RESEARCH ARTICLE OPEN ACCESS

Dynamics of a system of two coupled third-order MEMS oscillators

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ABSTRACT
In this work we present a systematic review of novel and interesting behaviour we have observed in a simplified model of a MEMS oscillator. The model is third order and nonlinear, and we expressit as a single ODE for a displacement variable. We find that a single oscillator exhibits limitcycles whose amplitude is well approximated by perturbation methods. Two coupled identicaloscillators have in-phase and out-of-phase modes as well as more complicated motions.Bothof the simple modes are stable in some regions of the parameter space while the bifurcationstructure is quite complex in other regions. This structure is symmetric; the symmetry is brokenby the introduction of detuning between the two oscillators. Numerical integration of the fullsystem is used to check all bifurcation computations.

Each individual oscillator is based on a MEMS structure which moves within a laser-driven interference pattern. As the structure vibrates, it changes the interference gap, causing the quantity of absorbed light to change, producing a feedback loop between the motion and the absorbed light and resulting in a limit cycle oscillation. A simplified model of this MEMS oscillator, omitting parametric feedback and structural damping, is investigated using Lindstedt's perturbation method. Conditions are derived on the parameters of the model for a limit cycle to exist.

The original model of the MEMS oscillator consists of two equations: a second order ODE which describes the physical motion of a microbeam, and a first order ODE which describes the heat conduction due to the laser. Starting with these equations, we derive a single governing ODE which is of third order and which leads to the definition of a linear operator called the MEMS operator. The addition of nonlinear terms in the model is shown to produce limit cycle behavior.

The differential equations of motion of the system of two coupled oscillators are numerically integrated for varying values of the coupling parameter. It is shown that the in-phase mode loses stability as the coupling parameter is reduced below a certain value, and is replaced by two new periodic motions which are born in a pitchfork bifurcation. Then as this parameter is further reduced, the form of the bifurcating periodic motions grows more complex, with yet additional bifurcations occurring. This sequence of bifurcations leads to a situation in which the only periodic motion is a stable out-of-phase mode. The complexity of the resulting sequence of bifurcations is illustrated through a series of diagrams based on numerical integration.

Date of Submission: 28-10-2020 Date of Acceptance: 09-11-2020

I. INTRODUCTION

This work involves models of MEMS limit cycle (LC) oscillators. These are characterized by the presence of a laser light source which plays a dual role in the dynamics: it both illuminates the vibrating specimen, permitting us to detect how it is moving, and simultaneously heats the specimen, producing thermal stresses which influence the motion.The present authors have produced several papers on this topic [1,2,3,4,5].The present work constitutes an overview and compendium of the foregoing works in which a new MEMS operator is introduced, emphasizing the third order character of the associated dynamical system.

The Basic MEMS LC Oscillator

In a JMEMS paper of 2004 [6] the following model of a MEMS oscillator was presented:

$$\ddot{z}+\frac{1}{Q}(\dot{z}-D\dot{T})+(1+CT)(z-DT)+\beta(z-DT)^3=0 \qquad (1)$$

$$\dot{T} + BT = AP(\alpha + \gamma \sin^2 2\pi(z - z_0)) \quad (2)$$

Here *z* is the displacement of a mechanical oscillator and *T* is its temperature due to laser illumination.
In the mechanical equation *Q* is the quality factor, *C* is the stiffness change due to temperature, *D* is the displacement due to temperature and *β* is the coefficient of the cubic nonlinearity. In the thermal equation the quantities *α* and *γ* represent the average and contrast of the absorption of laser power,
P is the laser power, *A* and *B* represent the thermal mass and heat loss rate. The offset, z_0, models the equilibrium position of the oscillator with respect to the interference field created by the oscillator/gap/substrate stack.
This sophisticated model, which includes effects of damping, stiffness change due to heating, periodic dependence of light absorption on interferometric gap, and nonlinearity, was shown to support limit cycle oscillations.
The present work was motivated by the question, what is the simplest version of the above MEMS oscillator which supports limit cycle oscillations?Our candidate is the following system which omits damping and various other effects [1]:

$$\ddot{z} + z = T \quad (3)$$

$$\dot{T} + T = z^2 - zz_0 \quad (4)$$

For simplicity, all constants have been taken equal to unity. Numerical integration shows that this system supports a limit cycle, see Fig.1.

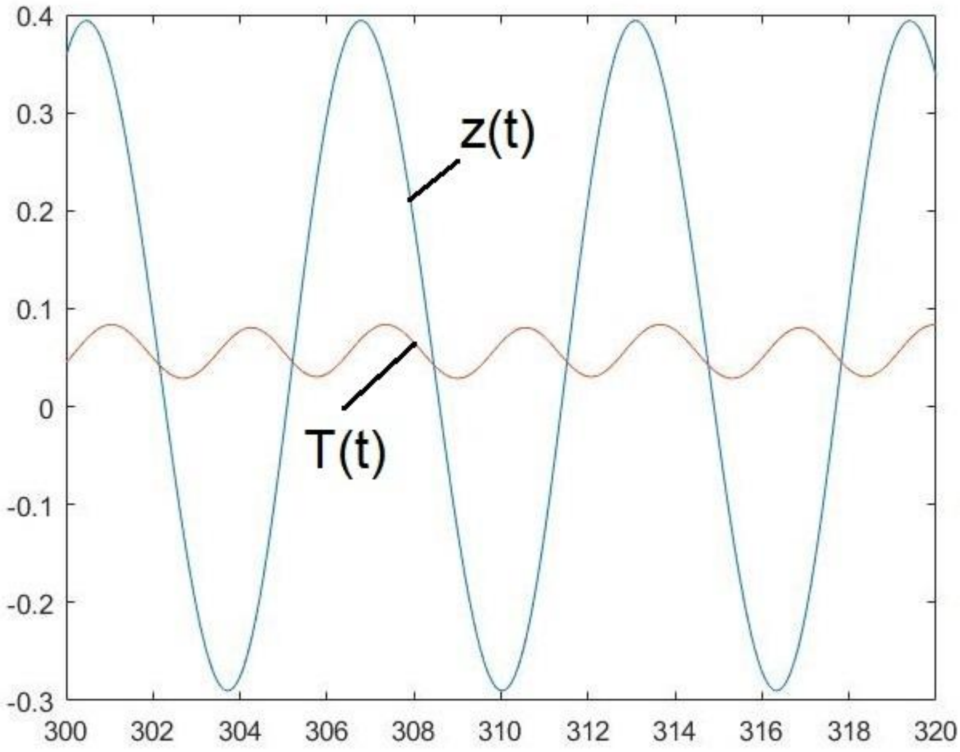


Figure 1 : Results of numerical integration of eqs.(3),(4) for z_0=0.1.Note that the variable *T*(*t*) appears to have twice the frequency of *z*(*t*) [1].

In this paper we are asking how special this system is, i.e., if it is generalized in the form:

$$\ddot{z} + z = T \quad (5)$$

$$\dot{T} + T = f(z) \quad (6)$$

where *f*(*z*) is a general function of *z*, what conditions on *f*(*z*) give a LC at the origin?

A Third order ODE model

We begin by differentiating eq.(5):

$$\dddot{z} + \dot{z} = \dot{T} \tag{7}$$

Next we add equations (5) and (7) to get:

$$\dddot{z} + \ddot{z} + \dot{z} + z = \dot{T} + T \tag{8}$$

Finally we substitute eq.(6) in eq.(8), obtaining:

$$\dddot{z} + \ddot{z} + \dot{z} + z = f(z) \tag{9}$$

In the case that $f(z)$=0, we set z=exp(λt) giving

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0 \tag{10}$$

$$\Rightarrow (\lambda+1)(\lambda^2+1) = 0 \tag{11}$$

$$\Rightarrow \lambda = -1, \pm i \tag{12}$$

For a stable LC at the origin, we require the origin to be an unstable equilibrium point. Let us take $f(z)$ in the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \tag{13}$$

In order for z=0 to be an equilibrium point, we take

$$a_0 = 0 \tag{14}$$

Using eqs.(13),(14), the eigenvalues λ of eq.(9) satisfy

$$\lambda^3 + \lambda^2 + \lambda + (1 - a_1) = 0 \tag{15}$$

Eq.(15) can be solved in closed form and then Taylor-expanded for small real values of a_1. The result is that the real parts of eigenvalue λ can be written (valid for small real values of a_1):

$$Re(\lambda) \sim -0.25 a_1, -0.25 a_1, -1 + 0.5 a_1 \tag{16}$$

Thus we see that for the origin z=0 to be unstable, $a_1<0$.

So now we have derived the following equation:

$$\dddot{z} + \ddot{z} + \dot{z} + z = a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \qquad a_1 < 0 \tag{17}$$

We ask for which values of a_1, a_2 and a_3 does eq.(17) exhibit a stable LC?

Lindstedt's Method [1]

In what follows we use Lindstedt's method to determine conditions on the a_i coefficients such that eq.(17) exhibits a LC.

To begin with, we rescale a_1 to involve a small parameter ε :

$$a_1 = -p\varepsilon^2, \qquad p > 0 \tag{18}$$

Next we rescale time t:

$$\tau = \omega t, \quad where \quad \omega = 1 + k\varepsilon^2 + \cdots \tag{19}$$

So that eq.(17) becomes

$$\omega^3 z''' + \omega^2 z'' + \omega z' + z = -pz\varepsilon^2 + a_2 z^2 + a_3 z^3 + \cdots \tag{20}$$

where now $z' = dz/d\tau$.

Next we expand z in a power series in ε:

$$z = \varepsilon z_1 + z_2 \varepsilon^2 + z_3 \varepsilon^3 + \cdots \tag{21}$$

Substituting eqs.(19,21)into eq.(20), collecting terms and setting the coefficient of ε^i to zero, we get:

$$i = 1, \qquad L(z_1) = 0 \tag{22}$$

$$i = 2, \qquad L(z_2) = a_2 z_1^2 \tag{23}$$

$$i = 3, \qquad L(z_3) + kM(z_1) = -p\varepsilon^2 z_1 + 2a_2 z_1 z_2 + a_3 z_1^3 \tag{24}$$

where we have used the abbreviations:

$$L(z_i) = z_i''' + z_i'' + z_i' + z_i = \frac{d^3 z_i}{d\tau^3} + \frac{d^2 z_i}{d\tau^2} + \frac{dz_i}{d\tau} + z_i \tag{25}$$

$$M(z_i) = 3z_i''' + 2z_i'' + z_i' = 3\frac{d^3 z_i}{d\tau^3} + 2\frac{d^2 z_i}{d\tau^2} + \frac{dz_i}{d\tau} \tag{26}$$

We choose to satisfy eq.(22) with

$$z_1 = A\cos\tau \tag{27}$$

Substituting eq.(27) into eq.(23), we obtain the following expression for z_2:

$$z_2 = a_2 A^2 \left(\frac{1}{2} - \frac{1}{30}\cos 2\tau - \frac{1}{15}\sin 2\tau \right) \tag{28}$$

Next we substitute eqs.(27),(28) into eq.(24) and eliminate secular terms, giving:

$$k = -\frac{2a_2^2\, p}{54a_2^2 + 45a_3} \tag{29}$$

$$A = 2\sqrt{\frac{5}{3}}\sqrt{\frac{p}{6a_2^2 + 5a_3}} \tag{30}$$

The condition for a limit cycle to exist is that A be real in eq.(30). As a check, if a_3=0 we get

$$A = \frac{\sqrt{10p}}{3a_2} \tag{31}$$

which agrees with previous work in [1].

Coupled MEMS Oscillators

In the previous work [1],authors have modelled a single MEMS LC oscillator by the eqs, see eqs.(3),(4) above:

$$z'' + z = T \tag{32}$$

$$T' + T = z^2 - zp \tag{33}$$

Now we start with two such identical limit cycle oscillators which are coupled:

$$z_1'' + z_1 = T_1 + \alpha(z_2 - z_1) \tag{34}$$

$$T_1' + T_1 = z_1^2 - z_1 p \tag{35}$$

$$z_2'' + z_2 = T_2 + \alpha(z_1 - z_2) \tag{36}$$

$$T_2'' + T_2 = z_2^2 - z_2 p \tag{37}$$

Taking the derivative of equations (34) and (36):

$$z_1''' + z_1' = T_1' + \alpha(z_2' - z_1') \tag{38}$$

$$T_1' + T_1 = z_1^2 - z_1 p \tag{39}$$

$$z_2''' + z_2' = T_2' + \alpha(z_1' - z_2') \tag{40}$$

$$T_2' + T_2 = z_2^2 - z_2 p \tag{41}$$

Replace (38) by (38)+(34), and replace (40) by (40)+(38):

$$z_1''' + z_1'' + z_1' + z_1 = T_1' + T_1 + \alpha(z_2 - z_1) + \alpha(z_2' - z_1') \tag{42}$$

$$T_1' + T_1 = z_1^2 - z_1 p \tag{43}$$

$$z_2''' + z_2'' + z_2' + z_2 = T_2' + T_2 + \alpha(z_1 - z_2) + \alpha(z_1' - z_2') \quad (44)$$

$$T_2' + T_2 = z_2^2 - z_2 p \quad (45)$$

Plug (43) into (42) and plug (45) into (44):

$$z_1''' + z_1'' + z_1' + z_1 = z_1^2 - z_1 p + \alpha(z_2 - z_1) + \alpha(z_2' - z_1') \quad (46)$$

$$z_2''' + z_2'' + z_2' + z_2 = z_2^2 - z_2 p + \alpha(z_1 - z_2) + \alpha(z_1' - z_2') \quad (47)$$

We define the MEMS operator $L(z)$ as follows:

$$L(z) = z''' + z'' + z' + z \quad (48)$$

Eqs.(46) and (47) become:

$$L(z_1) = z_1^2 - z_1 p + \alpha(z_2 - z_1) + \alpha(z_2' - z_1') \quad (49)$$

$$L(z_2) = z_2^2 - z_2 p + \alpha(z_1 - z_2) + \alpha(z_1' - z_2') \quad (50)$$

Next we define variables u and v:

$$u = z_1 - z_2 \quad (51)$$

$$v = z_1 + z_2 \quad (52)$$

Replace (49) by (49)-(50) and replace (50) by (49)+(50):

$$L(u) = (z_1^2 - z_2^2) - (z_1 - z_2)p + 2\alpha(z_2 - z_1) + 2\alpha(z_2' - z_1') \quad (53)$$

$$L(v) = z_1^2 + z_2^2 - (z_1 + z_2)p \quad (54)$$

Now we use $(z_1^2 - z_2^2) = (z_1 - z_2)(z_1 + z_2) = uv$ and $z_1^2 + z_2^2 = (1/2)(u^2 + v^2)$ whereupon eqs.(53) and (54) become:

$$L(u) = uv - up - 2\alpha(u' + u) \quad (55)$$

$$L(v) = (1/2)(u^2 + v^2) - vp \quad (56)$$

Note that once the variables u and v are solved for (for example by numerical integration), the original variables z_1 and z_2 can be recovered by the inverse transformation of eqs. (51), (52):

$$z_1 = \frac{u + v}{2} \quad (57)$$

$$z_2 = \frac{v - u}{2} \quad (58)$$

Bifurcations via numerical integration

Appendix I contains Figures 2 through 19 which show the results of numerically integrating the MEMS equations (55), (56). Results are displayed in the z_1-z_2 plane as the coupling parameter $\alpha$is varied. The Matlab program used to produce these figures is displayed in Appendix II.

II. CONCLUSION

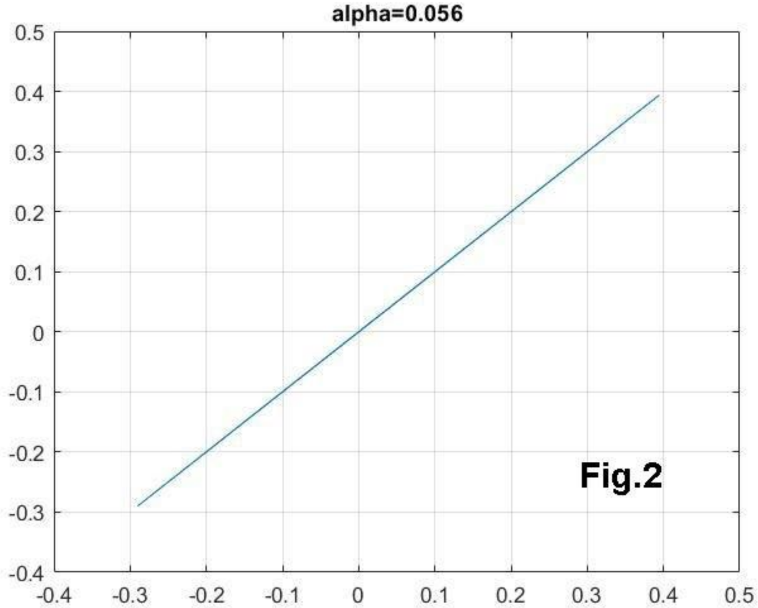
Thus we see that the third order MEMS systems are of considerable interest from the nonlinear dynamical point of view. In future works we hope to extend our study to more general third order systems. We also plan to manufacture the oscillators in the laboratory and verify these theoretical predictions.

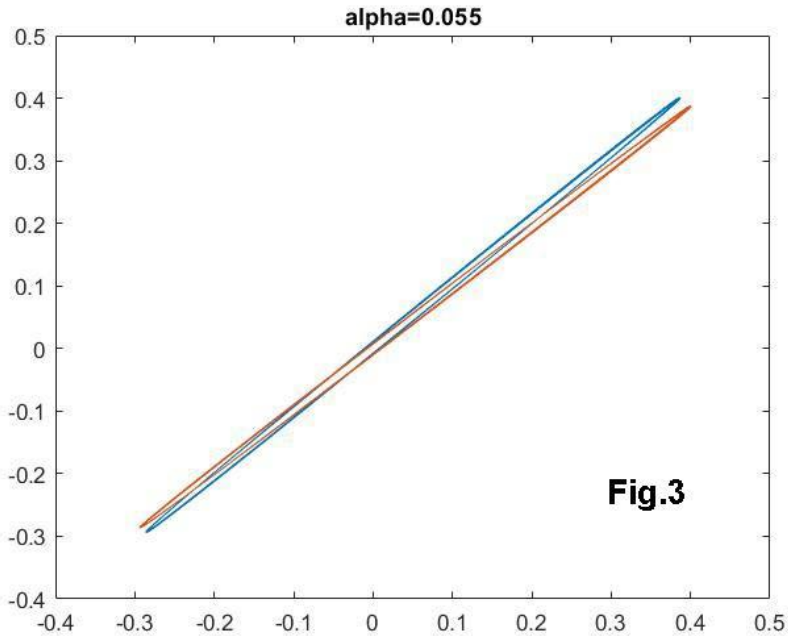
ACKNOWLEDGEMENT

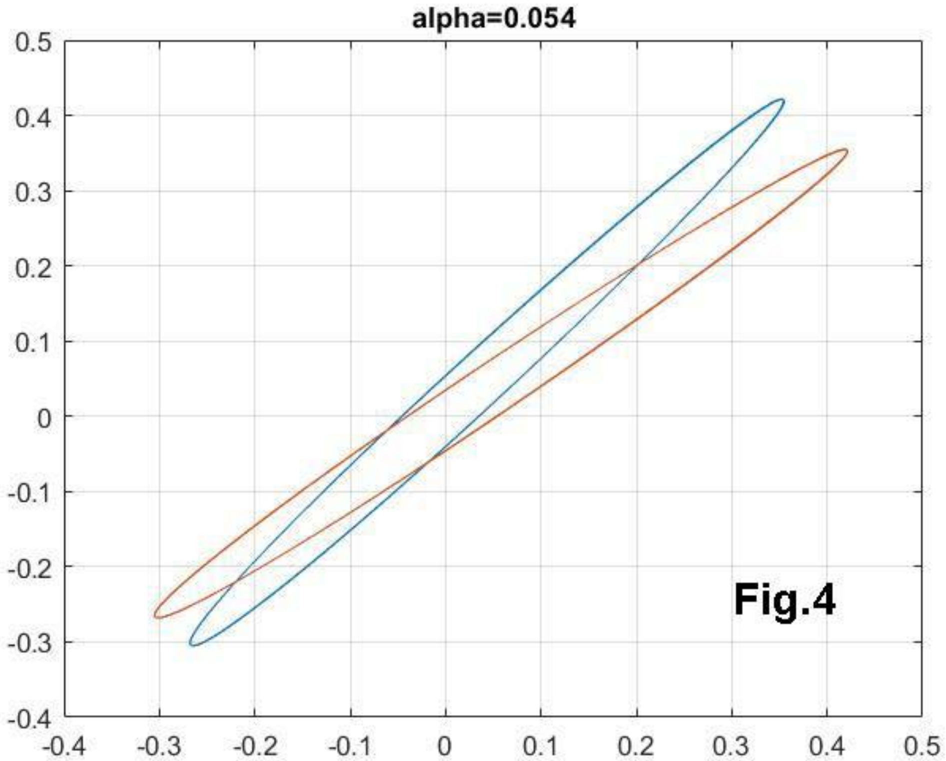
This work was supported by the National Science Foundation under grant number CMMI-1634664.

Appendix 1. Results of numerical integration

As shown in Fig.2, the in-phase mode is stable for α>0.057. As α is decreased below 0.057, the in-phase mode loses stability and is replaced by two new periodic motions which are born in a pitchfork bifurcation. These may be seen in Figs.3 through 11. Then asα is further decreased through 0.046, the form of the bifurcating periodic motions grows more complex, see Figs.12 through 17. Another bifurcation occurs when α decreases through about 0.04, in which the two periodic motions join, see Fig.18. Then for α<0.039this joined periodic motion merges with an unstable motion and disappears, leaving only the out-of-phase mode, see Fig.19. This sequence of bifurcations confirms the results obtained by perturbation methods in previous work [2,4].



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0.4
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Fig.4

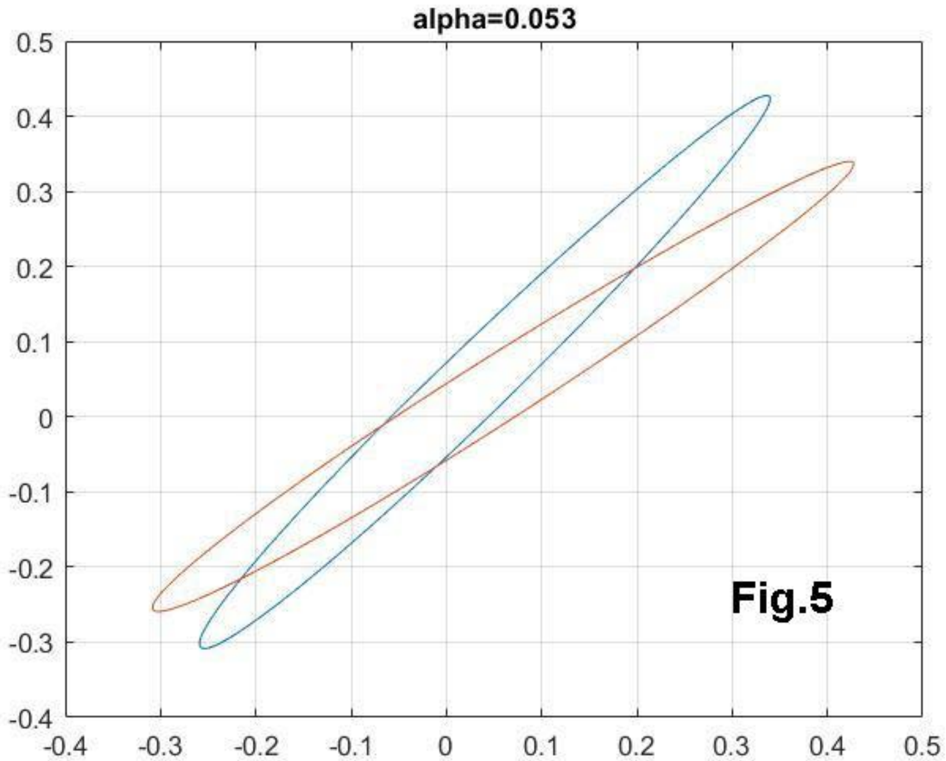
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Fig.5

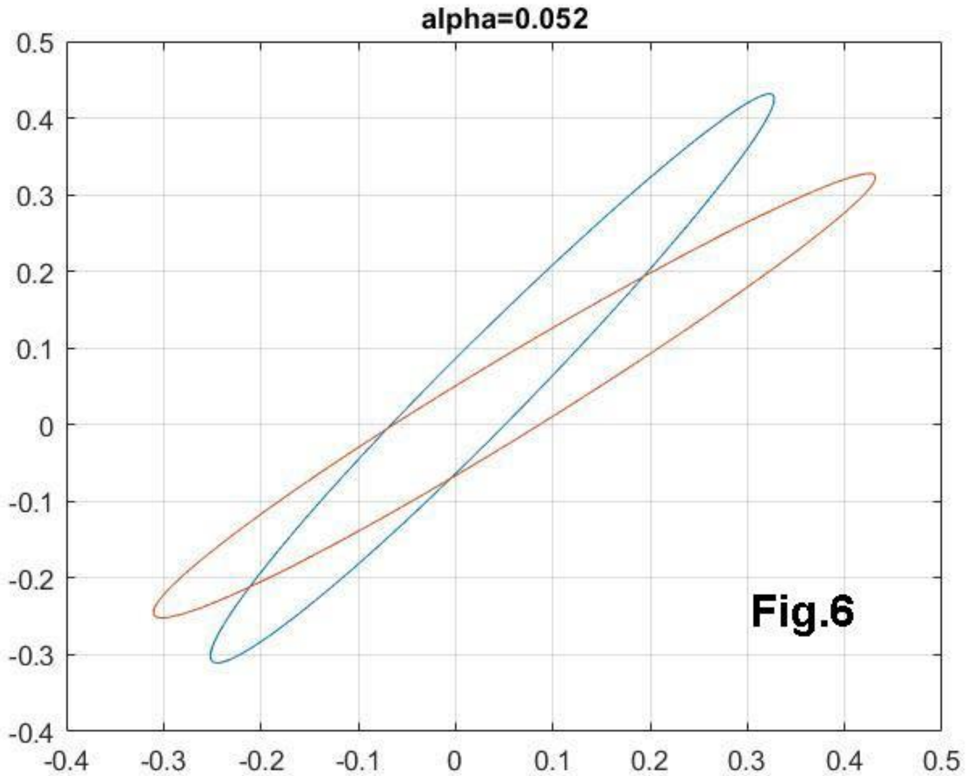
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Fig.6

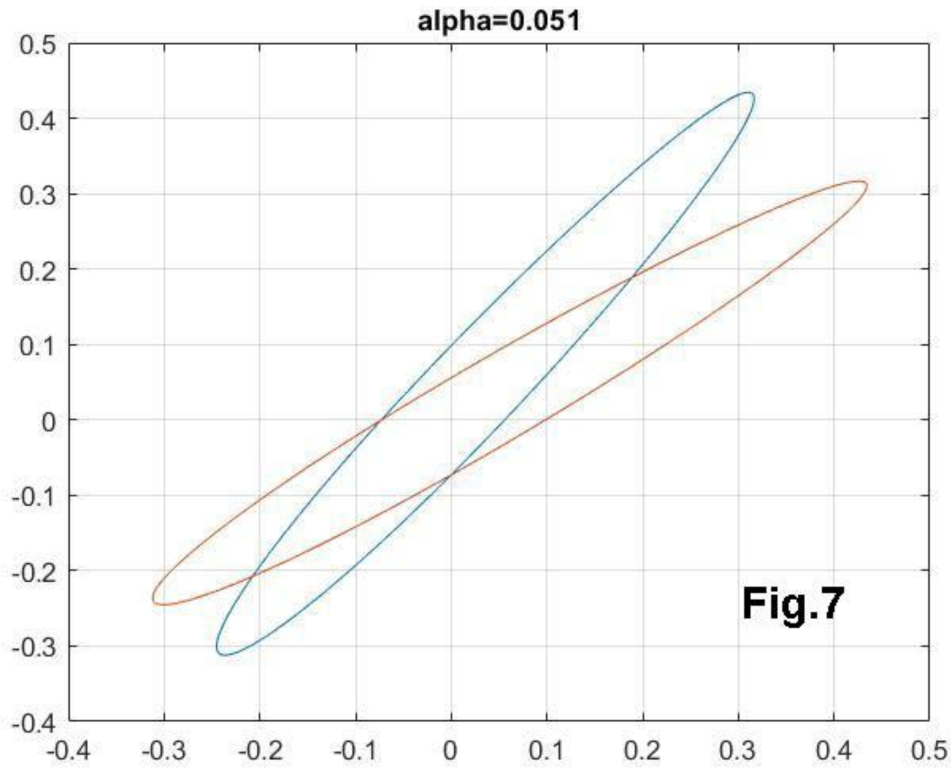
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Fig.7

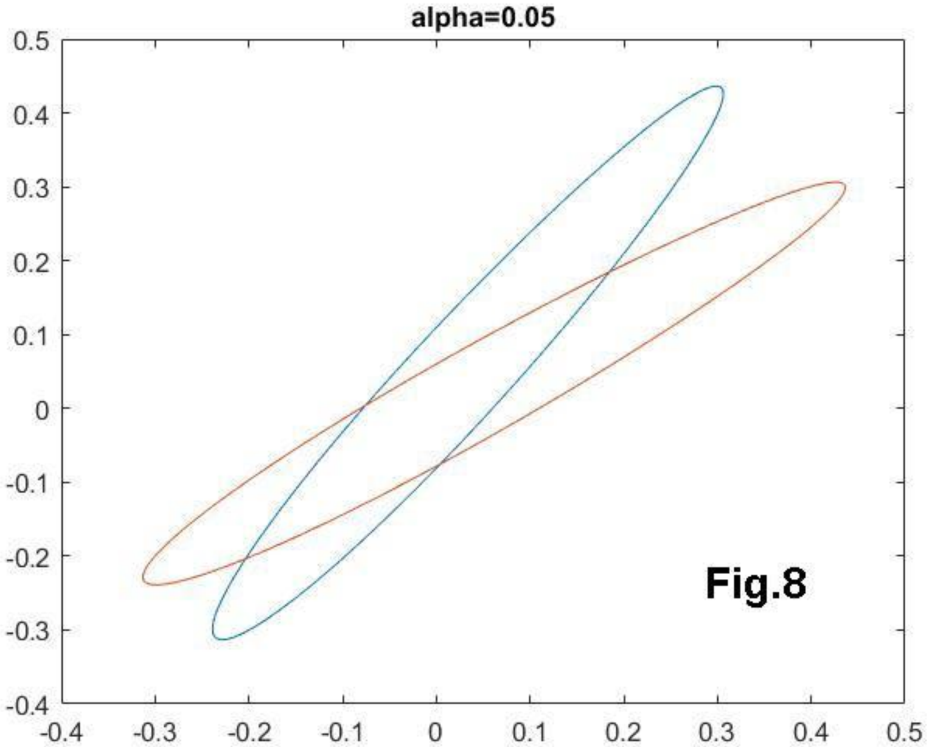
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Fig.8

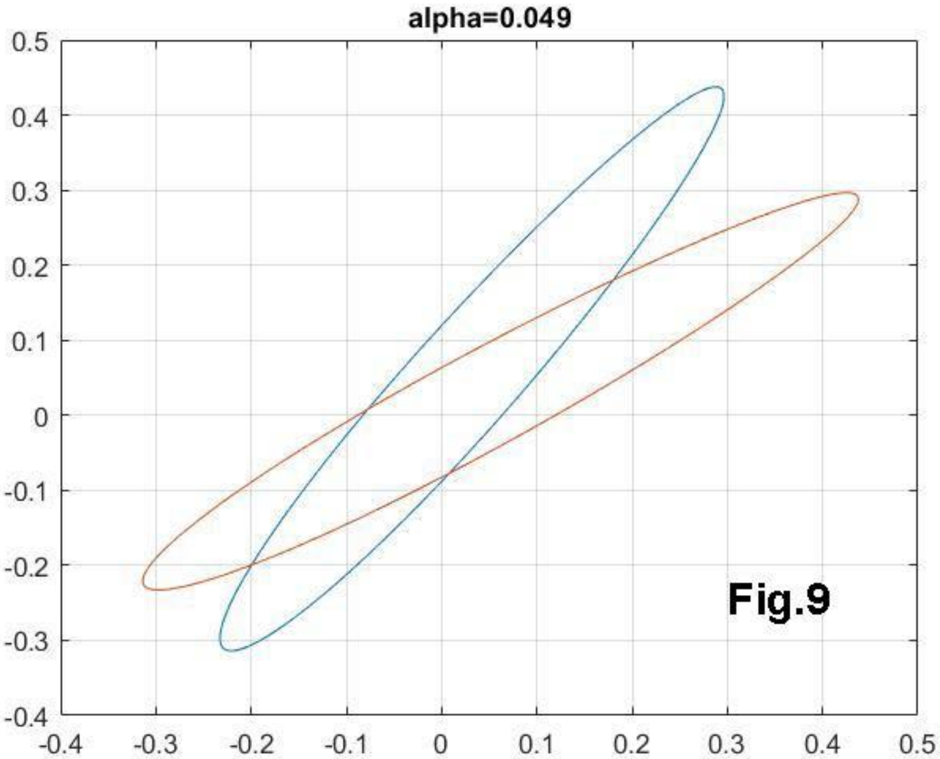
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Fig.9

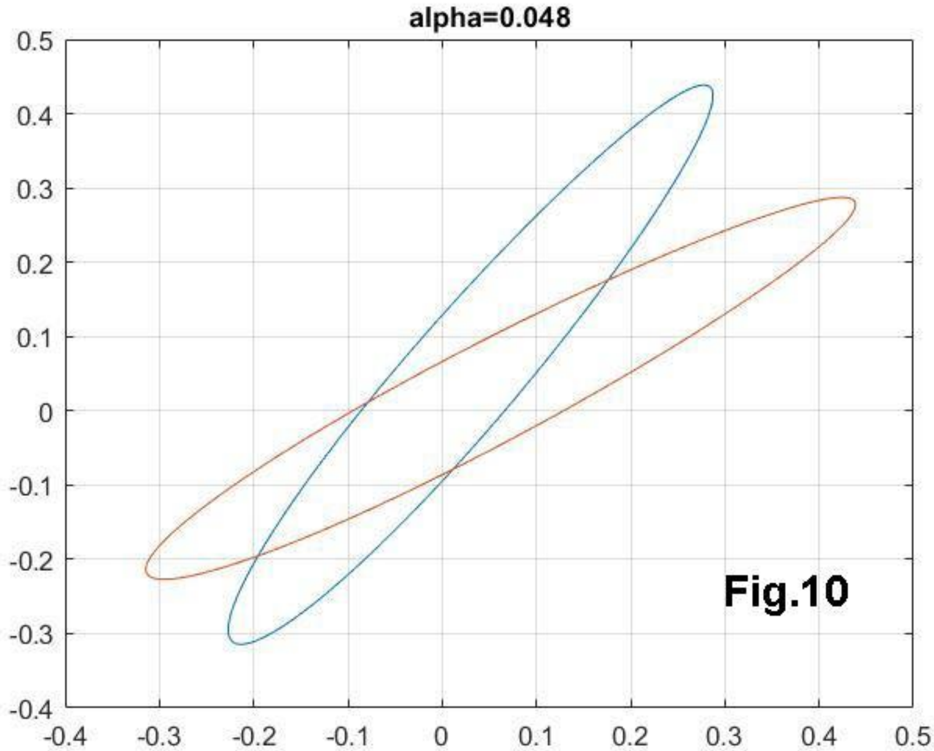
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Fig.10

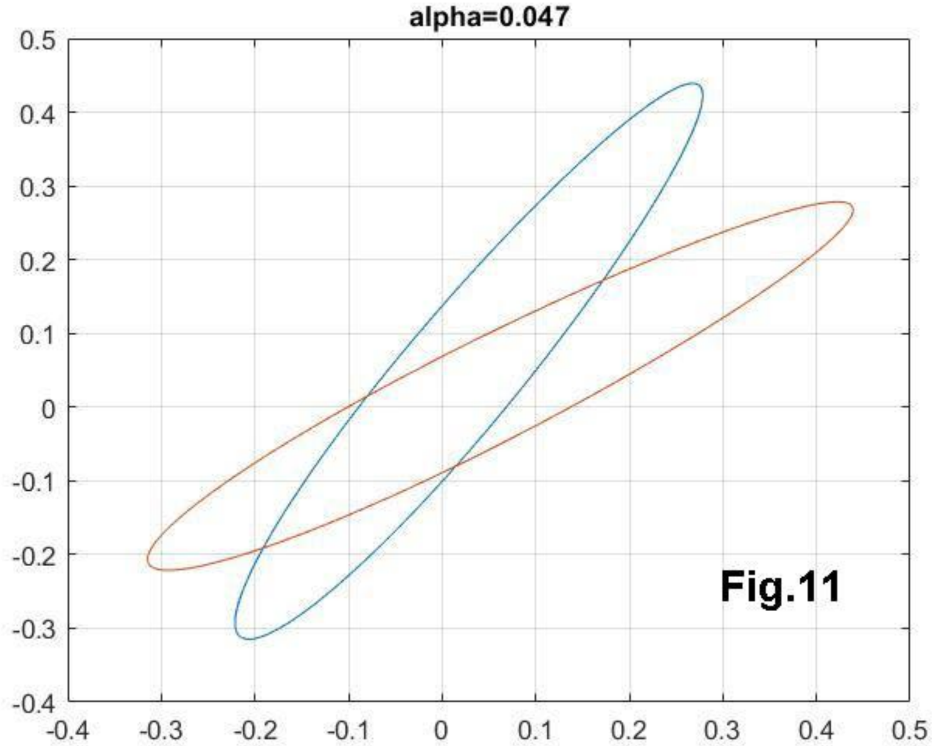
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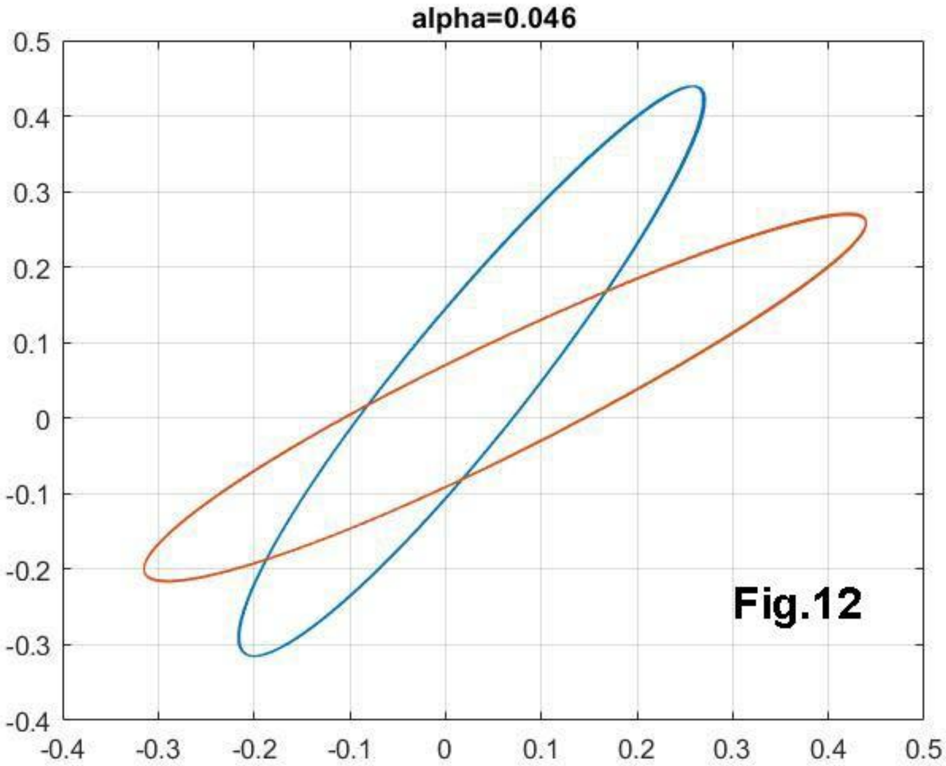
Fig.11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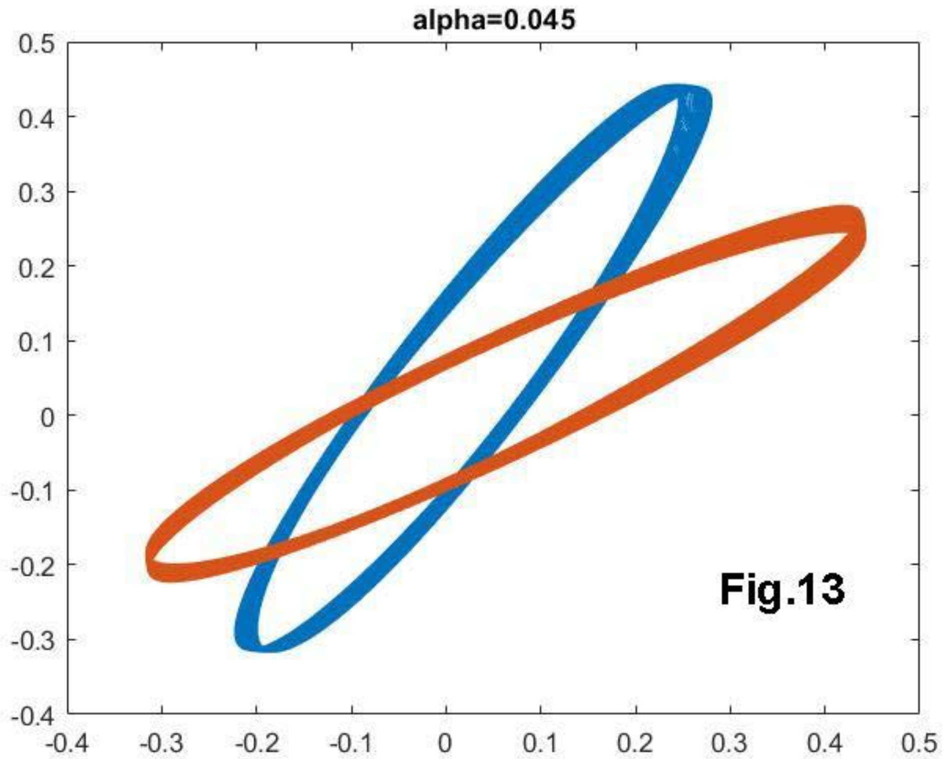
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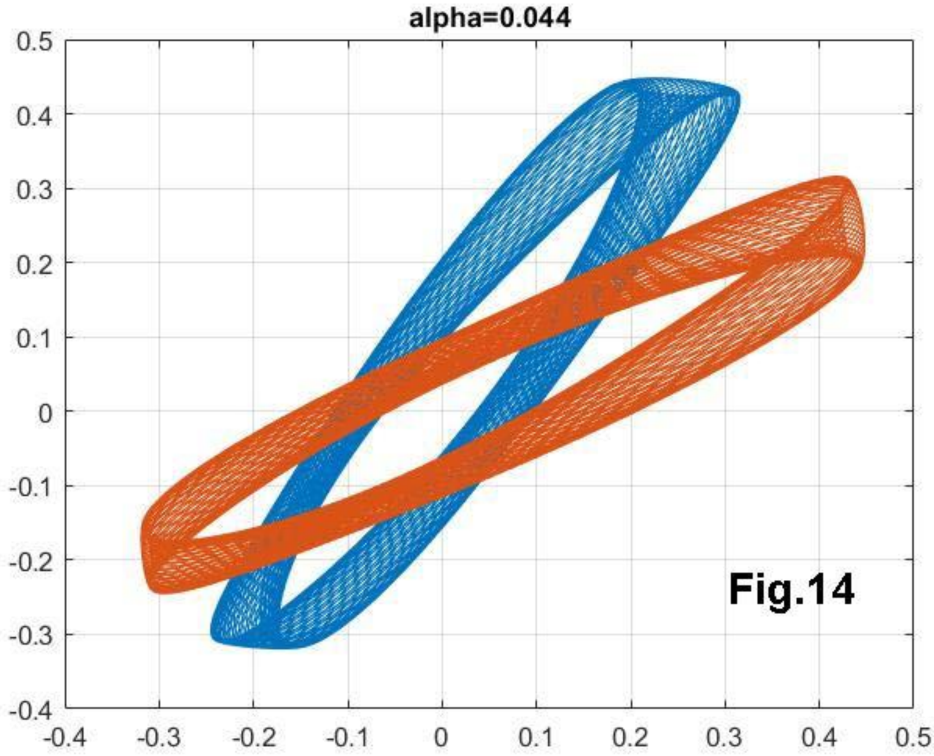
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Fig.14

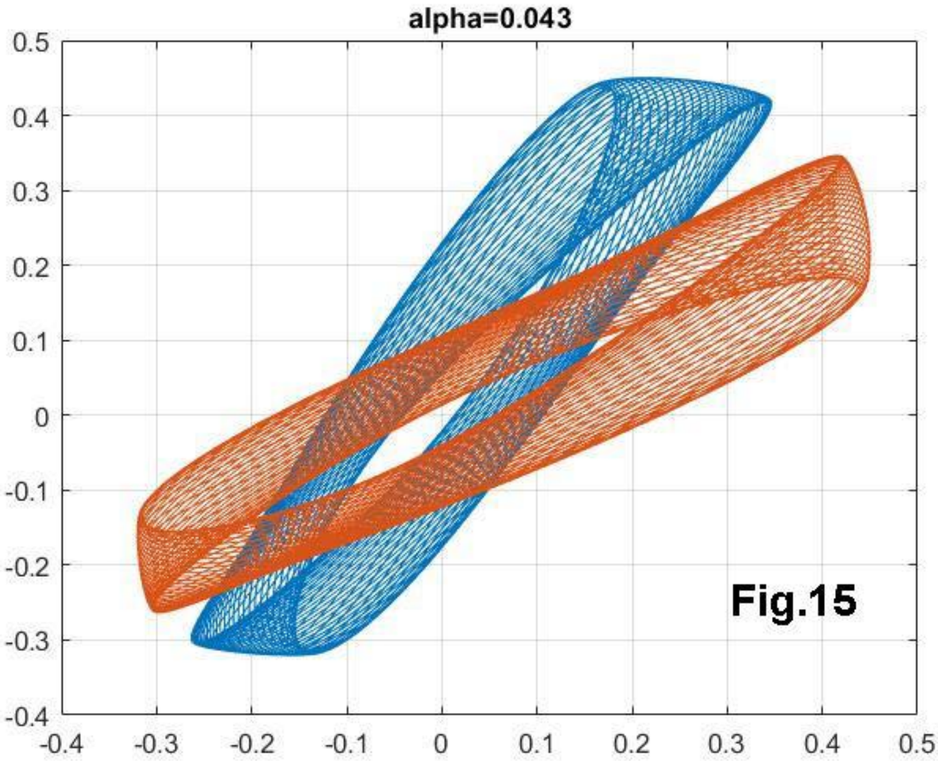
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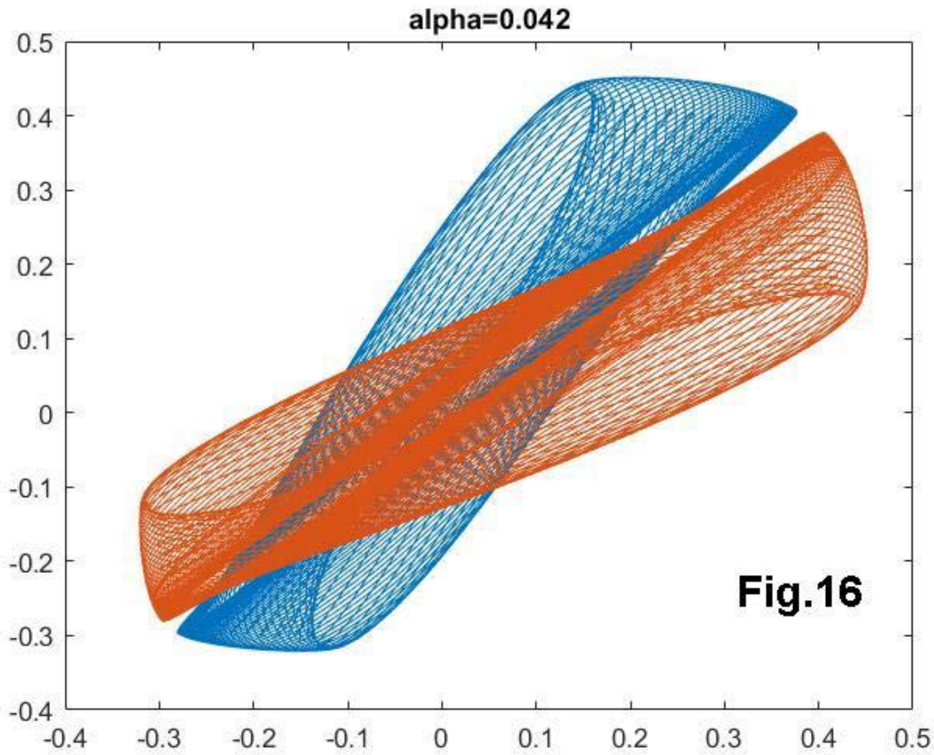
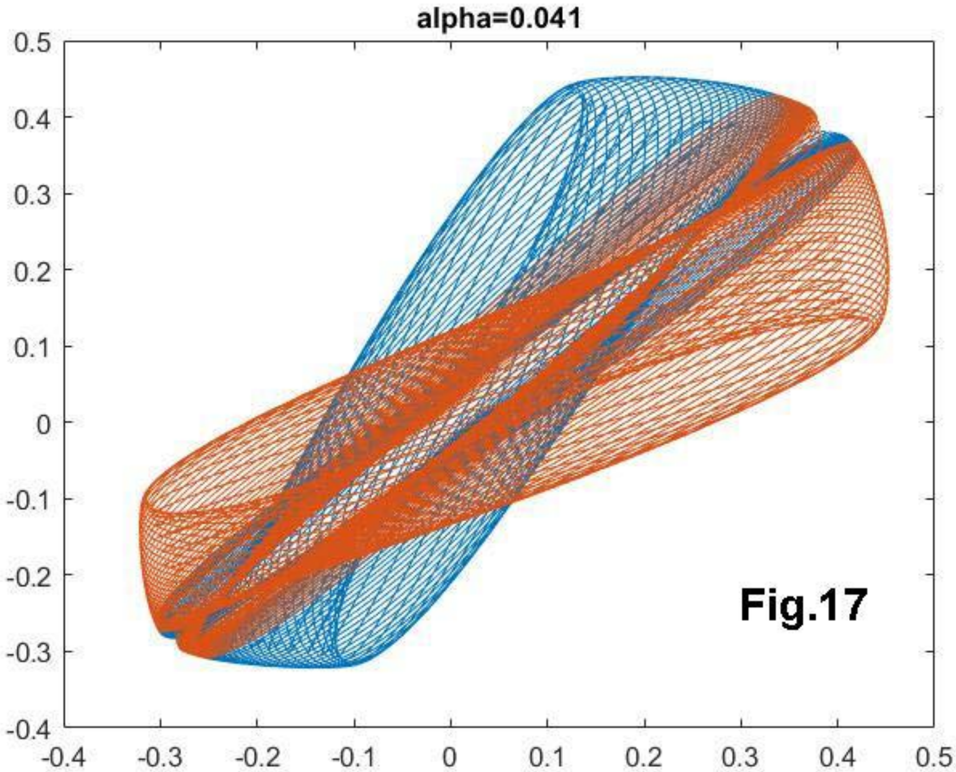
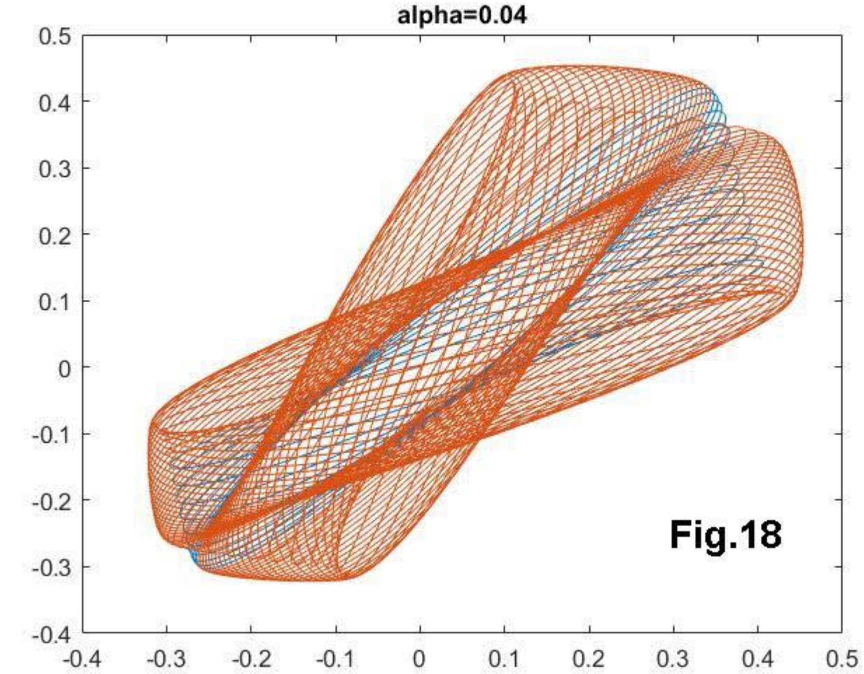
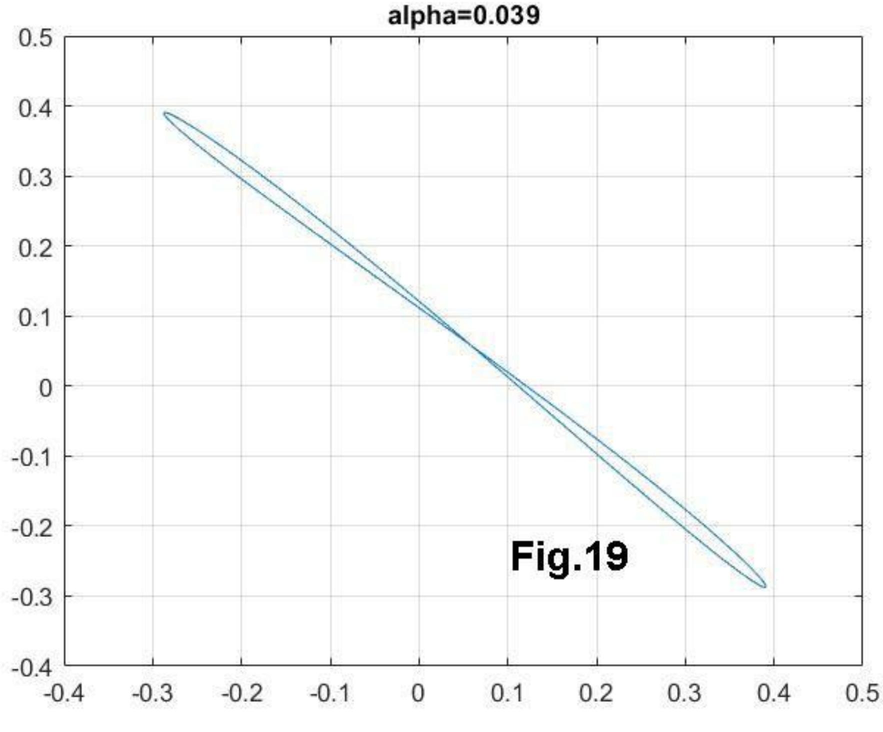
Fig.15

ISSN: 2248-9622, Vol. 10, Issue 11, (Series-I) November 2020, pp. 38-52









Appendix II : Matlab program
filename: numint_uv_eqs.m

```
function anyname
%call with filename
tspan=0:0.01:3000;
w0=[0.08 0 0 0.1 0 0];
%w0=[0.08 0 0 -0.1 0 0];
[t,w]=ode45(@foo,tspan,w0,odeset('MaxStep',.01));
u=w(:,1);
ud=w(:,2);
udd=w(:,3);
v=w(:,4);
vd=w(:,5);
```

```
vdd=w(:,6);
x=(u+v)/2;
y=(v-u)/2;
plot(x(end-40000+1:end),y(end-40000+1:end));
grid;
alpha=.05;
title(['alpha=' num2str(alpha)]);
axis([-.4 .5 -.4 .5]);
%--------------------------------
function any=foo(t,w);
p=.1;
alpha=.05;
u=w(1);
ud=w(2);
udd=w(3);
v=w(4);
vd=w(5);
vdd=w(6);
any=[ud;
udd;
   -u-ud-udd+u*v-p*u-2*alpha*(ud+u);
vd;
vdd;
    -v-vd-vdd+.5*(u^2+v^2)-p*v];
```

REFERENCES

[1]. Analysis of a simplified MEMS oscillator,Richard H. Rand, Alan T. Zehnder and B.Shayak Proceedings of 9th European Nonlinear Dynamics Conference (ENOC 2017), June 25-30, 2017, Budapest, Hungary

[2]. Dynamics of a System of Two Coupled MEMS OscillatorsRichard H. Rand, Alan T. Zehnder, B.Shayak and A.BhaskarChapter 20 in IUTAM Symposium on Exploiting Nonlinear Dynamics in Engineering Systems, Springer 2020

[3]. Phase Locking of Electrostatically Coupled Thermo-optically Driven MEMS Limit Cycle Oscillators
Alan T.Zehnder, Richard H. Rand and Slava KrylovInt. J. of Non-Linear Mechanics 102:92-100 (2018)

[4]. Simplified model and analysis of a pair of coupled thermo-optical MEMS oscillatorsRichard H. Rand, Alan T. Zehnder, B. Shayak and Aditya BhaskarNonlinear Dynamics, volume 99, pages73–83 (2020)

[5]. Coexisting modes and bifurcation structure in a pair of coupled detuned third order oscillators
B.Shayak, Aditya Bhasker, Alan T.Zehnder, Richard H.RandInt. J. Non-Linear Mechanics, volume 122 (2020) 103464

[6]. Limit cycle oscillations in CW laser-driven NEMSK. Aubin, M. Zalalutdinov, T. Alan, R.B. Reichenbach, R. Rand, A. Zehnder, J. Parpia, H. CraigheadJ. Microelectromechanical Syst., vol.13, 1018-1026 (2004)