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Hurst exponent and Multifractal Properties in the Time Series of Bitcoin Trading Volume

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ABSTRACT
We investigate the time series properties of the Bitcoin trading volume and compare them with the properties of realized volatility. The Hurst exponent of the trading volume time series is greater than 1/2, which indicates that the time series is persistent. The generalized Hurst exponent of trading volume is not constant, indicating that the time series is multifractal. We find that the trading volume and realized volatility have similar time variations in the Hurst exponent. We also explore the properties of incremental time series and find that both the time series of trading volume and realized volatility exhibit a value smaller than 1/2, indicating that the time series are anti-persistent. We conclude that trading volume and realized volatility are correlated in time to some extent and have similar time series properties.

Keywords–Hurst Exponent, Multifractality, Multifractal Detrended Fluctuation Analysis, Realized Volatility, Rough Volatility, Trading Volume

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I. INTRODUCTION

In empirical finance, volatility which expresses the magnitude of variation in asset time series, is of great importance for financial risk management, which is necessary to avoid huge losses in the future. Various volatility estimation techniques have been developed. For instance, the generalized autoregressive conditional heteroscedasticity (GARCH) model [1] is widely accepted as a volatility model that successfully captures notable properties of asset time series, such as volatility clustering, classified as stylized facts [2]. The GARCH model is further extended to capture more asset properties and mimic real variations in asset time series, e.g. [3-7].

There also exists a model-free approach called the “realized volatility” [8,9], which is constructed as the sum of squared intraday returns and is proven to approach integrated volatility as the number of intraday returns increases. Realized volatility is shown to be promising by an analysis that investigates the returns standardized by realized volatility. Provided that the return time series r_t is given by $r_t = \sigma_t \epsilon_t$, where σ_t is the standard deviation and ϵ_t is the standard Gaussian random number, the standardized returns $\frac{r_t}{\sigma_t}$ should be the standard Gaussian random number, or equivalently, the distribution of the standardized returns should become gaussian. Using realized volatility as a proxy of σ_t , we find that the distributions of standardized returns are well approximated by the standard Gaussian distribution, which indicates that realized volatility is suitable as a proxy for real volatility [10-16].

Since the volatility time series has the property of volatility clustering, it exhibits long-time correlations, which results in Hurst exponents greater than 1/2. However, the time series of the volatility increment has a Hurst exponent smaller than 1/2, indicating that the time series is anti-persistent [17-19].

To some degree, the trading volume explains the volatility time variation [20]. The study of the GARCH model, including the trading models, finds that the GARCH parameters become small when the trading volume is included in the model [21-25]. However, later studies argue that trading volume does not fully explain the effect of GARCH volatility [26-31].

In this study, we focus on the trading volume of Bitcoin and investigate its time-series properties. If the trading volume and volatility are correlated with each other in time, we can observe similar properties for both time series. We investigated the multifractal properties of the time series using a multifractal detrended fluctuation analysis (MDFA). The MDFA developed in [32] can be applied to nonstationary time series and has been widely accepted as a tool to explore the multifractal properties of time series. We investigate the multifractal properties of both trading volume and volatility using MDFA and clarify whether both have similar time-series properties.

The remainder of this paper is organized as follows. Section 2 introduces the MDFA; section 3 describes the data used; section 4 presents the results; and section 5 presents the conclusions.

II. MULTIFRACTAL DETRENDED FLUCTUATION ANALYSIS

The MDFA developed in [32] is described as follows. First, we determine profile $Y(i)$ from the time series $x(i)$ we considered.

$$Y(i) = \sum_{k=1}^{i}(x(k) - \langle x \rangle) \ , \quad (1)$$

where $\langle x \rangle$ is the average of $x(i)$. We then divide profile $Y(i)$ into N_s non-overlapping segments of equal length s, where N_s is defined by $N_s \equiv int\left(\frac{N}{s}\right)$. In general, the length of the time series is not always a multiple of s, and a short time period may remain at the end of the profile. To utilize this part, the same procedure is repeated, starting from the end of profile $Y(i)$. Therefore, we obtain $2N_s$ segments.

Next, using the segments, we calculated the variance $F^2(\upsilon, s)$. For the forward direction $\upsilon = 1, \ldots, N_s$, we obtained

$$F^2(\upsilon, s) = \frac{1}{s}\sum_{i=1}^{s}(Y[(\upsilon - 1)s + i] - P_\upsilon(i))^2. \quad (2)$$

Similarly, for the backward direction: $\upsilon = N_s + 1, \ldots, 2N_s$,

$$F^2(\upsilon, s) = \frac{1}{s}\sum_{i=1}^{s}(Y[N - (\upsilon - N_s)s + i] - P_\upsilon(i))^2. \quad (3)$$

Here, $P_\upsilon(i)$ is the fitting polynomial to remove the local trend in segment υ and we use a cubic-order polynomial.

Finally, we averaged over all segments and obtained the q-th order fluctuation function

$$F_q(s) = \left\{\frac{1}{2N_s}\sum_{\upsilon=1}^{2N_s}\left(F^2(\upsilon, s)\right)^{q/2}\right\}^{1/q}. \quad (4)$$

$F_q(s)$ is expected to behave as a power law function as

$$F_q(s) \sim s^{h(q)}. \quad (5)$$

The scaling exponent $h(q)$ is called the generalized Hurst exponent. The Hurst exponent is given by $h(2)$. When $h(q)$ is constant for all q, the time series is said to be "monofractal". However, $h(q)$ varies as a function of q, and the time series is "multifractal". Using this $h(q)$, we explored the multifractal properties of the time series.

III. DATA

In this study, we use Bitcoin Tick data (in dollars) traded on Bitstamp from September 14, 2011, to September 1, 2021, and downloaded from Bitcoincharts. From the tick data, we construct daily returns r_t defined by the logarithmic price difference, as

$$r_t = logP_t - logP_{t-1}, \quad (6)$$

where P_t is the daily price at time *t*. Fig.1 displays the time series of the daily return constructed by Eq.(6).

We use realized volatility for the volatility time series. Fig.2 shows the time series of daily realized volatility constructed with 5-min intraday returns. We recognize that there are periods in which large realized volatilities are clustered, called volatility clustering. Volatility clustering generates long-term correlations in a volatility time series. The incremental time series of logarithmic realized volatilities exhibits a Hurst exponent smaller than 1/2, which indicates that the time series is ant-persistent and is also called rough volatility [17-19].

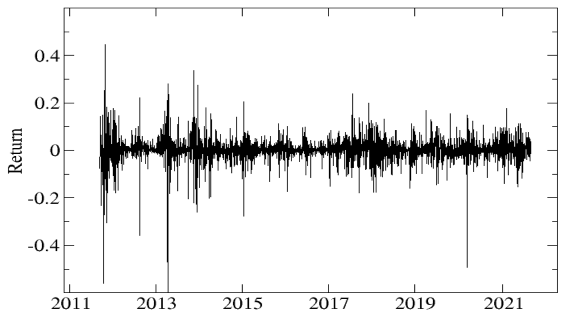

Fig.1 Daily return of Bitcoin.

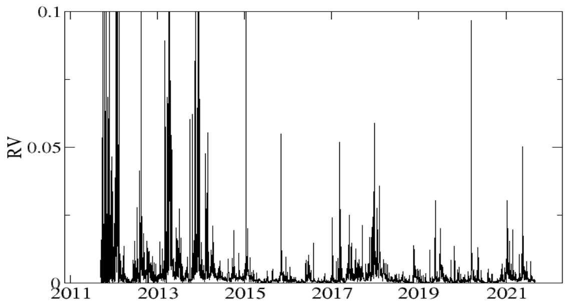

Fig.2 Daily realized volatility of Bitcoin.

IV. RESULTS

To investigate the time variation of multifractal properties, we employed the rolling window method. We set the window size to one year and performed the MDFA for the data in that window. Then, we determined the generalized Hurst exponent $h(q)$ by fitting the results of the fluctuation function to the power-law function of Eq.(5). We calculated $h(q)$ using $q = [-5,5]$ .To avoid the possibility that the fluctuation function diverges at a large q [33], we restrict q to this region.

Next, we shifted the window to one day and repeated the process.

We analyze the time series of the logarithmic realized volatility (log-RV) and logarithmic trading volume (log-Vol). Figs.3-4 display the time series of log-RV and log-Vol. It was found that the values of log-Vol before 2014 were smaller than those after 2014. This smaller log-Vol can be attributed to the early market stage, at which Bitcoin trading was inactive.

Fig.5 shows the time evolution of the Hurst exponent $h(2)$. We also plot the $h(2)$ of the return time series in Fig.5. The Hurst exponents of log-RV and log-Vol are mostly greater than 1/2, which indicates that both time series are persistent, and their time variations are similar. However, the Hurst exponent of returns fluctuates around 1/2, which indicates that the time series is close to the random walk. Before 2014, the Hurst exponent of returns was found to be smaller than 1/2, which is considered a sign of illiquidity in the early market [34-36]. For log-RV and log-Vol, we find smaller Hurst exponents in the early market (before 2014), and thus log-RV and log-Vol could also be affected by illiquidity.

Fig.6 displays a 3 dimensional (3D) plot of the generalized Hurst exponent $h(q)$ of log-RV. We find that the $h(q)$ is not constant for q, which indicates that the time series is multifractal, and moreover, we also find that the functional form of $h(q)$ varies over time. While the values of $h(q)$ in a range of $q = [-5,5]$ are mostly greater than 1/2, before 2014, at the early market, $h(q)$ takes smaller values than those in other periods.

Fig.7 shows the 3D-plot of the generalized Hurst exponent $h(q)$ of log-Vol. Similar to log-RV, $h(q)$ is not constant, showing multifractality, and varies over time. While the values of $h(q)$ are mostly greater than 1/2, they are smaller than those of log-RV.

Next, we calculate the Hurst exponent of the incremental time series of log-RV and log-Vol. Let x_t be a time series at time *t*. Then, the increment time series z_t is defined by $z_t = x_t - x_{t-1}$. Fig.8 displays the Hurst exponents of the increment time series of log-RV and log-Vol. The Hurst exponents of both time series were found to be less than 1/2, indicating that the time series were anti-persistent. The anti-persistency of log-RV has been known previously [17-19]. The Hurst exponents of log-RV and log-Vol vary over time. The time variations of the Hurst exponents of log-RV and log-Vol exhibit a similar variation pattern, which is also evidence that the two time series are correlated with each other. The Hurst exponents of the incremental time series at the early market stage before 2014 were found to be smaller than those at other periods, which can also be attributed to illiquidity in the early market.

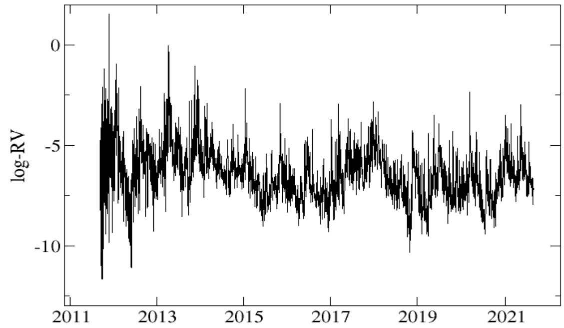

Fig.3 Time series of logarithmic RV.

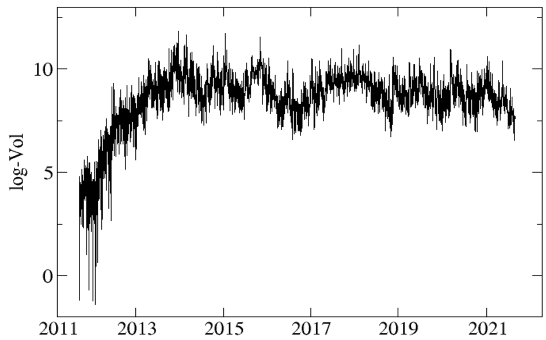

Fig.4 Time series of logarithmic trading volume.

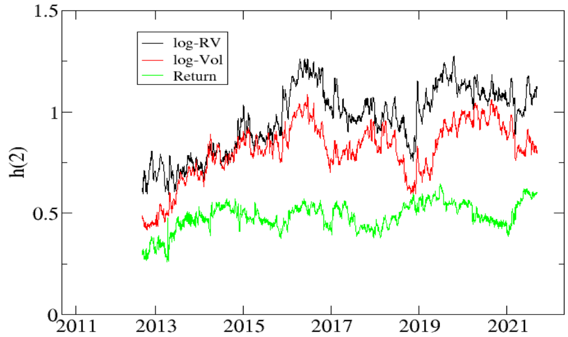

Fig.5 Hurst exponent $h(2)$ of log-RV, log-Vol and return.

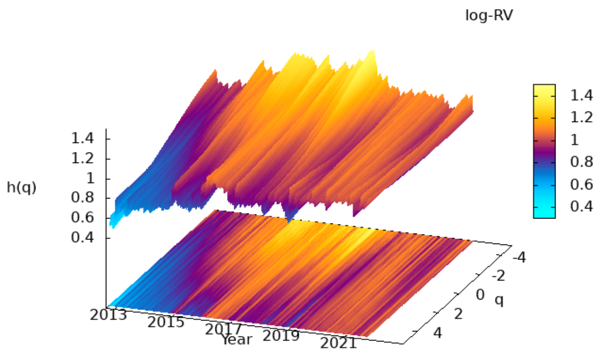

Fig.6 3D plot of $h(q)$ of log-RV.

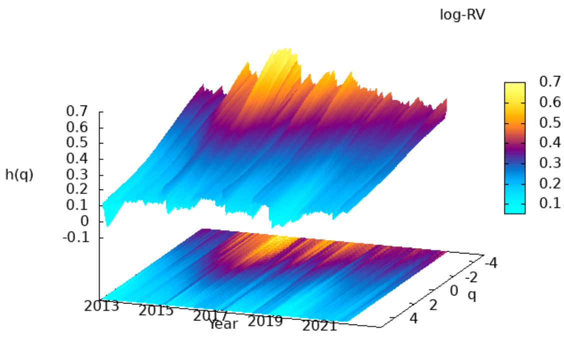

Fig.9 3D plot of $h(q)$ of increment log-RV.

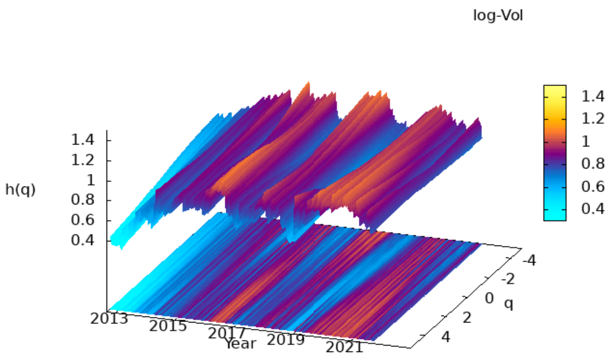

Fig.7 3D plot of $h(q)$ of log-Vol.

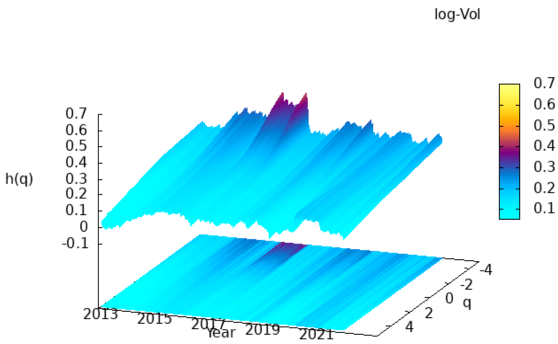

Fig.10 3D plot of $h(q)$ of increment log-Vol.

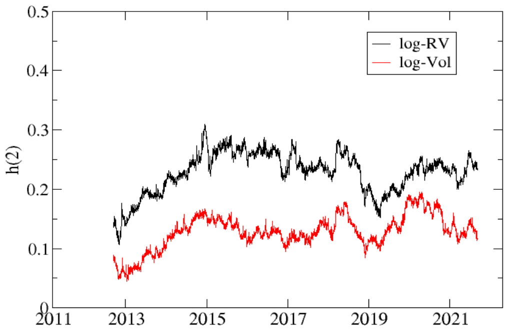

Fig.8 Hurst exponent $h(2)$ of increment time series for log-RV and log-Vol.

Figs.9-10 show $h(q)$ of increment time series of log-RV and log-Vol in 3D. It is found that $h(q)$'s of both time series vary with q, which means that the time series have a multifractal property. This is consistent with the previous results from Bitcoin and a stock return that show increment time series are multifractal [19].

V. CONCLUSION

To investigate the multifractal properties of the time series of log-RV and log-Vol, we performed the MDFA and obtained the generalized Hurst exponent $h(q)$. We find that $h(q)$'s of log-RV and log-Vol vary in q, indicating that the time series are multifractal. The time variations of the Hurst exponent $h(2)$ of log-RV and log-Vol are similar to each other, which suggests that log-RV and log-Vol are correlated in time and supports the view that the volatility time variation is explained by trading volume to some extent [20-21].

We also obtain $h(q)$'s of the increment time series of log-RV and log-Vol. The Hurst exponent $h(2)$ of log-RV is less than 1/2, and $h(q)$ is not constant in q, which indicates that the time series has anti-persistence and multifractal properties. These results are consistent with those obtained previously in [19].

While we focus on the trading volume in this paper, it might be informative to investigate other financial variables such as the number of transactions. The number of transactions is also found to correlate with the volatility [31]. Thus, it could be interesting to examine whether the number of transactions also exhibits the similar time series properties to the volatility.

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