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Solutions of Quadratic Diophantine Equation $x^2 - p(t)y^2 - (8p'(t) + 4)x + 16p(t)y = 0$

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ABSTRACT
One of the most famous equations in$\mathbb{Z}^2$ is that of Diophantine. In this paper, we contribute the resolution of the quadratic Diophantine equation of the type$\boldsymbol{D}$: $\boldsymbol{x^2 - p(t)y^2 - (8p'(t) + 4)x + 16p(t)y = 0}$. Our method consists in carryng out the transformation of the initial equation in order to obtain an auxiliary simple equation. The resolution of the auxiliary equation that we have between established finally allows us to find all the solutions of D. We also establish recurrence relation of these solutions. Our research results generalize some works of Amara Chandoulet **al** in 2011**.**

Keywords : Diophantine Equation, Recurrence Relation , Quadratique Equation,.

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I. INTRODUCTION.

A Diophante equationis a polynomial equationwithinteger coefficients to besolvedin $\mathbb{Z}^k$. It was Diophante Alexandre who made the first study of such an equation. There are different types of Diophantineequations in particular, quadraticDiophantineequationswithinteger coefficients of the form:

$ax^2 + bxy + cy^2 + dx + ey + f = 0.$ (1)

Without a doubt, one of the mostfamousDiophantineequationsisthat of Pell:

$x^2 - dy^2 = 1$ (2)

Where, the integer d is not a perfect square. The first study of Pell'sequationwas made by the IndianmathematiciancalledBrahmagupta (598-670). Hisworkwasthentaken over by BaskharaII(1114-1185).Ten centuries later, Europeanmathematicians have contributed to the study of the sameequationswhose objective is to respond to the challenge launched by Fermat in january 1657. Atthat time, Frénicle, Wallis, Brounker(1620-1684) and Euler (1707-1783) hadparticipated in solvingPell'sequation. It was Lagrange whoformailized the completetheory of solvingPell'sequationthrough the use of continued fractions [4, 5, 6].

Lately, somme researchers are interested in solving Diophante equation of type (1) thanks to the theoreticalresultsestablished by Lagrange. In [8,10,11], D. Sarath Sen Reddy et al and M. Somanath et al are interested in integer solutions of the quadraticDiophantineequation of type :

$x^2 + my^2 + nx + py + q = 0$(3)

Where, m, n, p et q are relatives integers.

Research has evolved, otherresearcherswanted to go further. Instead of solving the quadraticequationwith coefficients in $\mathbb{Z}$, they are rather interested in the equations with polynomial coefficients in $\mathbb{Z}[t] - \{0,1\}$. In 2010, Amet Tekcan**et al**solved the Diophantine equations: $x^2 - (t^2 \pm t)y^2 - (4t \pm 2)x + (4t^2 \pm 4t)y = 0$ [4,7]. In [1, 3], Amara Chandoul had found the solutions in $\mathbb{Z}[t] \times \mathbb{Z}[t]$ of equation $x^2 - (P^2 - P)y^2 - (4P - 2)x + (4P^2 - 4P)y = 0$ and also equation:

$x^2 - (t^2 - t)y^2 - (16t - 4)x + (16t^2 - 16t)y = 0$ (4)

In 2019, Amara Chandoul**et al**took over the work of the Amet Tekcan team by studying the general case [2]. A new resultappeared in litteraturein 2021 on solving the quadraticdiophantineequation$X^2 - p(t)Y^2 + 2K(t)X + 2p(t)L(t)Y = 0$, where P, K and L arepolynomials, thanks to Hasan Sankari et al [9].

Inspired by theseworks, we propose in thispaper the resolution in $\mathbb{Z}[t] \times \mathbb{Z}[t]$of the quadratic diophantine equation of type:

$D: x^2 - p(t)y^2 - (8p'(t) + 4)x + 16p(t)y = 0$ (5)

Where, p is non-square polynomial in $\mathbb{Z}[t] - \{0,1\}$.

The resultsthatwewill propose in thispaperwillgeneralize the study of equation (4) made by Amara Chandoul et al in 2011 [2].
First, in the first section, werecallsometheoriesrelated to the solving of Pell'sequation.In the next section, wetransform D in order to obtain an auxiliaryequation$\widetilde{D}$ associated.In the last section, we propose the resolution of equations$\widetilde{D}$ and D and wefind the resultsestablished by Amara Chandoul et al in [2].

II. PRELIMINARIES.

In this part, wereminder the theory on the resolution of the Pell equation, using of concept on the continued fraction. Wealsodescribe in this part, the methodweused to obtainequation (5) from (4).

2.1. Resolution of the Pell'sequation.

Let d be a positive integerwhichis not aperfect square.

Let us notice by$\sqrt{d} = [a_0, \overline{a_1, a_2, \ldots, a_n, 2a_0}]$, the continued fraction expansion of $\sqrt{d}$whichisperiodic. Le ℓ be the length of this period. The k^e convergent of $\sqrt{d}$ for $k \geq 0$ is given by:

$$r_k = \frac{p_k}{q_k} = [a_0; a_1; a_2; \ldots a_k] \qquad (6)$$

With, $\alpha_0 = \sqrt{d},\ a_k = E(\alpha_k)$ et $\alpha_{k+1} = \frac{1}{\alpha_k - a_k}, k = 0,1,2, \ldots$

Where$E(\alpha_k)$is the integer part of α_k.

Let (x_1, y_1), is the fundamental solution of $x^2 - dy^2 = 1$.

Théorème 2.1. (See[12])

The fundamental solution of $x^2 - dy^2 = 1$ is:

$$(x_1, y_1) = \begin{cases} (p_{\ell-1}, q_{\ell-1}) \ if\ \ell\ even; \\ (p_{2\ell-1}, q_{2\ell-1}), if\ \ell\ ood. \end{cases} \qquad (7)$$

Ahmet Tekcanshowed in thisworkin 2011 [6], the continued fraction expansion of $\sqrt{t^2 - t}$, as well as the fundamental solution of de $x^2 - (t^2 - t)y^2 = 1$ by the followingtheorem.

Théorème 2.2.(A.Tekcan [See 6])

(i) The continued fraction expansion of $\sqrt{t^2 - t}$ is:

$$\sqrt{t^2 - t} = \begin{cases} [1; 2], si\ t = 2 \\ [t - 1; \overline{2; 2t - 2}], si\ t > 2 \end{cases} \qquad (8)$$

(ii) The fundamental solution of $x^2 - (t^2 - t)y^2 = 1$is $(2t - 1,2)$.

2.2.Goingfrom Equation (4) to Equation (5).

Now, weconsider the equation (4): $x^2 - (t^2 - t)y^2 - (16t - 4)x + (16t^2 - 16t)y = 0$.

We have: $x^2 - (t^2 - t)y^2 - (8 \times (t^2 - t)' + 4)x + 16(t^2 - t)y = 0$.

For $t \geq 2$, we put$p(t) = t^2 - t$.

Then (4) isequivalentto $x^2 - p(t)y^2 - (8 \times p(t)' + 4)x + 16p(t)y = 0$.

Weget the mostgeneralform of (4).

In the next section, wewillsolvethis last equation.

II. Description of Method.

Weconsider the equation$D$: $x^2 - p(t)y^2 - (8p'(t) + 4)x + 16p(t)y = 0$.

Solvingthisequationdirectlyseems to beverydifficult. It is for thisreasonthatwetransformit by the transformation T defined by:

$$T: \begin{cases} x = u + \alpha \\ y = v + \beta \end{cases} \qquad (9)$$

By applying the transformation T to D, weget:

$$(u + \alpha)^2 - p(t)(v + \beta)^2 - (8p'(t) + 4)(u + \alpha) + 16p(t)(v + \beta) = 0 \qquad (10)$$

$$u^2 - p(t)v^2 + (2\alpha - 8p'(t) - 4)u + \big(-2\beta p(t) + 16p(t)\big)v + \alpha^2 - p(t)\beta^2 - (8p'(t) + 4)\alpha + 16p(t)\beta = 0$$

We are going to make the terms in u and v disappear.

Weget : $2\alpha - 8p'(t) - 4 = 0$ ***and*** $-2\beta p(t) + 16p(t) = 0$.
Let, $\alpha = 2 + 4p'(t)$***and*** $\beta = 8$.
Afterhavingsubstitutedα ***and*** β in D, we obtain an auxiliary equation which we have by $\widetilde{D}$ defined by :

$$\widetilde{D}: u^2 - p(t)v^2 = 16\left(p'(t)\right)^2 + 16p'(t) - 64p(t) + 4 \qquad (10)$$

We note by $(a(t); b(t))$ the fundamental solution of $u^2 - p(t)v^2 = 1$.

III. The Main Results of The quadraticequation$x^2 - p(t)y^2 - (8 \times p(t)' + 4)x + 16p(t)y = 0$.

3.1. Resolution of the Quadratic Equation $\widetilde{D}$.

Proposal 3.1.
(i) The fundamental solution of $\widetilde{D}$is : $(u_1; v_1) = (2 + 4p'(t); 8)$.
(ii) Definethe sequence(u_n) et (v_n) by :

$$\begin{cases} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 2 + 4p'(t) \\ 8 \end{pmatrix} \\ \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} a(t) & b(t)p(t) \\ b(t) & a(t) \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \end{cases} \qquad (15)$$

Then$(u_n; v_n)$ is a solution of $\widetilde{D}$.

(iii) For $n \geq 2$, the solution $(u_n; v_n)$satisfy the recurrence relations :

$$\begin{cases} u_n = a(t)u_{n-1} + b(t)p(t)v_{n-1} \\ v_n = b(t)u_{n-1} + a(t)v_{n-1} \end{cases} \qquad (16)$$

(iv) For $n \geq 4$, the solution $(u_n; v_n)$:

$$\begin{cases} u_n = (2a(t) - 1)(u_{n-1} + u_{n-2}) - u_{n-3} \\ v_n = (2a(t) - 1)(v_{n-1} + v_{n-2}) - v_{n-3} \end{cases} \qquad (17)$$

Proof (i).

Indeed, $u_1^2 - p(t)v_1^2 = \left(2 + 4p'(t)\right)^2 - p(t)(8)^2$

$$= 4 + 2 \times 2 \times 4p'(t) + \left(4p'(t)\right)^2 - 64p(t)$$

$$= 16\left(p'(t)\right)^2 + 16p'(t) - 64p(t) + 4$$

Thus, $(u_1; v_1) = (2 + 4p'(t); 8)$ is the fundamental solution of $\widetilde{D}$.
Proof (ii).
We prouve itusing the method of mathematical induction.
Let n=2, weget :

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} a(t) & b(t)p(t) \\ b(t) & a(t) \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

$$= \begin{pmatrix} a(t)u_1 + b(t)p(t)v_1 \\ b(t)u_1 + a(t)v_1 \end{pmatrix}$$

So, $u_2^2 - p(t)v_2^2 = (a(t)u_1 + b(t)p(t)v_1)^2 - p(t)(b(t)u_1 + a(t)v_1)^2$

$$= \left(\left(a(t)\right)^2 - p(t)(b(t))^2\right)u_1{}^2 - p(t)(\left(a(t)\right)^2 - p(t)(b(t))^2)v_1{}^2$$

$$= 1 \times u_1{}^2 - p(t) \times 1 \times v_1{}^2$$

$$= u_1^2 - p(t)v_1^2$$

$$= 16\left(p'(t)\right)^2 + 16p'(t) - 64p(t) + 4$$

Therefore$(u_2; v_2) = (a(t)u_1 + b(t)p(t)v_1; b(t)u_1 + a(t)v_1)$ is the solution of $\widetilde{D}$.

Now, for $n > 2$, we assume that $(u_n; v_n)$is solution of $\widetilde{D}$thenwe show that$(u_{n+1}; v_{n+1})$ is the solution of $\widetilde{D}$.

Indeed, $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} a(t) & b(t)p(t) \\ b(t) & a(t) \end{pmatrix}^{n+1-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$

$$= \begin{pmatrix} a(t) & b(t)p(t) \\ b(t) & a(t) \end{pmatrix} \begin{pmatrix} a(t) & b(t)p(t) \\ b(t) & a(t) \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

$$= \begin{pmatrix} a(t) & b(t)p(t) \\ b(t) & a(t) \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

$$= \begin{pmatrix} a(t)u_n + b(t)p(t)v_n \\ b(t)u_n + a(t)v_n \end{pmatrix}.$$

Henceweget, $u_{n+1}^2 - p(t)v_{n+1}^2 = (a(t)u_n + b(t)p(t)v_n)^2 - p(t)(b(t)u_n + a(t)v_n)^2$

$$= \left(\left(a(t)\right)^2 - p(t)(b(t))^2\right) {u_n}^2 - p(t)(\left(a(t)\right)^2 - p(t)(b(t))^2){v_n}^2$$

$$= 1 \times {u_n}^2 - p(t) \times 1 \times {v_n}^2$$

$$= u_n^2 - p(t)v_n^2$$

$$= 16\left(p'(t)\right)^2 + 16p'(t) - 64p(t) + 4$$

This, for $n > 2$, $(u_{n+1}; v_{n+1}) = (a(t)u_n + b(t)p(t)v_n; b(t)u_n + a(t)v_n)$ is the solution of $\widetilde{D}$.

Therefore, (u_n, v_n) is the solution of $\widetilde{D}$.

Proof (iii).

This is a direct consequence of proposal 3.1 (ii).

So, for $n \geq 2$ we get:

$$\begin{cases} u_n = a(t)u_{n-1} + b(t)p(t)v_{n-1} \\ v_n = b(t)u_{n-1} + a(t)v_{n-1} \end{cases} \qquad (18)$$

Proof (iv).

Westill use the method of mathematical induction.

To simplify the demonstration, wewillconsider the case where the polynomial p is constant that's to say, $p'(t) = 0$. In this case, the fundamental solution of $\widetilde{D}$ is $(u_1; v_1) = (2; 8)$.

Using the recurrence relation (iii) of the proposal 3.1, weobtain:

$u_1 = 2$.

$u_2 = 2a + 8pb$

$$u_3 = 2a^2 + 16abp + 2b^2p$$

And then, $u_4 = 2a^3 + 24a^2bp + 6ab^2p + 8(a(t))^2 - 8bp$. (19)

According to the recurrence relation (iv) of proposal 3.1, weget :

$$u_4 = (2a - 1)(u_3 + u_2) - u_1$$

$$= (2a - 1)(2a^2 + 16abp + 2b^2p + 2a + 8pb) - 2$$

$$= 4a^3 + 2a^2 + 32a^2b + 4ab^2p - 2b^2p - 8bp - 2a - 2$$

From (19), weget:

$$u_4 = 2a^3 + 24a^2bp + 6ab^2p + 8(a(t))^2 - 8bp$$

$$= 4a^3 + 2a^2 + 32a^2bp + 4ab^2p - 2b^2p - 8bp - 2(a^2 - bp^2) - 2a(a^2 - bp^2)$$

$$= 4a^3 + 2a^2 + 32a^2bp + 4ab^2p - 2b^2p - 8bp - 2a - 2$$

$$= (2a(t) - 1)(u_3 + u_2) - u_1.$$

Now, we assume that$n \geq 2$, $u_n = (2a - 1)(u_{n-1} + u_{n-2}) - u_{n-3}$. Then we show that $u_{n+1} = (2a - 1)(u_n + u_{n-1}) - u_{n-2}$.

Fromproposal (iii) weget:

$$u_{n+1} = au_n + bp(t)v_n$$

$$= a[(2a - 1)(u_{n-1} + u_{n-2}) - u_{n-3}] + bp[(2a - 1)(v_{n-1} + v_{n-2}) - v_{n-3}]$$

$$= (2a - 1)[a(u_{n-1} + u_{n-2}) + bp(v_{n-1} + v_{n-2})] - au_{n-3} - bpv_{n-3}$$

$$= (2a - 1)[au_{n-1} + bpv_{n-1} + av_{n-2} + bpv_{n-2}] - (au_{n-3} + bpv_{n-3})$$

$$= (2a - 1)(u_n + u_{n-1}) - u_{n-2}$$

Therefore, for $n > 4$, $u_{n+1} = (2a - 1)(u_n + u_{n-1}) - u_{n-2}$.

So, for $n > 4 u_n = (2a-1)(u_{n-1}+u_{n-2}) - u_{n-3}$.
In similaryway, itisdemonstrationthat, $v_n = (2a-1)(v_{n-1}+v_{n-2}) - v_{n-3}$.
Now, we have all the necessaryingredients to solveequation D.

3.2. Resolution of Qudratic Equation D.

Proposal 3.2.

(i) The fundamental solution of $D$is : $(x_1; y_1) = (2u_1; 2v_1)$.
(ii) Define by sequence$(u_n)$and (v_n) by:

$$\begin{cases} x_n = u_n + 2 + 4p'(t) \\ y_n = u_n + 8 \end{cases} \quad (20)$$

Then, $\{(u_n; v_n)\}_{n\geq 1}$is solution of D. So it has infinitymany solution in $\mathbb{Z}[t]\times\mathbb{Z}[t]$.
(iii) For $n \geq 2$, the solutions $(u_n; v_n)$satisfy the followingrecurrence relations :

$$\begin{cases} x_n = a(t)x_{n-1} + b(t)p(t)y_{n-1} - 4p^{'}(t)(a(t)-1) - 8b(t)p(t) - 2(a-1) \\ y_n = b(t)x_{n-1} + a(t)y_{n-1} - 2b(t)\left(1+2p^{'}(t)\right) - 8(a-1) \end{cases} \quad (21)$$

(iv) For $n \geq 4$, the solution $(u_n; v_n)$satisfy the followingrecurrence relation :

$$\begin{cases} x_n = (2a(t)-1)(x_{n-1}+x_{n-2}) - x_{n-3} - 4(a(t)-1)(1+2p^{'}(t)) + 4 + 8p'(t) \\ y_n = (2a(t)-1)(y_{n-1}+y_{n-2}) - y_{n-3} - 16(2a(t)-1) + 16 \end{cases} \quad (22)$$

Proof (i)
Let us put:

$$H = x^2 - py^2 - (8p^{'}+4)x + 16py \quad (23)$$

Let $x_1 = 2u_1$and $y_1 = 2v_1$. We will substitute x_1 and y_1 in (23).
Weget:

$$\begin{aligned} H &= (2u_1)^2 - p(2v_1)^2 - (8p^{'}+4)2u_1) + 16p(2v_1) \\ &= 4(u_1^2 - pv_1^2) - 16u_1p^{'} - 8u_1 + 32v_1p \\ &= 4(16p'^2 + 16p^{'} - 64p + 4) - 16(2+4p^{'})p^{'} - 8(2+4p^{'}) + 32\times 8\times p \\ &= 4(16p'^2 + 16p^{'} - 64p + 4) - 4[4(2+4p^{'})p^{'} + 2(2+4p^{'}) - 8\times 8p] \\ &= 4(16p'^2 + 16p^{'} - 64p + 4) - 4(16p'^2 + 16p^{'} - 64p + 4) \\ &= 0 \end{aligned}$$

Then, $(2u_1)^2 - p(t)(2v_1)^2 - (8p^{'}(t)+4)2u_1) + 16p(t)(2v_1) = 0$.
Thus, $(x_1; y_1) = (2u_1; 2v_1)$ is the fundamental solution of D.
Proof (ii).
According to proposal 3.2(i), for $n = 1$, then $(x_1; y_1) = (2u_1; 2v_1)$ is the fundamental solution of D. For $n > 1$, we assume that $(x_n; y_n)$ is the solution of D then we show that $(x_{n+1}; y_{n+1})$ is solution of D.

Let $x_{n+1} = u_{n+1} + u_1$and $y_{n+1} = v_{n+1} + v_1$. We will substitute x_{n+1} and y_{n+1} in (23).

Weget:

$$\begin{aligned} H &= (u_{n+1}+u_1)^2 - p(v_{n+1}+v_1)^2 - (8p^{'}+4)(u_{n+1}+u_1) + 16p(v_{n+1}+v_1) \\ &= u_{n+1}^2 - pv_{n+1}^2 + u_1^2 - pv_1^2 - (8p^{'}+4)(2+4p^{'}) + 16p\times 8 \\ &= u_n^2 - pv_n^2 + u_1^2 - pv_1^2 - 2(u_1^2 - pv_1^2) \\ &= 2(u_1^2 - pv_1^2) - 2(u_1^2 - pv_1^2) \end{aligned}$$

$= 0$.

So,$(u_{n+1}+u_1)^2 - p(t)(v_{n+1}+v_1)^2 - (8p^{'}(t)+4)(u_{n+1}+u_1) + 16p(t)(v_{n+1}+v_1) = 0$.

So then, $(u_{n+1} + 2 + 4p'(t), v_{n+1} + 8)$ is the solution of D: $x^2 - p(t)y^2 - (8p^{'}(t)+4)x + 16p(t)y = 0$.

Therefore, for $n \geq 1$, $(x_n; y_n)$is the solution of D in $\mathbb{Z}[t] \times \mathbb{Z}[t]$.

Proof (iii).

According to proposal 3.2 (ii), for $n \geq 1$, we have, $\begin{cases} x_n = u_n + 2 + 4p'(t) \\ y_n = u_n + 8 \end{cases}$.

According to proposal 3.2 (iii), we have: $\begin{cases} u_n = a(t)u_{n-1} + b(t)p(t)v_{n-1} \\ v_n = b(t)u_{n-1} + a(t)v_{n-1} \end{cases}$.

So then,

$x_n = u_n + 2 + 4p'(t)$

$$= a(t)u_{n-1} + b(t)p(t)v_{n-1} + 2 + 4p'(t)$$

$$= a(t)(x_{n-1} - 2 - 4p^{'}(t)) + b(t)p(t)(y_{n-1} - 8) + 2 + 4p'(t)$$

$$= a(t)x_{n-1} + b(t)p(t)y_{n-1} + a(t)\left(-2 - 4p^{'}(t)\right) - 8b(t)p(t) + 2 + 4p'(t)$$

$$= a(t)x_{n-1} + b(t)p(t)y_{n-1} - 4p^{'}(t)(a(t) - 1) - 8b(t)p(t) - 2(a(t) - 1)$$

Then, $x_n = a(t)x_{n-1} + b(t)p(t)y_{n-1} - 4p^{'}(t)(a(t) - 1) - 8b(t)p(t) - 2(a(t) - 1)$.

In similaryway, thenwe show that:

$y_n = (2a(t) - 1)(y_{n-1} + y_{n-2}) - y_{n-3} - 16(2a(t) - 1) + 16,$ for$n \geq 2$.

Proof (iv).

According to proposal 3.2 (iv), we have: $u_n = (2a(t) - 1)(u_{n-1} + u_{n-2}) - u_{n-3}$, for $n \geq 4$.

So,

$x_n = u_n + 2 + 4p'$

$$= (2a - 1)(u_{n-1} + u_{n-2}) - u_{n-3} + 2 + 4p'$$

$$= (2a - 1)((x_{n-1} - 2 - 4p^{'} + (x_{n-2} - 2 - 4p^{'}) - (x_{n-3} - 2 - 4p^{'}) + 2 + 4p'$$

$$= (2a - 1)(x_{n-1} + x_{n-2}) - x_{n-3} - 4(a - 1)(1 + 2p^{'}) + 4 + 8p'$$

So that, $(2a(t) - 1)(x_{n-1} + x_{n-2}) - x_{n-3} - 4(a(t) - 1)(1 + 2p^{'}(t)) + 4 + 8p'(t), n \geq 4$.

Now, let'sseesomeexample.

For $t \geq 2$, let $p(t) = t^2 - t$.

So, D : $x^2 - (t^2 - t)y^2 - (16t - 4)x + (16t^2 - \mathbf{16}t)y = 0$ (24)

Wefind the equationstudied by Amara Chandoul et al in 2011 in [2].

Itsauxiliaryequation:

$$\widetilde{D}: x^2 - (t^2 - t)y^2 = 32t + 4 \qquad (25)$$

According to theorem 2.2, the fundamental solution of $x^2 - (t^2 - t)y^2 = 1$ is $(a(t); b(t)) = (2t - 1; 2)$.

This, wefind the holes first points of the theoremannounced by Amara Chandoul in [2].

For the auxiliaryequation$\widetilde{D}$, we have the following results:

According to proposal 3.1 (i), $(u_1; v_1) = (8t - 2; 8)$is the fundamental solution of $\widetilde{D}$.

According to proposal 3.1 (ii), for $n \geq 2$, we get:

$$\begin{cases} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 8t - 2 \\ 8 \end{pmatrix} \\ \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 2t - 1 & 2(t^2 - t) \\ 2 & 2t - 1 \end{pmatrix}^{n-1} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \end{cases} \quad (26)$$

Next, for $n \geq 2$, we get:

$$\begin{cases} u_n = (2t - 1)u_{n-1} + 2(t^2 - t)v_{n-1} \\ v_n = 2u_{n-1} + (2t - 1)v_{n-1} \end{cases} \quad (27)$$

And, for $n \geq 2$, we get:

$$\begin{cases} u_n = (2t - 3)(u_{n-1} + u_{n-2}) - u_{n-3} \\ v_n = (2t - 3)(v_{n-1} + v_{n-2}) - v_{n-3} \end{cases} \quad (28)$$

For the D equation, we have the followingresults:

According to proposal 3.2 (i), the fundamental solution of D is$(x_1; y_1) = (16t - 4; 16)$.

Next, thisinfinitymany solutions is:

$$S = \{(x_n, y_n) \in \mathbb{Z}[t] \times \mathbb{Z}[t],\ x_n = u_n + 8t - 2,\ y_n = v_n + 8\} \quad (29)$$

According to proposal 3.2, weobtain the followingrecurrence relation:

(i) For $n \geq 2$, we get : $\begin{cases} x_n = (2t - 1)x_{n-1} + 2(t^2 - t)y_{n-1} - 32t^2 + 36t - 4 \\ y_n = 2x_{n-1} + (2t - 1)y_{n-1} - 32t + 20 \end{cases}$ (30)

(ii) For $n \geq 4$, we get : $\begin{cases} x_n = (4t - 3)(x_{n-1} + x_{n-2}) - x_{n-3} - 64t^2 + 80t - 16 \\ y_n = (4t - 3)(y_{n-1} + y_{n-2}) - y_{n-3} - 64t + 64 \end{cases}$ (31)

IV. CONCLUSION.

In thispaper, wepresent the results of ourresearch on the resolution of the Diophantineequationwith polynomial coefficients. Our resultswillgeneralize the work of Amara Chandoul et al in 2011. In future work, wewillconsiderdeepeningourresearch on the resolution of the sameequation in finitefield$(\mathbb{Z}/p\mathbb{Z})$, where p is a prime greater that or equal to 5.

REFERENCES

[1]. A. Chandoul, D. Marques and S.S.Albrbar, *The QuadraticDiophantineEquation*$x^2 - P(t)y^2 + ((P'(t))^2 - 4P(t) - 1 = 0$, Journal of Mathematics Research, vol 11, No 2 ; 2019.

[2]. A. Chandoul, *The QuadraticDiophantineEquation* $x^2 - (t^2 - t)y^2 - (16t - 4)x + (16t^2 - 16t)y = 0$, International MathematicalFourum, Vol. 6, No 36 ; 2011, 1777-1782.

[3]. A. Chandoul, *On Polynomials Solutions of QuadraticDiophantineEquatios*,Advances in Pure Mathematics, 2011,1,155-159.

[4]. A. Tekcan and A Ozkoç, *The Diophantineequation*$x^2 - (t^2 + t)y^2 - (4t + 2)x + (4t^2 + 4t)y = 0$, Rev Mat Complit (2010) 23 : 251-260.

[5]. A. Tekan, *The Pell equation*$x^2 - Dy^2 = \pm 4$, *Applied Mathematical Sciences*, Vol 1, No 8 ; 2007, 363-369.

[6]. A. Tekan, *Continued Fractions Expansions of* $\sqrt{D}$ *and Pell equation* $x^2 - Dy^2 = 1$, Mathematica Moravica, 15(2)(2011), 19-27.

[7]. A. Özkoç and A.Tekcan, *QuadraticDiophantineEquation* $x^2 - (t^2 - t)y^2 - (4t - 2)x + (4t^2 - 4t)y = 0$, Bull.Malays.Math.Sci.Soc(2)33(2)(2010), 273-280.

[8]. D. Sarath Sen Reddy, *On Polynomial Solutions of QuadraticDiophantine Equation*, Acta CienciaIndica.XLIV M, No 1, 39 (2018).

[9]. H. Sankari and A.Abdo, *The Polynomial Solution of QuadraticDiophantine Equation* $X^2 - p(t)Y^2 + 2K(t)Y + 2p(t)L(t) = 0$, Hindawi Journal of Mathematics, Vo 2021, Article ID 7464950, 6 pages.

[10]. M. Somanath, K. Raja , J. Kannan and M. Mahalakshmi, *On a Class of Solution for a QuadraticDiophantine Equation*, Advances and Applications in Mathematical Sciences, Vol 19 Issue 11, 2020, pages 1097-1103.

[11]. M. Somanath, K. Raja and J.Kannan, *On a Polynomial Solutions of a Diophantine Equation*,International Journal of Engineering, Science and Mathematics, Vol 6 Issue 6, 2017.

[12]. Ö. Özer and Y. Ceven, *Solution To The Pell Equation* $x^2 - Dy^2 = 2^k$ *Where* $D = r^2s^2 + 2s$ *and Recurrences, Int. Journal of Engineering Research and Application, Vol. 3, Issue 6, Nov-Dec 2013, pp.574-580.*