_4(f_0, f_0, f_0, f_0) + B_4(f_0, f_0, f_0, f_0) + B_4(g_0, g_0, g_0, f_0)$

$\Rightarrow B_4(f_0, f_0, f_0, g_0) \leq 0 + 0 + 0 + B_4(g_0, g_0, g_0, f_0)$

$\Rightarrow B_4(f_0, f_0, f_0, g_0) \leq B_4(g_0, g_0, g_0, f_0)$ (1.1)

Similarly,

$B_4(g_0, g_0, g_0, f_0) \leq B_4(f_0, f_0, f_0, g_0)$ (1.2)

From(1.1) and (1.2), $B_4(f_0, f_0, f_0, g_0) = B_4(g_0, g_0, g_0, f_0)$.

Lemma 02: *$f_{0_n} \rightarrow f_0$ if and only if $B_4(f_0, f_0, f_0, f_{0_n}) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof: Suppose(Y, B_4) is a B_4-metric space, $\{f_{0_n}\}$is a sequence in Y and $f_0 \in Y$.

By definition 3(i), $f_{0_n} \rightarrow f_0$

$\Leftrightarrow B_4(f_{0\,n}, f_{0\,n}, f_{0\,n}, f_0) \rightarrow 0$

$\Leftrightarrow B_4(f_0, f_0, f_0, f_{0\,n}) \rightarrow 0$ (by lemma 1)

The following Lemma shows that a convergent sequence has unique limit.

Lemma 03: $f_{0\,n} \rightarrow f_0,\ f_{0\,n} \rightarrow g_0 \Rightarrow f_0 = g_0$.

Proof: Suppose(Y, B_4) is a B_4- metric space.

$B_4(f_0, f_0, f_0, g_0) \leq B_4(f_0, f_0, f_0, f_{0\,n}) + B_4(f_0, f_0, f_0, f_{0\,n}) + B_4(f_0, f_0, f_0, f_{0\,n}) + B_4(g_0, g_0, g_0, g_n)$

$\longrightarrow 0 + 0 + 0 + 0$ (by hypothesis and lemma1)

Therefore $B_4(f_0, f_0, f_0, g_0) \leq 0$

$\Rightarrow B_4(f_0, f_0, f_0, g_0) = 0$

$\Rightarrow f_0 = g_0$.

Lemma 04: *$f_{0\,n} \rightarrow f_0 \Rightarrow \{f_{0\,n}\}$ is a Cauchy Sequence.*

Proof: Suppose$\varepsilon > 0$.Then there exist $n_0 \in N$ such that, for all $n \geq n_0$

$B_4(f_{0\,n}, f_{0\,n}, f_{0\,n}, f_0) < \varepsilon$.

We have, for m,$n \geq n_0$.

$B_4\ (f_{0\ n}, f_{0\ n}, f_{0\ n}, f_{0\ m})$
$\leq\ B_4(f_{0\ n}, f_{0\ n}, f_{0\ n}, f_0) + B_4(f_{0\ n}, f_{0\ n}, f_{0\ n}, f_0) + B_4(f_{0\ n}, f_{0\ n}, f_{0\ n}, f_0)$
$+ B_4(f_{0\ m}, f_{0\ m}, f_{0\ m}, f_0)$
$< \varepsilon + \varepsilon + \varepsilon + \varepsilon = 4\,\varepsilon$
Hence $\{f_{0\ n}\}$is a Cauchy Sequence.
In the following Lemma we show that an S-metric gives rise to a B_4-metric.

***Lemma 05**: Let(Y, S) be an S-metric space and define the function $B_S: Y^4 \rightarrow R$ by*
$B_S(f_0, g_0, h_0, t_0) = S(g_0, h_0, t_0) + S(f_0, h_0, t_0) + S(f_0, g_0, t_0) + S(f_0, g_0, h_0)$ *for all* $f_0, g_0, h_0, t_0 \in Y$.
Then B_S is a B_4-metricon Y.
Proof: Suppose (Y, S) is an S-metric space,
Then $B_4(f_0, g_0, h_0, t_0) = 0$
$\Leftrightarrow S(g_0, h_0, t_0) + S(f_0, h_0, t_0) + S(f_0, g_0, t_0) + S(f_0, g_0, h_0) = 0$
$\Leftrightarrow S(g_0, h_0, t_0) = 0, S(f_0, h_0, t_0) = 0, S(f_0, g_0, t_0) = 0, S(f_0, g_0, h_0) = 0$
$\Leftrightarrow g_0 = h_0 = t_0, \quad f_0 = h_0 = t_0, \quad f_0 = g_0 = t_0, \quad f_0 = g_0 = h_0$
$\Leftrightarrow f_0 = g_0 = h_0 = t_0$.
Now we show that, for all$f_0, g_0, h_0, t_0, a \in Y$,
$B_s(f_0, g_0, h_0, t_0) \leq B_s(f_0, f_0, f_0, a) + B_s(g_0, g_0, g_0, a) + B_s(h_0, h_0, h_0, a) + B_s(t_0, t_0, t_0, a)$ (5.1)
Now $B_s(f_0, f_0, f_0, a) + B_s(g_0, g_0, g_0, a) + B_s(h_0, h_0, h_0, a) + B_s(t_0, t_0, t_0, a)$
$= 3(S(f_0, f_0, a) + S(g_0, g_0, a) + S(h_0, h_0, a) + S(t_0, t_0, a))$
(5.2)
Also $B_s(f_0, g_0, h_0, t_0) = S(g_0, h_0, t_0) + S(f_0, h_0, t_0) + S(f_0, g_0, t_0) + S(f_0, g_0, h_0)$
$\leq (S(g_0, g_0, a) + S(h_0, h_0, a) + S(t_0, t_0, a) + S(f_0, f_0, a) + S(h_0, h_0, a) + S(t_0, t_0, a)$
$+ S(f_0, f_0, a) + S(g_0, g_0, a) + S(h_0, h_0, a) + S(f_0, f_0, a) + S(g_0, g_0, a) + S(t_0, t_0, a))$
$=3(S(f_0, f_0, a) + S(g_0, g_0, a) + S(h_0, h_0, a) + S(t_0, t_0, a))$ (5.3)
Therefore from(5.2) and (5.3), (5.1) holds.
Therefore (Y, B_s) is a B_4-metricon Y.
We call B_s as the B_4-metric generated by S.

***Definition04**:* Let $Y \neq \phi$, and $\lambda \geq 1$. Suppose $S: Y^3 \rightarrow R$ be a function meeting the criteria below:
for allf_0,g_0,h_0, $a \in Y$
1. $S\ (f_0, g_0, h_0) = 0$ if and only if $f_0 = g_0 = h_0$
2.$S(f_0, g_0, h_0) \leq \lambda(S(f_0, f_0, a) + S(g_0, g_0, a) + S(h_0, h_0, a))$.
Then, S is called a S -metric on Y and the pair (Y, S) is called a S - metric space with index λ.
Note: If $\lambda = 1$ we get the usual S - metric space (by definition 1)

Lemma 06: *Let (Y, B_4) be any B_4-metric space. Define $S_b: Y^3 \rightarrow R$ as follows:*
$S_b(f_0, g_0, h_0) = B_4(f_0, f_0, g_0, h_0) + B_4(f_0, g_0, g_0, h_0) + B_4(f_0, g_0, h_0, h_0)$, *for all* $f_0, g_0, h_0 \in Y$.
Then (Y, S_b) is a S-metricspace with index 2. ($S_b$is called S_b-metricspace with index 2)
Proof: Suppose $(Y, B_4)$is a B_4-metric space
Now$S(f_0, g_0, h_0) = 0$
$\Leftrightarrow B_4(f_0, f_0, g_0, h_0) + B_4(f_0, g_0, g_0, h_0) + B_4(f_0, g_0, h_0, h_0) = 0$
$\Leftrightarrow B_4(f_0, f_0, g_0, h_0) = 0, \quad B_4(f_0, g_0, g_0, h_0) = 0, \quad B_4(f_0, g_0, h_0, h_0) = 0$
$\Leftrightarrow f_0 = g_0 = h_0$.
(6.1)
Now we show that $S_b(f_0, g_0, h_0) \leq 2(S(f_0, f_0, a) + S(g_0, g_0, a) + S(h_0, h_0, a))$ for all$f_0, g_0, h_0, a \in Y$.
$S_b(f_0, g_0, h_0) = B_4(f_0, f_0, g_0, h_0) + B_4(f_0, g_0, g_0, h_0) + B_4(f_0, g_0, h_0, h_0)$
$\leq B_4(f_0, f_0, f_0, a) + B_4(f_0, f_0, f_0, a) + B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a) + B_4(f_0, f_0, f_0, a)$
$+ B_4(g_0, g_0, g_0, a) + B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a) + B_4(f_0, f_0, f_0, a)$
$+ B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a) + B_4(h_0, h_0, h_0, a)$
$\leq 4(B_4(f_0, f_0, f_0, a) + B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a))$ (6.2)
It can be easily shown that $S_b(f_0, f_0, a)$+$S_b(g_0, g_0, a)$+$S_b(h_0, h_0, a)$
=
2($B_s(f_0, f_0, f_0, a) + B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a) + B_4(f_0, f_0, f_0, a) +$

$B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a))$
(6.3)

From(6.2) and (6.3) we get,

$S_b(f_0, g_0, h_0) = 4(B_4(f_0, f_0, f_0, a) + B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a))$
$\leq 4(B_4(f_0, f_0, f_0, a) + B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a))$
$\leq 2((2((B_4(f_0, f_0, f_0, a) + B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a)) + B_4(f_0, f_0, a, a) +$
$B_4(g_0, g_0, a, a) + B_4(h_0, h_0, a, a))$
$= 2((2(B_4(f_0, f_0, f_0, a) + B_4(f_0, f_0, f_0, a)) + (2B_4(g_0, g_0, g_0, a) + B_4(g_0, g_0, a, a)) +$
$B4h0,h0,h0,a + B4h0,h0,a,a))$

$= 2(S_b(f_0, f_0, a) + S_b(g_0, g_0, a) + S_b(h_0, h_0, a))$

Hence S_b is a S-metric on Y with index 2.

Definition05: Suppose $Y \neq \phi$ and $\lambda \geq 1$. Suppose $d: Y^2 \to R$ satisfies

(i) $d(f_0, g_0) = 0$ if and only if $f_0 = g_0$

(ii) $d(f_0, g_0) = d(g_0, f_0)$

(iii) $d(f_0, g_0) \leq \lambda (d(f_0, h_0) + d(h_0, g_0)$, for all $f_0, g_0, h_0 \in Y$.

Then (Y, d) is called an d-metric space index $\lambda \geq 1$.

Lemma 07: *Let (Y, S) be any S-metric space. Define $d_s: Y^2 \to R$ as follows:*
$d_s(f_0, g_0) = S(f_0, f_0, g_0) + S(f_0, g_0, g_0), \textit{for all } f_0, g_0 \in Y.$
Then $d_s(f_0, g_0) \leq \frac{3}{2}\left(d_s(f_0, g_0) + d_s(g_0, h_0)\right), \textit{ for all } f_0, g_0, h_0 \in Y.$
Thus (Y, d_s) is a d-metricspace with index 3/2.

Example 04: Let $R = Y$ and define the function $S: Y^3 \to R$ by
$S(f_0, g_0, h_0) = |f_0 - h_0|^2 + |f_0 + h_0 - 2g_0|^2$, for all $f_0, g_0, h_0 \in R$
Then, (Y, S) is a S-metricspacewith index 4.
Now we introduce the notion of B_4-metric space with index λ.

Definition 06: Let $Y \neq \phi$, $\lambda \geq 1$ and let $B_4: Y^4 \to R$ be a function meeting the criteria below:
for all $f_0, g_0, h_0, t_0, a \in Y$
(i) $B_4(f_0, g_0, h_0, t_0) = 0$ if and only if $f_0 = g_0 = h_0 = t_0$,
(ii) $B_4(f_0, g_0, h_0, t_0) \leq B_4(f_0, f_0, f_0, a) + B_4(g_0, g_0, g_0, a) + B_4(h_0, h_0, h_0, a) + B_4(t_0, t_0, t_0, a))$.
Then, B_4 is called B_4-metricon X and the pair (Y, B_4) is called a B_4-metric space with index λ.
Note: If $\lambda = 1$, clearly (Y, B_4) is a B_4-metric space on Y.

Example 05: Let $Y \neq \phi$, $\mu > 0$ and define the function $B_4: Y^4 \to R$ as

$$B_4(f_0, g_0, h_0, t_0) = \begin{cases} 0, & if\ f_0 = g_0 = h_0 = t_0 \\ \mu, & otherwise \end{cases}$$

Then (Y, B_4) is a B_4-metric space with index 3.

Definition 07: Suppose $\mu > 0$, and (Y, d) is metricspace. Amap $\boldsymbol{T}: \boldsymbol{Y} \to Y$ is said to be a contraction,
if $d(T(f_0), T(g_0)) \leq \mu\, d(f_0, g_0)$, for all $f_0, g_0 \in Y$.

Theorem01: *([2], Proposition 1)Suppose (Y, d) is metric space and a map $\boldsymbol{T}: \boldsymbol{Y} \to Y$. If d is a contraction with constant contraction μ, that is $d(T(f_0), T(g_0)) \leq \mu\, d(f_0, g_0)$,*
then S_d (generated S - metric) is a contraction with constant contraction μ.

Definition 08: Let $(\boldsymbol{Y}, \boldsymbol{S})$ be a S-metricspace, $d_s(f_0, g_0) = d(f_0, g_0) = S(f_0, f_0, g_0) + S(g_0, g_0, f_0)$.
Suppose $\boldsymbol{T}: \boldsymbol{Y} \to Y$. Then T is called a contraction with constant contraction μ,
if $S(T(f_0), T(g_0), T(h_0)) \leq \mu\, S(f_0, g_0, h_0)$, for all $f_0, g_0, h_0 \in Y$.

Theorem02: *Suppose (Y, S) is a S-metric space and a map $\boldsymbol{T}: \boldsymbol{Y} \to Y$ is a contraction with constant contraction μ.*
Then $\boldsymbol{T}$ is a contraction with respect to d_s (the metric induced by S) with constant contraction μ.

Proof: $d_s(Tf_0,Tg_0) = S(Tf_0,Tf_0,Tg_0) + S(Tg_0,Tg_0,Tf_0)$
$\leq \mu\,(S(f_0,f_0,g_0) + S(g_0,g_0,f_0))$
$= \mu\, d_s(f_0,g_0)$
Therefore $\boldsymbol{T}$ is a contraction with respect to d_s with constant contraction μ.

Definition 09: Suppose $\mu > 0$ and (Y, B_4) is a B_4-metric space. $\boldsymbol{T}: \boldsymbol{Y} \to Y$ is called a contraction with constant contraction μ, if $B_4(Tf_0,Tg_0,Th_0,Tt_0) < \mu\, B_4(f_0,g_0,h_0,t_0)$, for all $f_0,g_0,h_0,t_0 \in Y$.

Theorem03: *Suppose $\boldsymbol{T}$ is a contraction with contraction constant μ on (Y, B_4). A map $\boldsymbol{T}: \boldsymbol{Y} \to Y$. Then $\boldsymbol{T}$ is a S_b-contraction on Y with constant contraction μ.*
Proof: $S_b(Tf_0,Tg_0,Th_0) = B_4(Tf_0,Tf_0,Tg_0,Th_0) + B_4(Tf_0, Tg_0,Tg_0,Th_0) + B_4(Tf_0,Tg_0,Th_0,Th_0)$
$\leq \mu\,(B_4(f_0,f_0,g_0,h_0) + B_4(f_0,g_0,g_0,h_0) + B_4(f_0,g_0,h_0,h_0))$
$= \mu\, S_b(f_0,g_0,h_0)$
Therefore $\boldsymbol{T}$ is a contraction with respect to S_b with constant contraction μ.

Theorem04: *([2], Proposition 1) "Suppose $(\boldsymbol{Y}, \boldsymbol{d})$ is a metric space and a map $\boldsymbol{T}: \boldsymbol{Y} \to Y$. Define $S: Y^3 \to R$ by*
$S(f_0,g_0,h_0) = d(f_0,h_0) + d(h_0,g_0)$ for all $f_0,g_0,h_0 \in Y$. That is S is the S-metric on $\boldsymbol{Y}$ induced by d.
Suppose $d(Tf_0,Tg_0) < max\{d(f_0,g_0), d(f_0,Tf_0), d(g_0,Tg_0), d(f_0,Tg_0), d(g_0,Tf_0)\}$ for all $f_0,g_0 \in Y$. Then
$S(Tf_0,Tf_0,Tg_0) < max\{S(f_0,f_0,g_0), S(Tf_0,Tf_0,f_0), S(Tf_0,Tf_0,g_0), S(Tg_0,Tg_0,f_0), S(Tg_0,Tg_0,g_0)\}$,
for all $f_0,g_0 \in Y$."

Theorem05: *([2], Proposition 2) "Suppose $(\boldsymbol{Y}, \boldsymbol{S})$ is a $\boldsymbol{S}$-metric spaceand a map $\boldsymbol{T}: \boldsymbol{Y} \to Y$.*
Define $d: Y^2 \to R$ by $d(f_0,g_0) = S(f_0,f_0,g_0) + S(g_0,g_0,f_0)$ for all $f_0,g_0 \in Y$ (that is d is the induced metric).
Suppose $S(Tf_0,Tf_0,T g_0) <$
$max\{S(f_0,f_0,g_0), S(Tf_0,Tf_0,f_0), S(Tf_0,Tf_0,g_0), S(Tg_0,Tg_0,f_0), S(T g_0,Tg_0,g_0)\}$,
for all $f_0,g_0 \in Y$.
Then $d(Tf_0,Tg_0) < max\{ d(f_0,g_0), d(f_0,Tf_0), d(g_0,Tg_0), d(f_0,Tg_0), d(g_0,Tf_0)\}$, for all $f_0,g_0 \in Y$.

Now we prove the following theorem, which is more general than Theorem 5."

Theorem06: *Suppose $(\boldsymbol{Y}, \boldsymbol{S})$ is a $\boldsymbol{S}$-metric spaceand a map $\boldsymbol{T}: \boldsymbol{Y} \to Y$. Define $d: Y^2 \to R$ by*
$d(f_0,g_0) = S(f_0,f_0,g_0) + S(g_0,g_0,f_0)$, for all $f_0,g_0 \in Y$.
Suppose
$S(Tf_0,Tg_0,Th_0) < max\{S(f_0,g_0,h_0), S(Tf_0,Tf_0,f_0), S(Tg_0,Tg_0,g_0), S(Th_0,Th_0,h_0)\}$,
for all $f_0,g_0,h_0 \in Y$.
Then $d(Tf_0,Tg_0) < max\{d(f_0,g_0), d(Tf_0,f_0), d(Tg_0,g_0)\}$ for all $f_0,g_0 \in Y$, where d is the induced metric.
Proof: $S(Tf_0,Tg_0,Th_0) < \max\{S(f_0,g_0,h_0), S(Tf_0,Tf_0,f_0), S(Tg_0,Tg_0,g_0), S(Th_0,Th_0,h_0)\}$, for all $f_0,g_0,h_0 \in Y$.
$S(Tf_0,Tf_0,Tg_0) < \max\{S(f_0,f_0,g_0), S(Tf_0,Tf_0,f_0), S(Tf_0,Tf_0,f_0), S(Tg_0,Tg_0,g_0)\}$
Now we show that $d(Tf_0,Tg_0) < max\{ d(f_0,g_0), d(Tf_0,f_0), d(Tg_0,g_0)\}$, for all $f_0,g_0 \in Y$, where d is the induced metric.
LHS: $d(Tf_0,Tg_0) = S(Tf_0,Tf_0,Tg_0) + S(Tg_0,Tg_0,Tf_0) = 2S(Tf_0,Tf_0, Tg_0)$
RHS: $\max\{d(f_0,g_0), d(Tf_0,f_0), d(Tg_0,g_0)\}$
$= \max\{S(f_0,f_0,g_0), S(g_0,g_0,f_0), S(Tf_0,Tf_0,f_0), S(f_0,Tf_0,Tf_0), S(Tg_0,Tg_0,g_0), S(g_0,Tg_0,Tg_0)\}$
$= 2\, max\{S(f_0,f_0,g_0), S(Tf_0,Tf_0,f_0), S(Tg_0,Tg_0,g_0)\}$
Therefore $d(Tf_0,Tg_0) = 2S(Tf_0,Tf_0,Tg_0)$
$< 2\max\{S(f_0,f_0,g_0), S(Tf_0,Tf_0,f_0), S(Tf_0,Tf_0,f_0), S(Tg_0,Tg_0,g_0)\}$
$= \max\{2S(f_0,f_0,g_0), 2S(Tf_0,Tf_0,f_0), 2S(Tg_0,Tg_0,g_0)\}$
$= \max\{d(f_0,g_0), d(Tf_0,f_0), d(Tg_0,g_0)\}$
Hence $d(Tf_0,Tg_0) < \max\{ d(f_0,g_0), d(Tf_0,f_0), d(Tg_0,g_0)\}$, for all $f_0,g_0 \in Y$, where d is the induced metric.

Relation between B_4 contractions and S contractions

In this section we establish some relations between B_4 contractions and S contractions

Theorem07: *Let (Y, B_4) be a B_4-metric spaceand a map $\boldsymbol{T}: \boldsymbol{Y} \to Y$. Define $S: Y^3 \to R$ by*
$S(f_0, g_0, h_0) = B_4(f_0, f_0, g_0, h_0) + B_4(f_0, g_0, g_0, h_0) + B_4(f_0, g_0, h_0, h_0), for\ all\ f_0, g_0, h_0 \in Y.$
Suppose
$B_4(Tf_0, Tg_0, Th_0, Tt_0)$
$< max\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, Tg_0, g_0), B_4(Th_0, Th_0, Th_0, h_0), B_4(Tt_0, Tt_0, Tt_0, t_0)\},$
$for\ all\ f_0, g_0, h_0, t_0 \in Y.$
Further suppose $B_4(Tf_0, Tf_0, Tf_0, f_0) \leq B_4(Tf_0, Tf_0, f_0, f_0), for\ all\ f_0 \in Y.$
Then $S(Tf_0, Tg_0, Th_0) < max\{S(Tf_0, Tf_0, f_0), S(Tg_0, Tg_0, g_0), S(Th_0, Th_0, h_0)\}, for\ all\ f_0, g_0, h_0 \in Y.$

Proof: Suppose
$B_4(Tf_0, Tg_0, Th_0, Tt_0)$
$< max\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, Tg_0, g_0), B_4(Th_0, Th_0, Th_0, h_0), B_4(Tt_0, Tt_0, Tt_0, t_0)\},$
for all $f_0, g_0, h_0, t_0 \in Y.$
Then we show that
$S(Tf_0, Tg_0, Th_0) <$ max$\{S(Tf_0, Tf_0, f_0), S(Tg_0, Tg_0, g_0), S(Th_0, Th_0, h_0)\}$, for all $f_0, g_0, h_0 \in Y.$
L.H.S: $S(Tf_0, Tg_0, Th_0) = B_4(Tf_0, Tf_0, Tg_0, Th_0) + B_4(Tf_0, Tg_0, Tg_0, Th_0) + B_4(Tf_0, Tg_0, Th_0, Th_0)$
Now
$B_4(Tf_0, Tf_0, Tg_0, Th_0)$
$< max\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, Tg_0, g_0), B_4(Th_0, Th_0, Th_0, h_0)\}$
$= B_4(Tf_0, Tf_0, Tf_0, f_0)$ (say)
$B_4(Tf_0, Tg_0, Tg_0, Th_0)$
$< max\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, Tg_0, g_0), B_4(Tg_0, Tg_0, Tg_0, g_0), B_4(Th_0, Th_0, Th_0, h_0)\}$
$= B_4(Tf_0, Tf_0, Tf_0, f_0)$ (say)
$B_4(Tf_0, Tg_0, Th_0, Th_0)$
$< max\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, Tg_0, g_0), B_4(Th_0, Th_0, Th_0, h_0), B_4(Th_0, Th_0, Th_0, h_0)\}$
$= B_4(Tf_0, Tf_0, Tf_0, f_0)$ (say)
Also (i) $S(Tf_0, Tf_0, f_0) = B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, f_0, f_0)$
(ii) $S(Tg_0, Tg_0, g_0) = B_4(Tg_0, Tg_0, Tg_0, g_0) + B_4(Tg_0, Tg_0, Tg_0, g_0) + B_4(Tg_0, Tg_0, g_0, g_0)$
(iii) $S(Th_0, Th_0, h_0) = B_4(Th_0, Th_0, Th_0, h_0) + B_4(Th_0, Th_0, Th_0, h_0) + B_4(Th_0, Th_0, h_0, h_0)$
And $B_4(Tf_0, Tf_0, Tf_0, f_0) < S(Tf_0, Tf_0, f_0)$,
L.H.S: $S(Tf_0, Tg_0, Th_0) < B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, Tf_0, f_0)$
$\leq B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, f_0, f_0)$
$= S(Tf_0, Tf_0, f_0)$
$\leq max\{\{S(Tf_0, Tf_0, f_0), S(Tg_0, Tg_0, g_0), S(Th_0, Th_0, h_0)\}$ = R.H.S
Therefore $S(Tf_0, Tg_0, Th_0) <$ max$\{\{S(Tf_0, Tf_0, f_0), S(Tg_0, Tg_0, g_0), S(Th_0, Th_0, h_0)\}$

Theorem08: *Let (Y, B_4) be a B_4-metric spaceand a map $\boldsymbol{T}: \boldsymbol{Y} \to Y$. Define $S: Y^3 \to R$ by*
$S(f_0, g_0, h_0) = B_4(f_0, f_0, g_0, h_0) + B_4(f_0, g_0, g_0, h_0) + B_4(f_0, g_0, h_0, h_0), for\ all\ f_0, g_0, h_0 \in Y.$
Suppose $B_4(f_0, f_0, g_0, h_0) = B_4(f_0, g_0, g_0, h_0) = B_4(f_0, g_0, h_0, h_0) = (C)$ *(say), for all* $f_0, g_0, h_0 \in Y.$
Suppose $\boldsymbol{T}: \boldsymbol{Y} \to Y$ *satisfies at*
$B_4(Tf_0, Tg_0, Th_0, Tt_0)$
$< max\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, Tg_0, g_0), B_4(Th_0, Th_0, Th_0, h_0), B_4(Tt_0, Tt_0, Tt_0, t_0)\},$
$for\ all\ f_0, g_0, h_0, t_0 \in Y.$
Further suppose $B_4(Tf_0, Tf_0, Tf_0, f_0) \leq B_4(Tf_0, Tf_0, f_0, f_0), for\ all\ f_0 \in Y.$
Then
$S(Tf_0, Tg_0, Th_0) < max\{S(f_0, g_0, h_0), S(Tf_0, Tf_0, f_0), S(Tg_0, Tg_0, g_0), S(Th_0, Th_0, h_0)\}, for\ all\ f_0, g_0, h_0 \in Y.$

Proof: L.H.S: $S(Tf_0, Tg_0, Th_0) = B_4(Tf_0, Tf_0, Tg_0, Th_0) + B_4(Tf_0, Tg_0, Tg_0, Th_0) + B_4(Tf_0, Tg_0, Th_0, Th_0)$
Now (i) $S(f_0, g_0, h_0) = B_4(f_0, f_0, g_0, h_0) + B_4(f_0, g_0, g_0, h_0) + B_4(f_0, g_0, h_0, h_0) = 3(C)$
(ii) $S(Tf_0, Tf_0, f_0) = B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, Tf_0, f_0) + + B_4(Tf_0, Tf_0, f_0, f_0)$
$> 3\ B_4(Tf_0, Tf_0, Tf_0, f_0)$
(iii) $S(Tg_0, Tg_0, g_0) = B_4(Tg_0, Tg_0, Tg_0, g_0) + B_4(Tg_0, Tg_0, Tg_0, g_0) + B_4(Tg_0, Tg_0, g_0, g_0)$
$> 3B_4(Tg_0, Tg_0, Tg_0, g_0)$

$(iv)\ S(Th_0, Th_0, h_0) = B_4(Th_0, Th_0, Th_0, h_0) + B_4(Th_0, Th_0, Th_0, h_0) + B_4(Th_0, Th_0, h_0, h_0)$
$> 3B_4(Th_0, Th_0, Th_0, h_0)$
Also
$B_4(Tf_0, Tf_0, Tg_0, Th_0)$
$< max\{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Th_0, Th_0, h_0, h_0)\}$
$B_4(Tf_0, Tg_0, Tg_0, Th_0)$
$< max\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Th_0, Th_0, h_0, h_0)\}$
$B_4(Tf_0, Tg_0, Th_0, Th_0)$
$< max\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Th_0, Th_0, h_0, h_0), B_4(Th_0, Th_0, h_0, h_0)\}$
By hypothesis $B_4(f_0, f_0, g_0, h_0) = B_4(f_0, g_0, g_0, h_0) = B_4(f_0, g_0, h_0, h_0)$.
Without losing generality, we may imagine that
$maf\ \{B_4(Tf_0, Tf_0, Tf_0, f_0), B_4(Tg_0, Tg_0, Tg_0, g_0), B_4(Th_0, Th_0, Th_0, h_0)\} = B_4(Tf_0, Tf_0, Tf_0, f_0)$
Then $B_4(Tf_0, Tf_0, Tg_0, Th_0) < max\ \{(C), B_4(Tf_0, Tf_0, Tf_0, f_0)\}$
$B_4(Tf_0, Tf_0, Tg_0, Th_0) < max\{(C), B_4(Tg_0, Tg_0, Tg_0, g_0)\}$
$B_4(Tf_0, Tg_0, Th_0, Th_0) < max\ \{(C), B_4(Th_0, Th_0, Th_0, h_0)\}$
L.H.S $= S(Tf_0, Tg_0, Th_0) = B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tg_0, Tg_0, g_0, g_0) + B_4(Th_0, Th_0, Th_0, h_0)$
$< 3\ max\ \{(C), B_4(Tf_0, Tf_0, Tf_0, f_0)\}$
$= max\ \{3(C), 3B_4(Tf_0, Tf_0, Tf_0, f_0)\}$
$< max\ \{S(f_0, g_0, h_0), S(Tf_0, Tf_0, f_0), S(Tg_0, Tg_0, g_0), S(Th_0, Th_0, h_0)\}$
= R.H.S
Therefore $S(Tf_0, Tg_0, Th_0) < max\{S(f_0, g_0, h_0), S(Tf_0, Tf_0, f_0), S(Tg_0, Tg_0, g_0), S(Th_0, Th_0, h_0)\}$,
$for\ all\ f_0, g_0, h_0 \in Y$.

Theorem09: *Let (Y, B_4) be a B_4-metric space and a map $\boldsymbol{T}: \boldsymbol{Y} \to Y$. Define $S: Y^3 \to R$ by*
$S(f_0, g_0, h_0) = B_4(f_0, f_0, g_0, h_0) + B_4(f_0, g_0, g_0, h_0) + B_4(f_0, g_0, h_0, h_0), for\ all\ f_0, g_0, h_0 \in Y.$
Suppose $B_4(Tf_0, Tf_0, f_0, f_0) \le B_4(Tf_0, Tf_0, Tf_0, f_0)\ for\ all\ f_0 \in Y.$ *(9.1)*
and
$B_4(Tf_0, Tg_0, Th_0, Tt_0) <$
$max\ \{B_4(Tf_0, Tf_0, f_0, f_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Th_0, Th_0, h_0, h_0), B_4(Tt_0, Tt_0, t_0, t_0)\},$
$for\ all\ f_0, g_0, h_0, t_0 \in Y.$
Then
$S(Tf_0, Tg_0, Th_0) < max\{S(Tf_0, Tf_0, f_0),\ S(Tg_0, Tg_0, g_0),\ S(Th_0, Th_0, h_0)\}, for\ all\ f_0, g_0, h_0 \in Y.$ *(9.2)*

Proof: L.H.S:
$S(Tf_0, Tg_0, Th_0) = B_4(Tf_0, Tf_0, Tg_0, Th_0) + B_4(Tf_0, Tg_0, Tg_0, Th_0) + B_4(Tf_0, Tg_0, Th_0, Th_0)$ (9.3)
$(i)\ S(Tf_0, Tf_0, f_0) = B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, Tf_0, f_0) + B_4(Tf_0, Tf_0, f_0, f_0)$
$\le 3B_4(Tf_0, Tf_0, f_0, f_0)$ (9.4)
$(ii)\ S(Tg_0, Tg_0, g_0) = B_4(Tg_0, Tg_0, Tg_0, g_0) + B_4(Tg_0, Tg_0, Tg_0, g_0) + B_4(Tg_0, Tg_0, g_0, g_0)$
$(iii)\ S(Th_0, Th_0, h_0) = B_4(Th_0, Th_0, Th_0, h_0) + B_4(Th_0, Th_0, Th_0, h_0) + B_4(Th_0, Th_0, h_0, h_0)$
Also
$B_4(Tf_0, Tf_0, Tg_0, Th_0)$
$< max\ \{B_4(Tf_0, Tf_0, f_0, f_0), B_4(Tf_0, Tf_0, f_0, f_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Th_0, Th_0, h_0, h_0)\}$
$= B_4(Tf_0, Tf_0, f_0, f_0)$ (say)
$B_4(Tf_0, Tg_0, Tg_0, Th_0)$
$< max\ \{B_4(Tf_0, Tf_0, f_0, f_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Th_0, Th_0, h_0, h_0)\}$
$= B_4(Tf_0, Tf_0, f_0, f_0)$ (say)
$B_4(Tf_0, Tg_0, Th_0, Th_0)$
$< max\ \{B_4(Tf_0, Tf_0, f_0, f_0), B_4(Tg_0, Tg_0, g_0, g_0), B_4(Th_0, Th_0, h_0, h_0), B_4(Th_0, Th_0, h_0, h_0)\}$
$= B_4(Tf_0, Tf_0, f_0, f_0)$ (say)
Therefore $B_4(Tf_0, Tf_0, Tg_0, Th_0) < B_4(Tf_0, Tf_0, Tf_0, f_0)$
$B_4(Tf_0, Tg_0, Tg_0, Th_0) < B_4(Tf_0, Tf_0, Tf_0, f_0)$
$B_4(Tf_0, Tg_0, Th_0, Th_0) < B_4(Tf_0, Tf_0, Tf_0, f_0)$
$3B_4(Tf_0, Tf_0, f_0, f_0) < S(Tf_0, Tf_0, f_0)$
Therefore from (10.3) we have $S(Tf_0, Tg_0, Th_0) < 3B_4(Tf_0, Tf_0, Tf_0, f_0)$(from (9.4))
$< S(Tf_0, Tf_0, f_0)$
$\le max\{S(Tf_0, Tf_0, f_0),\ S(Tg_0, Tg_0, g_0),\ S(Th_0, Th_0, h_0)\}$

Thus (9.2) is established.

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