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On the Analysis of Block Lower Hess Enberg Numerical Iterative Methods for Stationary Distribution in the Structured Markov Chains

[1]Agboola, Sunday O. and [2]Ayoade Abayomi A.
[1]Department of Mathematics, Nigerian Army University Biu, Biu, Borno State, Nigeria,
[2]Department of Mathematics, University of Lagos, Lagos, Nigeria
**Corresponding Author: e-mail: agboolasunday70@gmail.com*

ABSTRACT
The modelled system is believed to have only one state at any given time, and its evolution is represented by transitions from one state to the next. This system's physical or mathematical behavior can also be depicted by defining all of the numerous states it can be in and demonstrating how it moves between them. In this study, the iterative solution methods for the stationary distribution of Markov chains which start with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the genuine solution with each step or iteration is examined on the Markov chains whose transition matrices have a special block structure, a block structure that arises frequently when modeling G/M/1 queueing systems which leaves the transition matrices unchanged and saves time has been investigated, in order to provide some insight into the solutions of stationary distribution of Markov chain. Our quest is to compute the solutions using block lower Hess Enberg numerical iterative methods on the structured Markov chain. Matrix operations, boundary condition and normalization constant are used with the help of some existing laws, theorems and formulas of Markov chain while the stationary distribution vector's $\pi = (\pi_0,\ \pi_1,\ \pi_{2,\dots})$ where $\pi_i = (\pi(i,1)\ \ \pi(i,1),\dots,\pi(i,K))$ for $i = 0,1,\dots,$ and $\pi(i,K)$ is the probability of finding the Markov chain in state (i,K) at statistical equilibrium are obtained. Also, this is demonstrated with illustrative example with the following parameters $\alpha_1 = 1,\ \ \alpha_2 = 0.5,\ \ \mu = 4,\ \ \delta_1 = 5,\ \ \delta_2 = 3,$ and additional transition parameters $\varphi_1 = 0.25$ and $\varphi_2 = 0.75$.

Keywords: infinitesimal generator matrix, logarithmic reduction algorithm, lower Hess Enberg matrix, statistical equilibrium, upper Hess Enberg matrix

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I. INTRODUCTION

In the discipline of numerical analysis, there are two types of solution methods: iterative solution methods and direct solution methods. Iterative approaches start with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the genuine solution with each step or iteration. It eventually converges on the true solution. If there is no known initial approximation, a guess is performed or an arbitrary initial vector is used instead. The solution must be computed when a specified number of well-defined stages have been completed. The most widely utilized methods for deriving the stationary probability vector from either the stochastic transition probability matrix or the infinitesimal generator are iterative methods of one form or another. This decision was made for a variety of reasons. First, a look at the conventional iterative approaches reveals that the matrices are only involved in one operation: multiplication with one or more vectors, which leaves the transition matrices unchanged. When the transition matrix is large and not banded, direct techniques are generally not preferred due to the volume of fill-in that can quickly overwhelm available storage capacity. Block iterative methods are generalizations of point iterative methods and can be particularly beneficial in Markov chain problems in which the state space can be meaningfully partitioned into subsets. In general, such block iterative methods require more computation per iteration, but this is offset by a faster rate of convergence. Romanovsky (1970) established the application and simulation of discrete Markov Chains, which was followed by Stewart (1994, 2009) with the development of Numerical Solutions of Markov Chains, while Pesch *et al*. demonstrated the appropriateness of the

Markov chain technique in the wind feed in Germany (2015). Uzun and Kiral (2017) used the Markov chain model of fuzzy state to anticipate the direction of gold price movement and to estimate the probabilistic transition matrix of gold price closing returns, whereas Aziza *et al.* (2019) used the Markov chain model of fuzzy state to predict monthly rainfall data (2019). Clement (2019) demonstrated the application of Markov chain to the spread of disease infection, demonstrating that Hepatitis B became more infectious over time than tuberculosis and HIV, while Vermeer and Trilling (2020) demonstrated the application of Markov chain to journalism. However, in this study, the analysis of block lower Hess Enberg numerical iterative methods for stationary distribution in the structured Markov chains is considered.

Notation

α arrival rate
μ departure rate
Q the infinitesimal generator
R rate matrix
$\emptyset$ normalization constant
$\varphi_i, i = 1,\ 2$ additional transition parameter
$\pi(i, K)$ the probability of finding the Markov chain in state (i, K) at statistical equilibrium.
QBD quasi birth-death
R

II. MATERIALS AND METHODS

Neuts (1981, 1989) pioneered a method for examining the numerical solution of Markov chains with certain block structures in their transition matrices, a block structure that frequently appears while investigating queueing systems. In the simplest case, these matrices have infinite block tridiagonal matrices, with the three diagonal blocks repeating after some initial period. For values of $j > i + 1$, all components b_{ij} in a lower Hess Enberg matrix B must be zero. In other words, if we identify the three diagonals of a tridiagonal matrix as the super diagonal, the diagonal, and the sub diagonal as we move from top right to bottom left, then a lower Hess Enberg matrix can only have nonzero entries on and below the super diagonal. The following matrix, for example, is a 6 6 lower Hess Enberg matrix:

$$B = \begin{pmatrix} b_{00} & b_{01} & & & & \\ b_{10} & b_{11} & b_{12} & & & \\ b_{20} & b_{21} & b_{22} & b_{23} & & \\ b_{30} & b_{31} & b_{32} & b_{33} & b_{34} & \\ b_{40} & b_{41} & b_{42} & b_{43} & b_{44} & \\ b_{50} & b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{pmatrix}$$

G/M/1-type stochastic processes have block lower Hess Enberg matrices. Block upper Hess Enberg matrices, which are seen in M/G/1-type stochastic processes, can be defined similarly as block matrices with nonzero blocks only on or above the diagonal blocks and along the sub diagonal block. The first is just a block transposition of the second. The nonzero blocks in block lower Hess Enberg matrices are numbered A_0, A_1, A_2,..., but in block upper Hess Enberg matrices they are numbered in the reverse sequence, from left to right. We choose to use the block upper Hess Enberg notation to describe the blocks in QBD processes (which are both upper and lower block Hess Enberg at the same time). The Markov chains whose infinitesimal generators Q have the following repeated block lower Hess Enberg structure are of relevance to us in this research:

$$\begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & B_{11} & A_0 & 0 & 0 & 0 & \cdots \\ B_{20} & B_{22} & A_1 & A_0 & 0 & 0 & \cdots \\ B_{30} & B_{31} & A_2 & A_1 & A_0 & 0 & \cdots \\ B_{40} & B_{41} & A_3 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

When states are sorted into levels, similar to QBD processes, Markov chains with this structure emerge, but transitions are no longer limited to interlevel and adjacent surrounding levels; transitions are now allowed from any level to any lower level. In this situation, we use two boundary columns (B_{i0} and B_{i1}, $i = 0, 1, 2, \ldots$). to perform the analysis. Depending on the application, there may be more than two boundary columns, necessitating a matrix reorganization.

Our goal is to compute the stationary probability vector from the system of equations $\pi Q = 0$, as usual. Let be partitioned conformally with Q, i.e. $\pi = (\pi_0,\ \pi_1,\ \pi_{2,\ldots})$ where $\pi_i = (\pi(i,1)\ \ \pi(i,1), \ldots, \pi(i,K))$ for $i = 0,1, \ldots$, and $\pi(i, K)$ is the probability of finding the Markov chain in state (i, K) at statistical equilibrium. Neuts (1981) demonstrated that this problem has a matrix geometric solution that is similar to a QBD process, namely that there is a positive matrix R such that

$$\pi_i = \pi_{i-1} R, \text{ for } i = 2, 3, \ \ldots,$$

i.e.,

$$\pi_i = \pi_1 R^{i-1}, \text{ for } i = 2, 3, \ \ldots,$$

Observe that from $\pi Q = 0$ we have

$$\sum_{k=0}^{\infty} \pi_{k+j} A_k = 0, \quad j = 1, 3, \ ...,$$

and, in particular,

$$\pi_1 A_0 + \pi_2 A_1 + \sum_{k=2}^{\infty} \pi_{k+1} A_k = 0$$

Substituting $\pi_i = \pi_1 R^{i-1}$

$$\pi_1 A_0 + \pi_1 R A_1 + \sum_{k=2}^{\infty} \pi_1 R^k A_k = 0$$

Or

$$\pi_1 (A_0 + R A_1 + \sum_{k=2}^{\infty} R^k A_k) = 0$$

provides the following relation from which the matrix R may be computed:

$$(A_0 + R A_1 + \sum_{k=2}^{\infty} R^k A_k) = 0 \qquad (5)$$

Notice that Equation (5) reduces to

$$(A_2 + R A_1 + R^2 A_0) = 0. \text{ when } A_k = 0, \text{ for } k > 2.$$

Rearranging, we find

$$R = (-A_0 A_1^{-1} + \sum_{k=2}^{\infty} R^k A_k A_1^{-1}),$$

which leads to the iterative procedure

$$R_0 = 0, \qquad R_{(l-1)} = A_0 A_1^{-1} - \sum_{k=2}^{\infty} R_l^k A_k A_1^{-1}, \quad l = 0, 1, 2, ...$$

This is nondecreasing and converges to the matrix R, as demonstrated by Neuts. In many circumstances, the infinitesimal generator's structure is such that the blocks $A_i$are zero for small values of i reducing the computational work required for each iteration. The number of iterations required for convergence is frequently huge, as previously, and the incredibly efficient logarithmic reduction technique is no longer applicable—it is built exclusively for QBD processes. Now we'll look at how to get the first sub-vectors π_0 and π_1.. We have $\pi Q = 0$, from the first equation.

$$\sum_{i=0}^{\infty} \pi_i B_{i0} = 0$$

and we may write

$$\pi_0 B_{00} + \sum_{i=1}^{\infty} \pi_i B_{i0} = \pi_0 B_{00} + \sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i0} = \pi_0 B_{00} + \pi_1 (\sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i0}) = 0, \qquad (6)$$

while from the second equation of $\pi Q = 0$,

$$\pi_0 B_{01} + \sum_{i=1}^{\infty} \pi_i B_{i1} = 0, \quad \text{i.e.,} \quad \pi_0 B_{01} + \pi_1 (\sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i1}) = 0 \qquad (7)$$

Putting Equations (6) and (7) together in matrix form, we see that we can compute π_0 and π_1 from

$$(\pi_0 \quad \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ \sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i0} & \sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i1} \end{pmatrix} = (0 \quad 0).$$

The computed values of π_0 and π_1 must now be normalized by dividing them by

$$\emptyset = \pi_0 e + \pi_1 \left(\sum_{i=1}^{\infty} R^{i-1} \right) e = \pi_0 e + \pi_1 (1 - R)^{-1} e$$

III. RESULTS AND DISCUSSION

This section discusses the numerical examples and solutions on the analysis of block lower Hess Enberg numerical iterative methods for stationary distribution in the structured Markov chains.

Illustrative example 1: Consider the Markov chain with the parameters $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\mu = 4$, $\delta_1 = 5$, $\delta_2 = 3$, and additional transition parameters $\varphi_1 = 0.25$ and $\varphi_2 = 0.75$ as illustrated in the state transition diagram below.

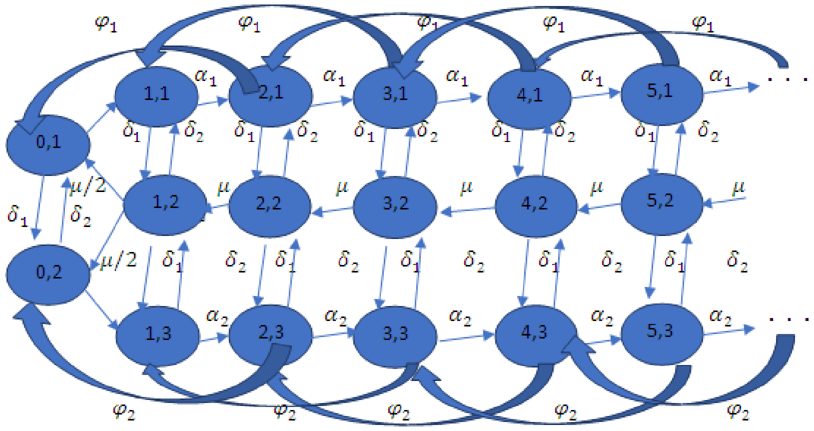

Figure 10.2. State transition diagram for a G/M/1-type process.

The infinitesimal generator transition matrix is

$$Q = \begin{pmatrix}
* & \delta_1 & & \alpha_1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\delta_2 & * & & \square & \alpha_2 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & * & \delta_1 & \square & \alpha_1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\mu/2 & \mu/2 & \delta_2 & * & \delta_1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \delta_2 & * & \square & \square & \alpha_2 & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & * & \delta_1 & \square & \alpha_1 & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \mu & \square & \delta_2 & * & \delta_1 & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \delta_2 & * & \square & \square & \alpha_2 & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & * & \delta_1 & \square & \alpha_1 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \delta_2 & * & \delta_1 & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \mu & \square & \square & \delta_2 & * & \square & \square & \alpha_1 & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & * & \delta_1 & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \mu & \square & \delta_2 & * & \delta_1 & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \delta_2 & * & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \ddots & \square & \ddots
\end{pmatrix}$$

The infinitesimal generator transition matrix is

$$Q = \begin{pmatrix}
-6 & 5 & & 1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
3 & -3.5 & & \square & \square & 0.5 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & -6 & 5 & \square & 1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
2 & 2 & 3 & -12 & 5 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & 3 & -3.5 & \square & \square & 0.5 & \square & \square & \square & \square & \square & \square & \square & \square \\
0.25 & \square & \square & \square & \square & -6 & 5 & \square & 1 & \square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & 4 & \square & 3 & -12 & 5 & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0,75 & \square & \square & \square & \square & 3 & -3.5 & \square & \square & 0.5 & \square & \square & \square & \square & \square \\
\square & \square & 0.25 & \square & \square & \square & \square & \square & -6 & 5 & \square & 1 & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & 4 & \square & 3 & -12 & 5 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & 0.75 & \square & \square & \square & \square & 3 & -3.5 & \square & \square & 0.5 & \square & \square \\
\square & \square & \square & \square & \square & \ddots & \square & \square & \ddots & \square & \square & \ddots & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square
\end{pmatrix} \square$$

Where

$$A_0 = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -(\delta_1+\alpha_1) & \delta_1 & 0 \\ \delta_2 & -(\mu+\delta_1+\delta_2) & \delta_1 \\ 0 & \delta_2 & -(\delta_2+\alpha_2) \end{pmatrix} = \begin{pmatrix} -6 & 5 & 0 \\ 3 & -12 & 5 \\ 0 & 3 & -3.5 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \varphi_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varphi_2 \end{pmatrix} = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.75 \end{pmatrix}.$$

and

$$B_{00} = \begin{pmatrix} -(\delta_1+\alpha_1) & \delta_1 \\ \delta_2 & -(\delta_2+\alpha_2) \end{pmatrix} = \begin{pmatrix} -6 & 5 \\ 3 & -3.5 \end{pmatrix}, \quad \text{etc.}$$

Therefore

$$A_1^{-1} = \begin{pmatrix} -0.2233 & -0.1318 & -0.1550 \\ -0.0791 & -0.1647 & -0.1938 \\ -0.0558 & -0.1163 & -0.3721 \end{pmatrix},$$

$$A_0A_1^{-1} = \begin{pmatrix} -0.2233 & -0.1318 & -0.1550 \\ 0 & 0 & 0 \\ -0.0279 & -0.0581 & -0.1860 \end{pmatrix},$$

$$A_2A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -0.3163 & -0.6589 & -0.7752 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_3A_1^{-1} = \begin{pmatrix} -0.0558 & -0.0329 & -0.0388 \\ 0 & 0 & 0 \\ -0.0419 & -0.0872 & -0.2791 \end{pmatrix},$$

The iterative process is

$$R = (-A_0A_1^{-1} + \sum_{k=2}^{\infty} R^k A_k A_1^{-1}),$$

$$R_{(k+1)} = \begin{pmatrix} 0.2233 & 0.1318 & 0.1550 \\ 0 & 0 & 0 \\ 0.0279 & 0.0581 & 0.1860 \end{pmatrix} + R_{(k)}^2 \begin{pmatrix} 0 & 0 & 0 \\ 0.3163 & 0.6589 & 0.7752 \\ 0 & 0 & 0 \end{pmatrix} + R_{(k)}^3 \begin{pmatrix} 0.0558 & 0.0329 & 0.0388 \\ 0 & 0 & 0 \\ 0.0419 & 0.0872 & 0.2791 \end{pmatrix}$$

and iterating successively, beginning with $R_{(0)} = 0$, we find

$$R_{(1)} = \begin{pmatrix} 0.2233 & 0.1318 & 0.1550 \\ 0 & 0 & 0 \\ 0.0279 & 0.0581 & 0.1860 \end{pmatrix},$$

$$R_{(2)} = \begin{pmatrix} 0.2370 & 0.1593 & 0.1910 \\ 0 & 0 & 0 \\ 0.0331 & 0.0686 & 0.1999 \end{pmatrix},$$

$$R_{(3)} = \begin{pmatrix} 0.2415 & 0.1684 & 0.2031 \\ 0 & 0 & 0 \\ 0.0347 & 0.0719 & 0.2043 \end{pmatrix},$$

After 27 iterations, successive differences are smaller than 10^{-12}, at which point

$$R_{(27)} = \begin{pmatrix} 0.2440 & 0.1734 & 0.2100 \\ 0 & 0 & 0 \\ 0.0356 & 0.0736 & 0.1669 \end{pmatrix},$$

The boundary conditions are now

$$(\pi_0 \quad \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ \sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i0} & \sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i1} \end{pmatrix} = (0 \quad 0).$$

$$= (\pi_0 \quad \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ B_{10} + RB_{20} & B_{11} + RB_{21} + R^2 B_{31} \end{pmatrix} = (0 \quad 0)$$

$$(\pi_0 \quad \pi_1) = \left(\begin{array}{cc|ccc} -6 & 5 & 1 & 0 & 0 \\ 3 & -3.5 & 0 & 0 & 0.5 \\ \hline 0.0610 & 0.1575 & -5.9832 & 5.6938 & 0.0710 \\ 2 & 2 & 3 & -12 & 5 \\ 0.0089 & 0.1555 & 0.0040 & 3.2945 & -3.4624 \end{array}\right) = (0 \quad 0)$$

As before, we solve this by replacing the last equation with the equation $\pi_{0_1} = 1$. The system of equation becomes

$$(\pi_0 \quad \pi_1) = \left(\begin{array}{cc|ccc} -6 & 5 & 1 & 0 & 1 \\ 3 & -3.5 & 0 & 0 & 0 \\ \hline 0.0610 & 0.1575 & -5.9832 & 5.6938 & 0 \\ 2 & 2 & 3 & -12 & 0 \\ 0.0089 & 0.1555 & 0.0040 & 3.2945 & 0 \end{array}\right) = (0 \quad 0| \quad 0 \quad 0 \quad 1)$$

With solution

$$(\pi_0 \quad \pi_1) = (1 \quad 1.7169| \quad 0.3730 \quad 0.4095 \quad 0.8470)$$

The normalization constant is

$$\emptyset = \pi_0 e + \pi_1 \left(\sum_{i=1}^{\infty} R^{i-1}\right) e = \pi_0 e + \pi_1 (1-R)^{-1} e$$

$$\emptyset = (1 \quad 1.7169)e + \quad (0.3730 \quad 0.4095 \quad 0.8470) \begin{pmatrix} 1.3395 & 0.2584 & 0.3546 \\ 0 & 1 & 0 \\ 0.06 & 0.1044 & 1.2764 \end{pmatrix} e$$

$$= 2.7169 + 2.3582 = 5.0751,$$

which allows us to compute

$$\frac{\pi_0}{\emptyset} = (0.1970 \quad 0.3383) \text{ and } \frac{\pi_1}{\emptyset} = (0.0735 \quad 0.0807 \quad 0.1669).$$

Successive subcomponents of the stationary distribution are now computed from $\pi_k = \pi_{k-1} R$. For example,

$$\pi_2 = \pi_1 R = (0.0735 \quad 0.0807 \quad 0.1669) \begin{pmatrix} 0.2440 & 0.1734 & 0.2100 \\ 0 & 0 & 0 \\ 0.0356 & 0.0736 & 0.1669 \end{pmatrix} = (0.0239 \quad 0.0250 \quad 0.0499)$$

$$\pi_3 = \pi_2 R = (0.0239 \quad 0.0250 \quad 0.0499) \begin{pmatrix} 0.2440 & 0.1734 & 0.2100 \\ 0 & 0 & 0 \\ 0.0356 & 0.0736 & 0.1669 \end{pmatrix} = (0.0076 \quad 0.0078 \quad 0.0135)$$

and so on.

When the initial B blocks have the same dimensions as the A blocks and the infinitesimal generator can be expressed in the following regularly occurring form, some simplifications occur:

$$Q = \begin{pmatrix} B_{00} & A_0 & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ B_{20} & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ B_{30} & A_3 & A_2 & A_1 & A_0 & 0 & \cdots \\ B_{40} & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

In this case,

$$\pi_i = \pi_0 R^i, \; for \;\; i = 1, 2, \ldots$$

Furthermore, the sub vector π_0 is the stationary probability vector of $\sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i0}$, and $\sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i0}$ is normalized to give $\pi_0 (1-R)^{-1} e = 1$. More than two border columns can occur in some situations, such as queueing systems with bulk arrivals. Consider the generator matrix, for example.

$$Q = \begin{pmatrix} B_{00} & B_{01} & B_{02} & A_0 & & & & & & & & \\ B_{10} & B_{11} & B_{12} & A_1 & A_0 & & & & & & & \\ B_{20} & B_{21} & B_{22} & A_2 & A_1 & A_0 & & & & & & \\ B_{30} & B_{31} & B_{32} & A_3 & A_2 & A_1 & A_0 & & & & & \\ B_{40} & B_{41} & B_{42} & A_4 & A_3 & A_2 & A_1 & A_0 & & & & \\ B_{50} & B_{51} & B_{52} & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & & \\ B_{60} & B_{61} & B_{62} & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & \\ B_{70} & B_{71} & B_{72} & A_7 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & \\ B_{80} & B_{81} & B_{82} & A_8 & A_7 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

This matrix is currently not block lower Hess Enberg. If, on the other hand, it is reorganized into the form

$$\bar{Q} = \begin{pmatrix} \overline{B_{00}} & \overline{A_0} & 0 & 0 & \cdots \\ \overline{B_{10}} & \overline{A_1} & \overline{A_0} & 0 & \cdots \\ \overline{B_{20}} & \overline{A_2} & \overline{A_1} & \overline{A_0} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$Q = \left(\begin{array}{ccc|ccc|ccc|ccc} B_{00} & B_{01} & B_{02} & A_0 & & & & & & & & \\ B_{10} & B_{11} & B_{12} & A_1 & A_0 & & & & & & & \\ B_{20} & B_{21} & B_{22} & A_2 & A_1 & A_0 & & & & & & \\ \hline B_{30} & B_{31} & B_{32} & A_3 & A_2 & A_1 & A_0 & & & & & \\ B_{40} & B_{41} & B_{42} & A_4 & A_3 & A_2 & A_1 & A_0 & & & & \\ B_{50} & B_{51} & B_{52} & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & & \\ \hline B_{60} & B_{61} & B_{62} & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & \\ B_{70} & B_{71} & B_{72} & A_7 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & \\ B_{80} & B_{81} & B_{82} & A_8 & A_7 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 \\ \hline \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right)$$

With

$$\overline{A_0} = \begin{pmatrix} A_0 & & \\ A_1 & A_0 & \\ A_2 & A_1 & A_0 \end{pmatrix}$$

$$\overline{A_1} = \begin{pmatrix} A_3 & A_2 & A_1 \\ A_4 & A_3 & A_2 \\ A_5 & A_4 & A_1 \end{pmatrix}$$

$$\overline{B_{00}} = \begin{pmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{pmatrix}$$

The matrix $\bar{Q}$ has the desired block lower Hess Enberg form, therefore the procedures discussed in this section can be applied to it successfully. In the case of discrete-time Markov chains, rather than the continuous-time case, replacing $-A_1^{-1}$ with $(1 - A_1)^{-1}$ as it was applicable for QBD processes, is sufficient.

IV. CONCLUSION

the iterative solution methods for the stationary distribution of Markov chains which start with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the genuine solution with each step or iteration is examined on the Markov chains whose transition matrices have a special block structure, a block structure that arises frequently when modeling G/M/1 queueing systems which leaves the transition matrices unchanged and saves time has been investigated, in order to provide some insight into the solutions of stationary distribution of Markov chain. Our quest is to compute the solutions using block lower Hess Enberg numerical iterative methods on the Structured Markov. Matrix operations, boundary condition and normalization constant are used with the help of some existing laws, theorems and formulas of Markov chain while the stationary distribution vector's $\pi = (\pi_0,\ \pi_1,\ \pi_{2,\ldots})$ where $\pi_i = (\pi(i,1)\ \ \pi(i,1), \ldots, \pi(i,K))$ for $i = 0,1, \ldots,$ and $\pi(i,K)$ is the probability of finding the Markov chain in state (i, K) at statistical equilibrium are obtained. Also, this is demonstrated with illustrative example with the following parameters $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\mu = 4$, $\delta_1 = 5$, $\delta_2 = 3$, and additional transition parameters $\varphi_1 = 0.25$ and $\varphi_2 = 0.75$.

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