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Extra Geometry I. Extra lines

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ABSTRACT
Using the theory of exploded numbers by the axiom – systems of real numbers and Euclidean geometry, we introduce a the concept of extra - line of the three – dimensional space. The extra - lines are the visible subsets of super – lineswhich are the explodeds of the Euclidean lines. We investigate the main properties of extra – lines. We prove more similar properties of Euclidean lines and extra – lines, but with respect to the paralellity there is an essentialdifference among them.
Keywords: exploded and compressed numbers ,super – line, extra – line, border points, extra – prallelity.

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I. INTRODUCTION

We imagine our universe as the familiar three dimensional Euclidean space

$$\mathbb{R}^3 = \left\{ P = (x, y, z) \middle| \begin{Bmatrix} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{Bmatrix} \right\},$$

with its well known apparatus, among others

- the ordered field $(\mathbb{R}, <, +, \cdot)$ of real numbers,
- the vector algebra of $\mathbb{R}^3$: the multiplication $c \cdot P = (cx. cy, cz), c \in \mathbb{R}, P \in \mathbb{R}^3$ the addition $P_1 + P_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$, the inner product $P_1 \cdot P_2 = x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2$, the norm $\|P\| = \sqrt{P \cdot P}$ and distance $d(P_1, P_2) = \|P_1 - P_2\|$.

The apparatus of exploded and compressed numbers is described in [1]. (See Chapter 2.). Here we collect the some important informations:

- For any $x \in \mathbb{R}$ its exploded is denoted by $\breve{x}$.A two – dimensional model for $\breve{x}$ is the ordered pair $\left((sgn\, x) \cdot \tanh^{-1}\{|x|\}, (sgn\, x) \cdot [|x|]\right)$ (Here, $\tanh^{-1} x = \frac{1}{2} ln \frac{1+x}{1-x}, -1 < x < 1$; $[x]$ is the greatest integer number, which is less than or equal to x and $\{x\} = x - [x]$.) The mapping $x \to \breve{x}$ is mutually unambiguous. (See [1] Theorem 3.2.6.)The set of exploded numbers is denoted by $\breve{\mathbb{R}}$. If $x \in]-1,1[$then instead of $\left((sgn\, x) \cdot \tanh^{-1}\{|x|\}, 0\right)$ we write
$$\breve{x} = \tanh^{-1} x .$$
So, the set $\mathbb{R}$ is a proper subset of $\breve{\mathbb{R}}$.
- The set $\breve{\mathbb{R}}$ has an algebraic structure with the super – operations
$$\breve{x} \overline{\oplus} \breve{y} = \widetilde{x + y} \quad ; x, y \in \mathbb{R}$$
and
$$\breve{x} \overline{\odot} \breve{y} = \widetilde{x \cdot y} \quad ; x, y \in \mathbb{R}$$
are called super – addition and super – multiplication, respectively.
- For any pair $\breve{x}, \breve{y} \in \breve{\mathbb{R}}$ we say that $\breve{x} = \breve{y}$ if and only if $x = y$ and $\breve{x} < \breve{y}$ if and only if $x < y$. (Monotonity of explosion.) Hence, we have that $\left(\breve{\mathbb{R}}, <, \overline{\oplus}, \overline{\odot}\right)$ is an ordered field which is isomorphic with $(\mathbb{R}, <, +, \cdot)$.
- For any $u \in \breve{\mathbb{R}}$ the real number $\underline{u}$ is called the compressed of u defined by the first inversion formula
$$\widetilde{(\underline{u})} = u \quad , \quad u \in \breve{\mathbb{R}}.$$

Hence, for any $u \in \mathbb{R}$ we have

$$\underline{u} = \tanh u \left(= \frac{e^u - e^{-u}}{e^u + e^{-u}} \right) \quad , \quad u \in \mathbb{R}.$$

- The first inversion formula yields the second inversion formula

$$\underline{(\breve{x})} = x \quad , \quad x \in \mathbb{R}.$$

- Using the inversion formulas we give the super – operations other forms

$$u \overline{\oplus} v = \widetilde{\underline{u} + \underline{v}} \quad ; u, v \in \breve{\mathbb{R}},$$

and

$$u \overline{\odot} v = \widetilde{\underline{u} \cdot \underline{v}} \quad ; u, v \in \breve{\mathbb{R}} ,$$

Moreover, we use the super – subtraction and super - division

$$u \overline{\ominus} v = \widetilde{\underline{u} - \underline{v}} \quad ; u, v \in \breve{\mathbb{R}},$$

and

$$u \overline{\oslash} v = \widetilde{\underline{u} : \underline{v}} \quad ; u, v (\neq 0) \in \breve{\mathbb{R}},$$

respectively.

Extending the concept of additive inverse element for exploded number u we denote by $(-u)$ the exploded number for which $u\overline{\oplus}(-u) = 0$. Moreover,

$$\underline{(-u)} = -\underline{u} \qquad , u \in \breve{\mathbb{R}}.$$

is obtained. The super absolute value of the exploded number u, denoted by $]u[$, such that

$$]u[= \begin{cases} u \; if \; u > 0 \\ 0 \; if \; u = 0 \\ -u \; if \; u < 0 \end{cases}.$$

Clearly, if u is a real number then $|u| =]u[$.

The exploded numbers $\widetilde{(-1)}$ and $\breve{1}$ are not real numbers. $\widetilde{(-1)}$is the greatest exploded number which is smaller than each real number and $\breve{1}$ is the smallest exploded number which is greater than each real number. $\widetilde{(-1)}$ and $\breve{1}$ are called negative and positive discriminators, respectively. Clearly, the exploded number $u(\in \breve{\mathbb{R}})$ is a real number that is $-\infty < u < \infty$ if and only if $-1 < \underline{u} < 1$.

■

If $P = (x, y, z) \in \mathbb{R}^3$ then its exploded is denoted by $\breve{P} = (\breve{x}, \breve{y}, \breve{z})$. Clearly, the mapping $P \to \breve{P}$ is mutually unambiguous. If $\mathcal{P} = (u, v, w) \in \breve{\mathbb{R}^3}$ then its compressed is denoted by $\underline{\mathcal{P}} = (\underline{u}, \underline{v}, \underline{w})$.So, we have the inversion formulas for points

(0.1)

$$\underline{(\breve{P})} = P(\in \mathbb{R}^3) \quad and \quad \breve{(\underline{\mathcal{P}})} = \mathcal{P}(\in \breve{\mathbb{R}^3}).$$

■

In this paper the model of the Multiverse

$$\breve{\mathbb{R}^3} = \left(\mathcal{P} = \breve{P} \middle| P \in \mathbb{R}^3\right)$$

is created by the poinweise explosion of our universe. On the analogy of the euclidean space $\mathbb{R}^3$, for the Multiverse we use the following concepts. Let

$$\gamma, \gamma_1, \gamma_2$$

be exploded numbers and let

$\mathcal{P} = (u, v, w), \mathcal{P}_1 = (u_1, v_1, w_1), \mathcal{P}_2 = (u_2, v_2, w_2)$ ... and so on, be the points of Multiverse. On the analogy of vector – addition we give

$$\mathcal{P}_1 \overline{\oplus} \mathcal{P}_2 = (u_1 \overline{\oplus} u_2, v_1 \overline{\oplus} v_2, w_1 \overline{\oplus} w_2).$$

Using the other form of super- addition we can prove

(0.2) $\mathcal{P}_1 \overline{\oplus} \mathcal{P}_2 = \widetilde{(\underline{\mathcal{P}_1} + \underline{\mathcal{P}_2})}$.

Moreover, the properties

$$\begin{cases} \mathcal{P}_1 \overline{\oplus} \mathcal{P}_2 = \mathcal{P}_2 \overline{\oplus} \mathcal{P}_1 \\ (\mathcal{P}_1 \overline{\oplus} \mathcal{P}_2) \overline{\oplus} \mathcal{P}_3 = \mathcal{P}_1 \overline{\oplus} (\mathcal{P}_2 \overline{\oplus} \mathcal{P}_3) \\ \mathcal{P} \overline{\oplus} \mathcal{O} = \mathcal{P} \\ \mathcal{P} \overline{\oplus} (-\mathcal{P}) = \mathcal{O} \quad , where - \mathcal{P} = (-u, -v, -w) \end{cases},$$

are valid.

In the case of super – subtraction we can write

(0.3) $\mathcal{P}_1 \overline{\ominus} \mathcal{P}_2 = \widetilde{(\underline{\mathcal{P}_1} - \underline{\mathcal{P}_2})}$.

On the analogy of multiplying by scalar $\gamma \in \breve{\mathbb{R}}$, we give

$$\gamma \overline{\odot} \mathcal{P} = (\gamma \overline{\odot} u, \gamma \overline{\odot} v, \gamma \overline{\odot} w).$$

Using the other form of super- multiplication we can prove

(0.4) $\gamma \overline{\odot} \mathcal{P} = \widetilde{(\underline{\gamma} \cdot \underline{\mathcal{P}})}$.

Moreover, the properties

$$\begin{cases} \gamma \overline{\odot} (\mathcal{P}_1 \overline{\oplus} \mathcal{P}_2) = (\gamma \overline{\odot} \mathcal{P}_1) \overline{\oplus} (\gamma \overline{\odot} \mathcal{P}_2) \\ (\gamma_1 \overline{\oplus} \gamma_2) \overline{\odot} \mathcal{P} = (\gamma_1 \overline{\odot} \mathcal{P}) \overline{\oplus} (\gamma_2 \overline{\odot} \mathcal{P}) \\ (\gamma_1 \overline{\odot} \gamma_2) \overline{\odot} \mathcal{P} = \gamma_1 \overline{\odot} (\gamma_2 \overline{\odot} \mathcal{P}) \\ \breve{1} \overline{\odot} \mathcal{P} = \mathcal{P} \end{cases},$$

are valid. We may observe that $-\mathcal{P} = \widetilde{-1} \overline{\odot} \mathcal{P}$.

Although it is a little bit funny, we use the identity

(0.5) $\mathcal{P} \overline{\oslash} \gamma = \widetilde{(\underline{\mathcal{P}} : \underline{\gamma})}$, $\gamma \neq 0$,

too.

On the analogy of the inner product we give

$$\mathcal{P}_1 \overline{\odot} \mathcal{P}_2 = (u_1 \overline{\odot} u_2) \overline{\oplus} (v_1 \overline{\odot} v_2) \overline{\oplus} (, w_1 \overline{\odot} w_2).$$

Using the definition of super addition and the other form of super- multiplication we can prove

(0.6) $\mathcal{P}_1 \overline{\odot} \mathcal{P}_2 = \widetilde{(\underline{\mathcal{P}_1} \cdot \underline{\mathcal{P}_2})}$.

Moreover, we have the familar properties of the traditional inner product

$$\begin{cases} \mathcal{P}_1 \overline{\odot} \mathcal{P}_2 = \mathcal{P}_2 \overline{\odot} \mathcal{P}_1 \\ (\gamma \overline{\odot} \mathcal{P}_1) \overline{\odot} \mathcal{P}_2 = \gamma \overline{\odot} (\mathcal{P}_1 \overline{\odot} \mathcal{P}_2) \\ (\mathcal{P}_1 \overline{\oplus} \mathcal{P}_2) \overline{\odot} \mathcal{P}_3 = (\mathcal{P}_1 \overline{\odot} \mathcal{P}_3) \overline{\oplus} (\mathcal{P}_2 \overline{\odot} \mathcal{P}_3) \\ \mathcal{P} \overline{\odot} \mathcal{P} \geq 0 \; and \; \mathcal{P} \overline{\odot} \mathcal{P} = 0 \Leftrightarrow \mathcal{P} = \mathcal{O} \end{cases} \text{ and}$$

Having that $\|\underline{\mathcal{P}}\| = \sqrt{(\underline{\mathcal{P}} \cdot \underline{\mathcal{P}})}$ for the super – norm we give the definition

(0.7) $]|\mathcal{P}|[= \widetilde{(\|\underline{\mathcal{P}}\|)}$.

We can prove the following propertes

$$]|\mathcal{P}|[\; \geq 0 \; and \;]|\mathcal{P}|[= 0 \Leftrightarrow \mathcal{P} = \mathcal{O} = (0,0,0).$$

$$]|\gamma \overline{\odot} \mathcal{P}|[=]\gamma[\overline{\odot}]|\mathcal{P}|[$$

$$]\mathcal{P}_1 \overline{\odot} \mathcal{P}_2[\leq]|\mathcal{P}_1|[\overline{\odot}]|\mathcal{P}_2|[\quad \text{, (Cauchy - inequality)}$$

$$]|\mathcal{P}_1 \overline{\oplus} \mathcal{P}_2|[\; \leq \;]|\mathcal{P}_1|[\overline{\oplus}]|\mathcal{P}_2|[\quad \text{, (Minkowsky - inequality).}$$

We can say that the Multiverse $\breve{\mathbb{R}^3}$ is a normed (Euclidean) space. If we define the super – distance of the points a Multiverse such as $d_{\breve{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_2) =]|\mathcal{P}_1 \overline{\ominus} \mathcal{P}_2|[$ then (0.3), (0.7) and (0.1) yield

(0.8) $$d_{\breve{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_2) = \left(d\left(\widetilde{\underline{\mathcal{P}_1}, \underline{\mathcal{P}_2}}\right)\right).$$

By (0.8) we can prove the following properties

$$d_{\breve{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_2) \geq 0 \; and \; d_{\breve{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_2) = 0 \Leftrightarrow \mathcal{P}_1 = \mathcal{P}_2,$$
$$d_{\breve{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_2) = d_{\breve{\mathbb{R}^3}}(\mathcal{P}_2, \mathcal{P}_1).$$

Moreover, for any points $\mathcal{P}_1, \mathcal{P}_2 \, and \, \mathcal{P}_3$ of the Multiverse

$d_{\breve{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_3) \leq d_{\breve{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_2) \overline{\oplus} d_{\breve{\mathbb{R}^3}}(\mathcal{P}_2, \mathcal{P}_3)$,

(triangular – inequality)

is valid.■

Considering a set of points of our universe

$$\mathbb{S} = \{P = (x, y, z) | x, y, z \in \mathbb{R}\}$$

the set

$$\breve{\mathbb{S}} = \{\breve{P} | P \in \mathbb{S}\}$$

is called the exploded of $\mathbb{S}$.

So, the explodeds of Euclidean lines and planes are called super – lines and super – planes, respectively. They satisfy the rules of Euclidean geometry.

Of course, the Multiverse is the exploded set of our universe $\mathbb{R}^3$. The Multiverse was already denoted by $\breve{\mathbb{R}^3}$.

Considering a set of points of the Multiverse

$$\mathfrak{M} = \{\mathcal{P} = (u, v, w) | u, v, w \in \breve{\mathbb{R}}\}$$

the set

$$\underline{\mathfrak{M}} = \{\underline{\mathcal{P}} | \mathcal{P} \in \mathfrak{M}\}$$

is called the compressed of $\mathfrak{M}$.The formulas under (0.1) yield the inversion formulas for sets

(0.9)

$$\underline{(\breve{\mathbb{S}})} = \mathbb{S} (\subset \mathbb{R}^3) \quad and \quad \widetilde{(\underline{\mathfrak{M}})} = \mathfrak{M} (\subset \breve{\mathbb{R}^3}).$$

Clearly, the compressed of our universe $\mathbb{R}^3$is the open cube

$$\underline{\mathbb{R}^3} = \left\{P = (x, y, z) | \begin{Bmatrix} -1 < x < 1 \\ -1 < y < 1 \\ -1 < z < 1 \end{Bmatrix}\right\},$$

and

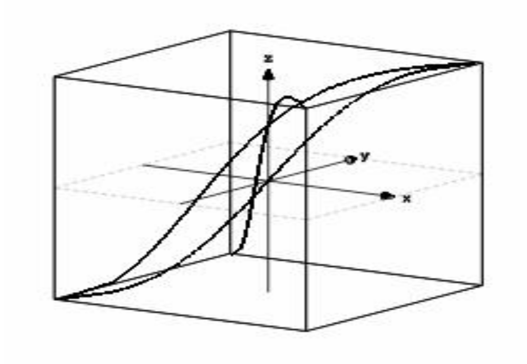

Fig. 0.10

(The Fig.0.10 shows some compressed lines in the compressed universe.)

Exploding $\underline{\mathbb{R}^3}$ we have our universe as a three – dimensional „big open cube".

(0.11)

$$\mathbb{R}^3 = \left\{\mathcal{P} = (u, v, w) | \begin{Bmatrix} \widetilde{(-1)} < u < \breve{1} \\ \widetilde{(-1)} < v < \breve{1} \\ \widetilde{(-1)} < w < \breve{1} \end{Bmatrix}\right\}.$$

1. The characterization of extra – lines

The points $P = (x, y, z)$ of the Euclidean lines$\mathbb{L}_{P_0;E}$ are described by the vector – equation

(1.1) $$P = P_0 + t \cdot E \quad , -\infty < t < \infty ,$$

where $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and $E = (e_x, e_y, e_z) \in \mathbb{R}^3$ are given such that $\|E\| = 1$ or by the set

(1.2) $$\mathbb{L}_{P_0;E} = \Bigg\{(x, y, z) \in$$

ℝ3x=x0+tex y=y0+tey z=z0+tez, −∞<t<∞ ,

Hence, the points $\mathcal{P} = (u, v, w)$ of the super – line $\widetilde{\mathbb{L}_{P_0;E}}$ are described by the equation

(1.3) $$\mathcal{P} = \mathcal{P}_0 \overline{\oplus} (\tau \overline{\odot} \mathcal{E}) \qquad , \tau \in \breve{\mathbb{R}} \quad ,$$

where $\mathcal{P}_0 = \widetilde{P_0}$, $\tau = \breve{t} \; and \; \mathcal{E} = \breve{E}$. Clearly, $]|\mathcal{E}|[= \breve{1}$. So, $\widetilde{\mathbb{L}_{P_0;E}}$ is a set of the Multiverse:

(1.4) $$\widetilde{\mathbb{L}_{P_0;E}} = \Bigg\{(u, v, w) \in$$

ℝ3u=u0⊕τ⊙εxv=v0⊕τ⊙εyw=w0⊕τ⊙εz, τ∈ℝ

such that $u_0 = \breve{x_0}$, $v_0 = \breve{y_0}$, , $w_0 = \breve{z_0}$ and $\varepsilon_x = \widetilde{e_x}$, $\varepsilon_y = \widetilde{e_y}$, $\varepsilon_z = \widetilde{e_z}$.If $\mathcal{P}_1, \mathcal{P}_2$and $\mathcal{P}_3 \in \widetilde{\mathbb{L}_{P_0;E}}$ then we say that $\mathcal{P}_2$ is an intermediate point between the points $\mathcal{P}_1$and $\mathcal{P}_3$ if on the line $\mathbb{L}_{P_0;E}$, the point $\underline{\mathcal{P}_2}$ is situated between $\underline{\mathcal{P}_1}$ and $\underline{\mathcal{P}_1}$

The extra - line $\mathcal{L}_{\mathcal{P}_0,\mathcal{E}}$ is defined by

(1.5) $$\mathcal{L}_{\mathcal{P}_0,\mathcal{E}} = \widetilde{\mathbb{L}_{P_0;E}} \cap \mathbb{R}^3.$$

We remark, that $\mathcal{L}_{\mathcal{P}_0,\mathcal{E}}$ is an open super passage of the super – line $\widetilde{\mathbb{L}_{P_0;E}}$ and the extra - line is visible in our universe. Clearly, $\mathcal{L}_{\mathcal{P}_0,\mathcal{E}}$ is coincident with $\mathcal{L}_{\mathcal{L}_0^*,\mathcal{E}^*}$ if and only if $\mathbb{L}_{P_0;E} \cong \mathbb{L}_{\mathcal{L}_0^*,\mathcal{E}^*}$. By (0.9) we can write that$\mathcal{L}_{\mathcal{P}_0,\mathcal{E}} \cong \mathcal{L}_{\mathcal{L}_0^*,\mathcal{E}^*} \Leftrightarrow \underline{\mathcal{L}_{\mathcal{P}_0,\mathcal{E}}} \cong \underline{\mathcal{L}_{\mathcal{L}_0^*,\mathcal{E}^*}}$.

Example 1.6. Let be $P_0 = \mathcal{O} = (0,0,0)$ and $E = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$. By (1.3) we have that $\widetilde{\mathbb{L}_{\mathcal{O};E}}$ is decribed by the equation $\mathcal{P} = \tau \overline{\odot} \mathcal{E}$, where $\tau \in \breve{\mathbb{R}}$ and $\mathcal{E} = \left(\widetilde{\left(\frac{1}{\sqrt{6}}\right)}, \widetilde{\left(\frac{1}{\sqrt{6}}\right)}, \widetilde{\left(\frac{2}{\sqrt{6}}\right)}\right)$. Considering, (1.4) , (1.5)and (0.11) the inequalities

(1.7) $$\begin{cases} \widetilde{(-1)} < \tau \overline{\odot} \widetilde{\left(\frac{1}{\sqrt{6}}\right)} < \breve{1} \\ \widetilde{(-1)} < \tau \overline{\odot} \widetilde{\left(\frac{1}{\sqrt{6}}\right)} < \breve{1} \\ \widetilde{(-1)} < \tau \overline{\odot} \widetilde{\left(\frac{2}{\sqrt{6}}\right)} < \breve{1} \end{cases}$$

are required for $\mathcal{L}_{\mathcal{O},\mathcal{E}}$. If $\widetilde{\left(-\frac{\sqrt{6}}{2}\right)} < \tau < \widetilde{\left(\frac{\sqrt{6}}{2}\right)}$ then the requirements fulfill.

Finally, using (1.2) by (1.4)

(1.8) $$\mathcal{L}_{\mathcal{O},\mathcal{E}} = \left\{ (u,v,w) \in \right.$$

$\mathbb{R}3u$=tanh$-1t6v$=tanh$-1t6w$=tanh$-12t6$, ,
where $-\frac{\sqrt{6}}{2} < t < \frac{\sqrt{6}}{2}$

is obtained.

The graph of the extra – line $\mathcal{L}_{\mathcal{O},\mathcal{E}}$

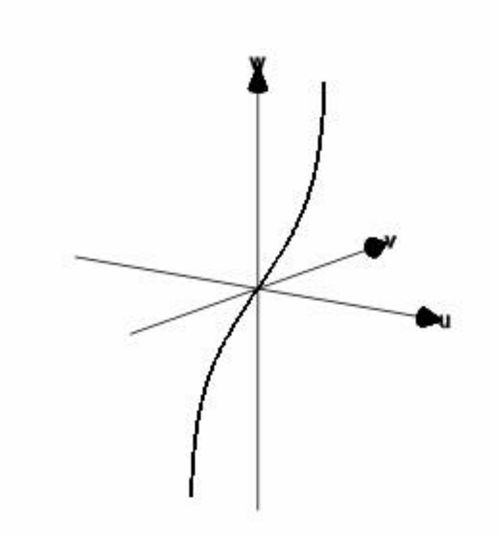

Fig. 1.9

Fig. 1.9 shows that the extra- line is not an Euclidean line, in general.■

As the our universe is subset of Multiverse we are able to close $\mathbb{R}^3$. Namely,

$$\overline{\mathbb{R}^3} = \left\{ P = (u,v,w) \in \breve{\mathbb{R}^3} \middle| \begin{cases} \widetilde{(-1)} \le u \le \breve{1} \\ \widetilde{(-1)} \le v \le \breve{1} \\ \widetilde{(-1)} \le w \le \breve{1} \end{cases} \right\}.$$

So, our universe is bordered by six super – planes, decribed by the equations

$Lower\ border: w = \widetilde{(-1)}$, $Upper\ border: w = \breve{1}$,
$Before\ border: v = \widetilde{(-1)}$, $Back\ border: v = \breve{1}$,
and
$Left\ border: u = \widetilde{(-1)}$, $Right\ border: u = \breve{1}$.

Of course, the border of $\overline{\mathbb{R}^3}$ is invisible from our universe. Considering (1.4), $P_0 \in \mathbb{R}^3$ yields that the super – line $\widetilde{\mathbb{L}_{P_0;E}}$ has two joint points with the border of $\overline{\mathbb{R}^3}$. These points are called the border – ponts of extra – line $\mathcal{L}_{\mathcal{P}_0,\mathcal{E}}$ (see (1.5)) and denoted by $\mathcal{B}_1(\mathcal{L}_{\mathcal{P}_0,\mathcal{E}})$ and $\mathcal{B}_2(\mathcal{L}_{\mathcal{P}_0,\mathcal{E}})$.

Theorem 1.10. If $\mathcal{B}_1 = (u_1, v_1, w_1)$ and $\mathcal{B}_2 = (u_2, v_2, w_2)$ are different points on the border of $\overline{\mathbb{R}^3}$ and they do not situate on the same super – plane then the points $\mathcal{P} = (u,v,w) \in \breve{\mathbb{R}^3}$ described by the equation

(1.11)
$$\mathcal{P} = \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2)\right) \overline{\oplus} \left(\tau \overline{\odot} \left((\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2) \overline{\oslash} \,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[\right)\right), \quad \tau \in \breve{\mathbb{R}},$$

form a super – line which contains $\mathcal{B}_1$and $\mathcal{B}_2$,such that

(1.12) $$\left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2)\right) \in \mathbb{R}^3$$

fulfills. Moreover, for the parameter – domain

(1.13) $$\widetilde{\left(-\frac{1}{2}\right)} \overline{\odot}]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[< \tau < \widetilde{\left(\frac{1}{2}\right)} \overline{\odot}]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[$$

the points of super – line give an extra – line with the border – points $\mathcal{B}_1$and $\mathcal{B}_2$.

Proof. As $\left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2)\right) \in \breve{\mathbb{R}^3}$it is obvious that $P_0 = \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2)\right) \in \mathbb{R}^3$ is valid. Similarly, $E = \overline{\left((\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2) \overline{\oslash} \,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[\right)} \in \mathbb{R}^3$, where $]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[\ne 0$,because$\mathcal{B}_1 \ne \mathcal{B}_2$.

Using (0.1) , (0.3) , (0.5) and (0.7)

$$(\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2) \overline{\oslash} \,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[= \left(\widetilde{\underline{\mathcal{B}_1} - \underline{\mathcal{B}_2}}\right) \overline{\oslash} \left(\left\|\widetilde{\underline{\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2}}\right\|\right) =$$

$$= \left(\widetilde{\underline{\mathcal{B}_1} - \underline{\mathcal{B}_2}}\right) \overline{\oslash} \left\|\widetilde{\underline{\mathcal{B}_1} - \underline{\mathcal{B}_2}}\right\| = \widetilde{\left(\left(\underline{\mathcal{B}_1} - \underline{\mathcal{B}_2}\right) : \left\|\underline{\mathcal{B}_1} - \underline{\mathcal{B}_2}\right\|\right)}.$$

By the first inversion formulas for points (see (0.1)) $E = \frac{\underline{\mathcal{B}_1}-\underline{\mathcal{B}_2}}{\left\|\underline{\mathcal{B}_1}-\underline{\mathcal{B}_2}\right\|}$ is obtained. Hence, $\|E\|$=1.

Considering (1.1) and (1.3) with $\mathcal{P}_0 = \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2)$ and $\mathcal{E} = (\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2) \overline{\oslash} \,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[$

we have that (1.11) determines a super – line. For the parameters $\tau = \widetilde{\left(-\frac{1}{2}\right)} \overline{\odot}]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[$

and $\tau = \widetilde{\left(\frac{1}{2}\right)} \overline{\odot}]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[$ (1.11) yields the points

$$\mathcal{P} = \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2)\right) \overline{\oplus} \left(\widetilde{\left(-\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2)\right) = \mathcal{B}_2$$

and

$$\mathcal{P} = \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2)\right) \overline{\oplus} \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2)\right) = \mathcal{B}_1,$$

respectively.

Because the border – points $\mathbb{B}_1$and $\mathbb{B}_2$do not situate on the same super – plane, we have

$$\widetilde{(-2)} < u_1 \overline{\oplus} u_2 < \check{2}\,;\ \widetilde{(-2)} < v_1 \overline{\oplus} v_2 < \check{2}\,;\ \widetilde{(-2)} < w_1 \overline{\oplus} w_2 < \check{2}$$

and so,

$$\begin{cases} \widetilde{(-1)} < \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (u_1 \overline{\oplus} u_2) < \check{1} \\ \widetilde{(-1)} < \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (v_1 \overline{\oplus} v_2) < \check{1} \\ \widetilde{(-1)} < \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (w_1 \overline{\oplus} w) < \check{1} \end{cases}$$

by (0.10) we can see, that (1.12) is fulfilled.

Considering the points $\mathcal{P} = (u, v, w)$ of super – plane, the vector – equation (1.11) is equivalent with the equation system

(1.14)

$$\begin{cases} u = \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (u_1 \overline{\oplus} u_2)\right) \overline{\oplus} \left(\tau \overline{\odot} \left((u_1 \overline{\ominus} u_2) \overline{\oslash} \,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[\right)\right) \\ v = \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (v_1 \overline{\oplus} v_2)\right) \overline{\oplus} \left(\tau \overline{\odot} \left((v_1 \overline{\ominus} v_2) \overline{\oslash} \,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[\right)\right) \\ w = \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (w_1 \overline{\oplus} w_2)\right) \overline{\oplus} \left(\tau \overline{\odot} \left((w_1 \overline{\ominus} w_2) \overline{\oslash} \,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[\right)\right) \end{cases}$$

, $\tau \in \check{\mathbb{R}}$.

If $u_1 = u_2$ then $u = \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (u_1 \overline{\oplus} u_2)$. If $v_1 = v_2$ then $v = \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (v_1 \overline{\oplus} v_2)$. If $w_1 = w_2$ then $w = \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (w_1 \overline{\oplus} w_2)$.

As $\mathcal{B}_1 \neq \mathcal{B}_2$,it is necessary that one among the inequations$u_1 \neq u_2$, $v_1 \neq v_2, w_1 \neq w_2$ is fulfilled, at least. Assuming that $u_1 > u_2$.by (1.13)

$$\left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (u_1 \overline{\oplus} u_2)\right) \overline{\oplus} \left(\widetilde{\left(-\frac{1}{2}\right)} \overline{\odot} (u_1 \overline{\ominus} u_2)\right) < u < \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (u_1 \overline{\oplus} u_2)\right) \overline{\oplus} \left(\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (u_1 \overline{\ominus} u_2)\right)$$

that is$u_2 < u < u_1$is obtained. (The assuption $u_1 < u_2$ results $u_1 < u < u_2$.) A similar argumentation used in the case of the other two inequations.

Finally, by (1.14) we have

$$u = \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (u_1 \overline{\oplus} u_2) \ or\ \widetilde{(-1)} \leq min(u_1, u_2) < u < max(u_1, u_2) \leq \check{1}$$

$$v = \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (v_1 \overline{\oplus} v_2) \ or\ \widetilde{(-1)} \leq min(v_1, v_2) < v < max(v, v_2) \leq \check{1}$$

$$w = \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (w_1 \overline{\oplus} w_2) \ or\ \widetilde{(-1)} \leq min(w_1, w_2) < w < max(w_1, w_2) \leq \check{1}.$$

Hemce, we can see that for the parameter – domain

$$\widetilde{\left(-\frac{1}{2}\right)} \overline{\odot}]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[< \tau < \widetilde{\left(\frac{1}{2}\right)} \overline{\odot}]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[$$

the super – passage of the super – line is an extra – line with the border – points

$\mathcal{B}_1 = (u_1, v_1, w_1)$and .$\mathcal{B}_2 = (u_2, v_2, w_2)$.■

Example 1.15. Let us discover the extra line determined by the border points

$\mathcal{B}_1 = \left(\widetilde{\left(-\frac{1}{2}\right)}, \widetilde{\left(-\frac{1}{2}\right)}, \widetilde{(-1)}\right)$and .$\mathcal{B}_2 = \left(\widetilde{\left(\frac{1}{2}\right)}, \widetilde{\left(\frac{1}{2}\right)}, \check{1}\right)$.

Solution.

We apply Theorem 1.10. Now,

$\widetilde{\left(\frac{1}{2}\right)} \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2) = \mathcal{O} = (0,0,0)$; $(\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2) = \left(\widetilde{(-1)}, \widetilde{(-1)}, \widetilde{(-2)}\right)$; $]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[= \widetilde{(\sqrt{6})}$.

$$(\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2) \overline{\oslash} \,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[= \left(\widetilde{\left(-\frac{1}{\sqrt{6}}\right)}, \widetilde{\left(-\frac{1}{\sqrt{6}}\right)}, \widetilde{\left(-\frac{2}{\sqrt{6}}\right)}\right);$$

$\widetilde{\left(-\frac{1}{2}\right)}\overline{\odot}]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[= \widetilde{\left(-\frac{\sqrt{6}}{2}\right)}$; $\widetilde{\left(\frac{1}{2}\right)}\overline{\odot}]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[= \widetilde{\left(\frac{\sqrt{6}}{2}\right)}$.

So, by (1.11) and (1.13) we have

(1.16)
$\mathcal{P} = \left(\widetilde{\left(-\frac{1}{\sqrt{6}}\right)}\overline{\odot}\tau, \widetilde{\left(-\frac{1}{\sqrt{6}}\right)}\overline{\odot}\tau, \widetilde{\left(-\frac{2}{\sqrt{6}}\right)}\overline{\odot}\tau\right)$, $\widetilde{\left(-\frac{\sqrt{6}}{2}\right)} < \tau < \widetilde{\left(\frac{\sqrt{6}}{2}\right)}$.

The vector equation (1.16) gives the equation system

(1.17) $\begin{cases} u = \widetilde{\left(-\frac{1}{\sqrt{6}}\right)}\overline{\odot}\tau = -\tanh^{-1}\frac{\tanh \tau}{\sqrt{6}} \\ v = \widetilde{\left(-\frac{1}{\sqrt{6}}\right)}\overline{\odot}\tau = -\tanh^{-1}\frac{\tanh \tau}{\sqrt{6}} \\ w = \widetilde{\left(-\frac{2}{\sqrt{6}}\right)}\overline{\odot}\tau = -\tanh^{-1}\frac{2\tanh \tau}{\sqrt{6}} \end{cases}$, $-\frac{\sqrt{6}}{2} < \tanh\tau < \frac{\sqrt{6}}{2}$.

Comparing (1.8) and (1.17) we can observe that our extra line is $\mathcal{L}_{\mathcal{O},E}$ with contrasted direction. The border point $\mathcal{B}_1$ with $\tau = \widetilde{\left(\frac{\sqrt{6}}{2}\right)}$. is situated on the lower- border, while the border point $\mathcal{B}_2$ with $\tau = \widetilde{\left(-\frac{\sqrt{6}}{2}\right)}$. is situated on the upper – border. Moreover, Fig. 1.9. shows the graph of the extra – line discovered now. ■

Remark 1.18. We return to the Example 1.6, where the extra – line $\mathcal{L}_{\mathcal{O},\mathcal{E}}$is determined by the point $\mathcal{O}$ and the „vector" $\mathcal{E} = \left(\widetilde{\left(\frac{1}{\sqrt{6}}\right)}, \widetilde{\left(\frac{1}{\sqrt{6}}\right)}, \widetilde{\left(\frac{2}{\sqrt{6}}\right)}\right)$, has the description (1.8). Considering

$\lim_{\substack{t\to -\frac{\sqrt{6}}{2} \\ t>-\frac{\sqrt{6}}{2}}} u(t) = -\tanh^{-1}\frac{1}{2}$,$\lim_{\substack{t\to -\frac{\sqrt{6}}{2} \\ t>-\frac{\sqrt{6}}{2}}} v(t) =$ −tanh−112 and lim𝑡⟶−62𝑡>−62𝑤𝑡=−∞,

the (invisible) border – point $\mathcal{B}_1(\mathcal{L}_{\mathcal{O},\mathcal{E}}) = \left(\widetilde{\left(-\frac{1}{2}\right)}, \widetilde{\left(-\frac{1}{2}\right)}, \widetilde{(-1)}\right)$ is obtained. Moreover, by

$\lim_{\substack{t\to \frac{\sqrt{6}}{2} \\ t<\frac{\sqrt{6}}{2}}} u(t) = \tanh^{-1}\frac{1}{2}$,$\lim_{\substack{t\to \frac{\sqrt{6}}{2} \\ t<-\frac{\sqrt{6}}{2}}} v(t) = \tanh^{-1}\frac{1}{2}$

and $\lim_{\substack{t\to -\frac{\sqrt{6}}{2} \\ t<-\frac{\sqrt{6}}{2}}} w(t) = \infty$,

we have (invisible) border – point $\mathcal{B}_2(\mathcal{L}_{\mathcal{O},\mathcal{E}}) = \left(\widetilde{\left(\frac{1}{2}\right)}, \widetilde{\left(\frac{1}{2}\right)}, \check{1}\right)$. ■

Nomination 1.19. If the extra – line $\mathcal{L}_{\mathcal{P}_o,\mathcal{E}}$ with $\mathcal{P}_o = \widetilde{\left(\frac{1}{2}\right)}\overline{\odot}(\mathcal{B}_1\overline{\oplus}\mathcal{B}_2)$ and $\mathcal{E} = (\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2) \overline{\oslash}\,]|\mathcal{B}_1 \overline{\ominus} \mathcal{B}_2|[$ is given by its border points $\mathcal{B}_1(\mathcal{L}_{\mathcal{P}_0,\mathcal{E}})$ and $\mathcal{B}_2(\mathcal{L}_{\mathcal{P}_0,\mathcal{E}})$ we write $\mathcal{L}_{\widetilde{\left(\frac{1}{2}\right)}\overline{\odot}(\mathcal{B}_1\overline{\oplus}\mathcal{B}_2),,(\mathcal{B}_1\overline{\ominus}\mathcal{B}_2)\overline{\oslash}]|\mathcal{B}_1\overline{\ominus}\mathcal{B}_2|[} = \mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[)$.

II. THEOREMS FOR EXTRA - COLLINEARITY

Two or more points are said to be extra–collinear (super – collinear) if there is an extra –line (super – line) that contains them.

If $\mathfrak{M}$ is a subset of the Multiverse $\widetilde{\mathbb{R}^3}$ then

(2.1) $\mathfrak{M}_{box} = \mathfrak{M} \cap \mathbb{R}^3$.

is called the box- phenomenon of $\mathfrak{M}$. Clearly, $\widetilde{\mathbb{R}^3}_{box} = \mathbb{R}^2$

Theorem 2.2. If $\mathcal{P}_1 = (u_1, v_1, w_1)$ and $\mathcal{P}_2 = (u_2, v_2, w_2)$ are different points of the three – dimensional space $\mathbb{R}^3$ then the box- phenomenon of the set, having the points given by the equation

(2.3)
$\mathcal{P} = \left(\widetilde{\left(\frac{1}{2}\right)}\overline{\odot}(\mathcal{P}_1\overline{\oplus}\mathcal{P}_2)\right)\overline{\oplus}\left(\tau\overline{\odot}\left((\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1) \overline{\oslash}\,]|\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1|[\right)\right)$, $\tau \in \widetilde{\mathbb{R}}$,

is an extra – line which contains the points $\mathcal{P}_1$ and $\mathcal{P}_2$. Shortly:$\mathcal{P}_1$ and $\mathcal{P}_2$ are extra–collinear.

Proof. Choosing $P_0 = \underline{\widetilde{\left(\frac{1}{2}\right)}\overline{\odot}(\mathcal{P}_1\overline{\oplus}\mathcal{P}_2)} = \frac{\mathcal{P}_1+\mathcal{P}_2}{2}$ (see (0.1), (0.2) and (0.4))

and $E = \underline{(\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1) \overline{\oslash}\,]|\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1|[}$, (where $]|\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1|[\neq 0$ because $\mathcal{P}_2 \neq \mathcal{P}_1$)

we consider the Euclidean line $\mathbb{L}_{P_0,E}$under (1.1)or under (1.2), where $t = \underline{\tau} \in \mathbb{R}$. It is easy to verify that by the parameters$\tau_1 = \widetilde{\left(-\frac{1}{2}\right)}\overline{\odot}]|\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1|[$ *and* $\tau_2 = \widetilde{\left(\frac{1}{2}\right)}\overline{\odot}]|\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1|[$the (2.3) gives the points $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.Hence, by (0.1),(0.3) (0.4) and (0.7) we can prove, that the ponts $\underline{\mathcal{P}_1}$and $\underline{\mathcal{P}_2}$ (see the parameters

$t_1 = \underline{\widetilde{\left(-\frac{1}{2}\right)}\overline{\odot}]|\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1|[} = -\frac{\|\mathcal{P}_2-\mathcal{P}_1\|}{2}$ and $t_2 = \underline{\widetilde{\left(\frac{1}{2}\right)}\overline{\odot}]|\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1|[} = \frac{\|\mathcal{P}_2-\mathcal{P}_1\|}{2}$,

respectively) situated on the line $\mathbb{L}_{P_0,E}$. By the axioms of Euclidean geometry the ponts $\underline{\mathcal{P}_1}$and $\underline{\mathcal{P}_2}$ determine one and only one line. Exploding this line we have that the points given by the equation (2.3) form a super – line $\widetilde{\mathbb{L}_{P_0,E}}$.Finally, by (1.5) and (2.1) we get that the requested extra- line is$\left(\widetilde{\mathbb{L}_{P_0,E}}\right)_{box}$.

Example 2.4. Let us discover the extra - line determined by the points

$\mathcal{P}_1 = \mathcal{O} = (0,0,0)$ and $\mathcal{P}_2 = (1,1,2)$.

Solution.

We apply Theorem 1.10. Now, considering (2.3) with $P_0 = \left(\frac{\tanh 1}{2}, \frac{\tanh 1}{2}, \frac{\tanh 2}{2}\right)$ we have:

$\overbrace{\left(\frac{1}{2}\right)}\overline{\odot}(\mathcal{P}_1\overline{\oplus}\mathcal{P}_2)$
$= \left(\tanh^{-1}\frac{\tanh 1}{2}, \tanh^{-1}\frac{\tanh 1}{2}, \tanh^{-1}\frac{\tanh 2}{2}\right)$; $\mathcal{P}_2\,\overline{\ominus}\,\mathcal{I}$
$= (1,1,2).$
$]|\mathcal{P}_2\,\overline{\ominus}\,\mathcal{P}_1|[$
$= \overbrace{\left(\sqrt{2(\tanh 1)^2 + (\tanh 2)^2}\right)}$; $(\mathcal{P}_2\,\overline{\ominus}\,\mathcal{P}_1)\,\overline{\oslash}\,]|\mathcal{P}_2\,\overline{\ominus}\,\mathcal{P}_1|[$
$= \mathcal{P}_2 \oslash\,]|\mathcal{P}_2|[\, =$

$=$
$\left(\tanh^{-1}\frac{\tanh\ 1}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}, \tanh^{-1}\frac{\tanh\ 1}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}, \text{tanl}\right.$

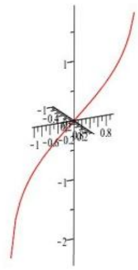

Fig.2.5*

Considering the points $\mathcal{P} = (u, v, w)$ of super – line $\widetilde{\mathbb{L}}_{P_0,E}$, the vector – equation (2.3) is equivalent with the equation system

$$\begin{cases} u = \left(\tanh^{-1}\frac{\tanh\ 1}{2}\right)\overline{\oplus}\left(\tau\overline{\odot}\tanh^{-1}\frac{\tanh\ 1}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right) \\ v = \left(\tanh^{-1}\frac{\tanh\ 1}{2}\right)\overline{\oplus}\left(\tau\overline{\odot}\tanh^{-1}\frac{\tanh\ 1}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right) \\ w = \left(\tanh^{-1}\frac{\tanh\ 2}{2}\right)\overline{\oplus}\left(\tau\overline{\odot}\tanh^{-1}\frac{\tanh\ 2}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right) \end{cases}$$

$,\ \ \tau \in \breve{\mathbb{R}},$

or denoting $t = \underline{\tau}$

$$\begin{cases} u = \overbrace{\left(\frac{\tanh\ 1}{2} + \frac{t\cdot\tanh\ 1}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right)} \\ v = \overbrace{\left(\frac{\tanh\ 1}{2} + \frac{t\cdot\tanh\ 1}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right)} \\ w = \overbrace{\left(\frac{\tanh\ 2}{2} + \frac{t\cdot\tanh\ 2}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right)} \end{cases} ,\quad t \in \mathbb{R}.$$

Finally, the points of the requested extra –line is described by the equation – system

(2.5)

$$\begin{cases} u = \tanh^{-1}\left(\frac{\tanh\ 1}{2} + \frac{t\cdot\tanh\ 1}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right) \\ v = \tanh^{-1}\left(\frac{\tanh\ 1}{2} + \frac{t\cdot\tanh\ 1}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right) \\ w = \tanh^{-1}\left(\frac{\tanh\ 2}{2} + \frac{t\cdot\tanh\ 2}{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}\right) \end{cases} ,$$

where

$$-\left(\frac{1}{\tanh 2} + \frac{1}{2}\right)\sqrt{2(\tanh 1)^2 + (\tanh 2)^2} < t < \left(\frac{1}{\tanh 2} - \frac{1}{2}\right)\sqrt{2(\tanh 1)^2 + (\tanh 2)^2}.$$

For the parameter $t_1 = -\frac{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}{2}$ the equation – system (2.5) yields the point $\mathcal{P}_1 = \mathcal{O}$ and for the parameter $t_2 = \frac{\sqrt{2(\tanh\ 1)^2+(\tanh\ 2)^2}}{2}$ the point $\mathcal{P}_2 = (1,1,2)$ is obtained.■

Remark 2.6. Despite of that both the extra – line described by the equation – system (2.5) and the Euclidean line described by the equation – system

(2.7) $$\begin{cases} u = \frac{s}{\sqrt{6}} \\ v = \frac{s}{\sqrt{6}} \\ w = \frac{2s}{\sqrt{6}} \end{cases} \quad -$$

$\infty < s < \infty$

contain the points $\mathcal{P}_1 = \mathcal{O}$ and $\mathcal{P}_2 = (1,1,2)$ (for the latter see the parameters $s_1 = 0$ and $s_2 = \sqrt{6}$, respectively) they are essentially different. To prove it is sufficient to mention that the extra – line has the border points

$\left(-\tanh^{-1}\frac{\tanh 1}{\tanh 2}, -\tanh^{-1}\frac{\tanh 1}{\tanh 2}, \widetilde{(-1)}\right)$ *and* $\left(\tanh^{-1}\frac{\tanh 1}{\tanh 2}, \tanh^{-1}\frac{\tanh 1}{\tanh 2}, \widetilde{1}\right)$,

while (2.7) yields that

$\lim_{s\to-\infty} u = -\infty$, $\lim_{s\to-\infty} v = -\infty$, $\lim_{s\to-\infty} w = -\infty$ and $\lim_{s\to\infty} v = \infty$, $\lim_{s\to\infty} w = \infty$.

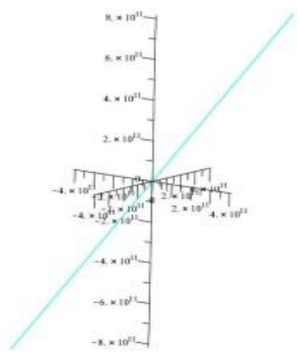

Fig. 2.7

The following two theorems are a simple consequence of Theorems 1.10 and 2.2.

Theorem 2.8. Every extra – line containss at least two distinc points situated in the closed three – dimensional space $\overline{\mathbb{R}^3}$.

Theorem 2.9. If the following cases, two extra – lines each contain

(i) the same two distinc points $\mathcal{P}_1$ and $\mathcal{P}_2$in the universe $\mathbb{R}^3$,

(ii) the same two distinc points, one of them$\mathcal{P}_*$in the universe $\mathbb{R}^3$, and the second one $\mathcal{B}_*$in the border of$\overline{\mathbb{R}^3}$,

(iii) the same two distinc points $\mathcal{B}_1$ and $\mathcal{B}_2$on the border of $\overline{\mathbb{R}^3}$ such that they do not situate on the same super – plane,

then the two extra – lines are equal.

Proof. Choosing in the cases

(i) $\mathcal{P}_0 = \mathcal{P}_1 \in \mathbb{R}^3$ and $\mathcal{E} = (\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1) \overline{\oslash} \,]|\mathcal{P}_2 \overline{\ominus} \mathcal{P}_1|[$,

(ii) $\mathcal{P}_0 = \mathcal{P}_* \in \mathbb{R}^3$ and $\mathcal{E} = (\mathcal{B}_* \overline{\ominus} \mathcal{P}_*) \overline{\oslash} \,]|\mathcal{B}_* \overline{\ominus} \mathcal{P}_*|[$,

(iii) $\mathcal{P}_0 = \left(\left(\widetilde{\frac{1}{2}}\right) \overline{\odot} (\mathcal{B}_1 \overline{\oplus} \mathcal{B}_2)\right) \in \mathbb{R}^3$ and$\mathcal{E} = (\mathcal{B}_2 \overline{\ominus} \mathcal{B}_1) \overline{\oslash} \,]|\mathcal{B}_2 \overline{\ominus} \mathcal{B}_1|[$

and denoting $P_0 = \underline{\mathcal{P}_0}$and $E = \underline{\mathcal{E}}$the super – line

$$\widetilde{\mathbb{L}_{P_0;E}} = \{\mathcal{P} \in \overline{\mathbb{R}^3} | \mathcal{P} = \mathcal{P}_0 \overline{\oplus} (\tau \overline{\odot} \mathcal{E}), \tau \in \widetilde{\mathbb{R}}\}$$

is obtained.

Let us consider the the cases (i) – (iii). The extra – line $\mathcal{L}_{\mathcal{P},\mathcal{E}} = \left(\widetilde{\mathbb{L}_{P_0;E}}\right)_{box}$ contains the point pairs $\mathcal{P}_1$ and $\mathcal{P}_2$; $\mathcal{P}_*$and $\mathcal{B}_*$; $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Denoting $P = \underline{\mathcal{P}}$ *and* $t = \underline{\tau}$ we have the line

$$\mathbb{L}_{P_0;E} = \{P \in \mathbb{R}^3 | \mathcal{P} = P_0 + (t \cdot E), t \in \mathbb{R}.\}$$

which contains the pairs of distinc points$\underline{\mathcal{P}_1}$ and $\underline{\mathcal{P}_2}$;$\underline{\mathcal{P}_*}$and $\underline{\mathcal{B}_*}$; $\underline{\mathcal{B}_1}$ and $\underline{\mathcal{B}_2}$. Having the Euclidean axiom „If two lines each contain the same two distinc points, then the two lines are equal." we have that $\mathbb{L}_{P_0;E}$ is unambiguously determined. Hence $\widetilde{\mathbb{L}_{P_0;E}}$ is unambiguously determined, too. Finally, having by (2.1) we have that $\mathcal{L}_{\mathcal{P},\mathcal{E}}$ is unambiguously determined. ∎

Theorem 2.10. Let be $\mathcal{P}_1, \mathcal{P}_2$and $\mathcal{P}_3$ pairwise different points of Multiverse.The equality

$$d_{\widetilde{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_3) = d_{\widetilde{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_2) \overline{\oplus} d_{\widetilde{\mathbb{R}^3}}(\mathcal{P}_2, \mathcal{P}_3)$$

fufills if and only if $\mathcal{P}_1, \mathcal{P}_2$and $\mathcal{P}_3$ are super – collinear and $\mathcal{P}_2$ is an intermediate point between the points $\mathcal{P}_1$and $\mathcal{P}_3$

Proof. Let us consider the points $\underline{\mathcal{P}_1}$,$\underline{\mathcal{P}_2}$ and $\underline{\mathcal{P}_3}$. They are pairwise different points of our universe. By the Euclidean geometry we have that the eqaulity

$$d_{\mathbb{R}^3}\left(\underline{\mathcal{P}_1}, \underline{\mathcal{P}_3}\right) = d_{\mathbb{R}^3}\left(\underline{\mathcal{P}_1}, \underline{\mathcal{P}_2}\right) + d_{\mathbb{R}^3}\left(\underline{\mathcal{P}_2}, \underline{\mathcal{P}_3}\right)$$

fufills if and only if $\underline{\mathcal{P}_1}$,$\underline{\mathcal{P}_2}$ and $\underline{\mathcal{P}_3}$. are collinear and $\underline{\mathcal{P}_2}$ is between $\underline{\mathcal{P}_1}$ and $\underline{\mathcal{P}_3}$ .So, having (0.8)

$$d_{\widetilde{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_3) = \left(\widetilde{d_{\mathbb{R}^3}\left(\underline{\mathcal{P}_1}, \underline{\mathcal{P}_2}\right) + d_{\mathbb{R}^3}\left(\underline{\mathcal{P}_2}, \underline{\mathcal{P}_3}\right)}\right) = \left(\widetilde{d_{\mathbb{R}^3}\left(\underline{\mathcal{P}_1}, \underline{\mathcal{P}_2}\right)}\right) \overline{\oplus} \left(\widetilde{d_{\mathbb{R}^3}\left(\underline{\mathcal{P}_2}, \underline{\mathcal{P}_3}\right)}\right)$$

proves our statement.

III. EXTRA PARALLELITY.

Theorem 3.1. If $\mathcal{P}_0 \in \mathbb{R}^3$ and $\mathcal{B}$ is a point situated on the border of $\overline{\mathbb{R}^3}$ then the set

(3.2) $\mathcal{L}([\mathcal{P}_0, \mathcal{B}[) = \{\mathcal{P} \in \mathbb{R}^3 | \mathcal{P} =$ *𝒫0⊕τ⊙ℬ⊖𝒫0⊘ℬ⊖𝒫0,*

where $0 \le \tau < \,]|\mathcal{B} \overline{\ominus} \mathcal{P}_0|[$, form a half extra – line with the border point $\mathcal{B}$.

Proof. Let be $P_0 = \underline{\mathcal{P}_0}$ and$E = \underline{(\mathcal{B} \overline{\ominus} \mathcal{P}_0) \overline{\oslash} \,]|\mathcal{B} \overline{\ominus} \mathcal{P}_0|[}$.Because of (3.2) we consider with $\underline{\tau} = t$, the vector equation

(3.3) $$\underline{\mathcal{P}} = P_0 + \frac{\underline{\mathcal{B}} - \underline{\mathcal{P}_0}}{\left\|\underline{\mathcal{B}} - \underline{\mathcal{P}_0}\right\|} \cdot t \quad , \quad 0 \le t < \left\|\underline{\mathcal{B}} - \underline{\mathcal{P}_0}\right\|.$$

Using (0.1) , (0.3) , (0.5) and (0.7) we can write

$$(\mathcal{B} \overline{\ominus} \mathcal{P}_0) \overline{\oslash} \,]|\mathcal{B} \overline{\ominus} \mathcal{P}_0|[= \left(\widetilde{\underline{\mathcal{B}} - \underline{\mathcal{P}_0}}\right) \overline{\oslash} \left(\widetilde{\left\|\underline{\mathcal{B} \overline{\ominus} \mathcal{P}_0}\right\|}\right) = \left(\widetilde{\underline{\mathcal{B}} - \underline{\mathcal{P}_0}}\right) \overline{\oslash} \widetilde{\left\|\underline{\mathcal{B}} - \underline{\mathcal{P}_0}\right\|} = \left(\widetilde{\left(\underline{\mathcal{B}} - \underline{\mathcal{P}_0}\right) : \left\|\underline{\mathcal{B}} - \underline{\mathcal{P}_0}\right\|}\right).$$

By the first inversion formulas for points (see (0.1)),$E = \frac{\underline{\mathcal{B}} - \underline{\mathcal{P}_0}}{\left\|\underline{\mathcal{B}} - \underline{\mathcal{P}_0}\right\|}$ is obtained. Hence, $\|E\|$=1.The points $\underline{\mathcal{P}}$ given under (3.3) forms a passage (closed from the left and open from the right) situared in $\underline{\mathbb{R}^3}$.

By the explosion the equation (3.3) gives that the points under (3.2) form a half extra – line, with the start – point $\mathcal{P}_0$. For the parameter $\tau =]|\mathcal{B} \overline{\ominus} \mathcal{P}_0|[$ the border point $\mathcal{B}$ is obtained. ■

Using the right hand part of (0.1) ,(0.2) , (0.4) and (0.7) by (3.2) the description

(3.4)$\mathcal{L}([\mathcal{P}_0, \mathcal{B}[) =$

$\left\{ (u, v, w) \in \right.$

ℝ3u=tanh−1u0+u0−uℬ·tℬ−𝒫0v=tanh−1v0+v0−vℬ·tℬ−𝒫0w=tanh−1w0+w0−wℬ·tℬ−𝒫0,0≤t<ℬ−𝒫0,

where $\mathcal{P}_0 = (u_0, v_0, w_0)$,$\mathcal{B} = (u_\mathcal{B}, v_\mathcal{B}, w_\mathcal{B})$ and $t = \underline{\tau}$, is obtained.

Theorem 3.5. If $\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[)$ is an extra – line and $\mathcal{P}_* = (u_*, v_*, w_*)$ is its arbitrary point, then

(3.6) $\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[) = \mathcal{L}([\mathcal{P}_*, \mathcal{B}_1[) \cup \mathcal{L}([\mathcal{P}_*, \mathcal{B}_1[)$

is valid.

Proof. Using the identity $d_{\widetilde{\mathbb{R}^3}}(\mathcal{P}_1, \mathcal{P}_2) =]|\mathcal{P}_1 \overline{\ominus} \mathcal{P}_2|[$by Theorem 1.10 we have

(3.7)$\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[) =$

$\{ \mathcal{P} \in \mathbb{R}^3 | \mathcal{P} =$

12⊙ℬ1⊕ℬ2⊕τ⊙ℬ1⊖ℬ2⊘dℝ3ℬ1,ℬ2,

where the parameter – domain

(3.8) $\widetilde{\left(-\frac{1}{2}\right)} \overline{\odot} d_{\widetilde{\mathbb{R}^3}}(\mathcal{B}_1, \mathcal{B}_2) < \tau < \widetilde{\left(\frac{1}{2}\right)} \overline{\odot} d_{\widetilde{\mathbb{R}^3}}(\mathcal{B}_1, \mathcal{B}_2)$, (see (1.13)).

Moreover, (3.2) yields

(3.9) $\mathcal{L}([\mathcal{P}_*, \mathcal{B}_1[) = \{ \mathcal{P} \in \mathbb{R}^3 | \mathcal{P} =$

𝒫∗⊕ϱ⊙ℬ1⊖𝒫∗⊘dℝ3ℬ1,𝒫∗

and

(3.10) $\mathcal{L}([\mathcal{P}_*, \mathcal{B}_2[) = \left\{ \mathcal{P} \in \mathbb{R}^3 | \mathcal{P} = \mathcal{P}_* \overline{\oplus} \left(\sigma \overline{\odot} \left(\left(\mathcal{B}_2 \overline{\ominus} \mathcal{P}_* \right) \overline{\oslash} d_{\widetilde{\mathbb{R}^3}}(\mathcal{B}_2, \mathcal{P}_*) \right) \right) \right\}$

with the parameter domains

(3.11) $0 \le \varrho < d_{\widetilde{\mathbb{R}^3}}(\mathcal{B}_1, \mathcal{P}_*)$ and $0 \le \sigma < d_{\widetilde{\mathbb{R}^3}}(\mathcal{B}_2, \mathcal{P}_*)$,

respectively.

Denoting $\underline{\tau} = t$, $\underline{\varrho} = r$ and $\underline{\sigma} = s$ and compessing the sets under (3.7) . (3.9) and (3.10) together their parameter – domains under (3.8) and under (3.11)

(3.12)$\underline{\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[)} = \left\{ \underline{\mathcal{P}} \in \underline{\mathbb{R}^3} | \underline{\mathcal{P}} = \frac{\underline{\mathcal{B}_1 + \mathcal{B}_2}}{2} + \right.$

ℬ1−ℬ2dℝ3ℬ1,ℬ2·t ;
−dℝ3ℬ1,ℬ22<t<dℝ3ℬ1,ℬ22,

(3.13) $\underline{\mathcal{L}([\mathcal{P}_*, \mathcal{B}_1[)} = \left\{ \underline{\mathcal{P}} \in \underline{\mathbb{R}^3} | \underline{\mathcal{P}} = \underline{\mathcal{P}_*} + \frac{\underline{\mathcal{B}_1 - \mathcal{B}_*}}{d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1}, \underline{\mathcal{P}_*}\right)} \cdot \right.$

r ; 0≤r<dℝ3ℬ1,𝒫∗

and

(3.14) $\underline{\mathcal{L}([\mathcal{P}_*, \mathcal{B}_2[)} = \left\{ \underline{\mathcal{P}} \in \underline{\mathbb{R}^3} | \underline{\mathcal{P}} = \underline{\mathcal{P}_*} + \frac{\underline{\mathcal{B}_2 - \mathcal{B}_*}}{d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_2}, \underline{\mathcal{P}_*}\right)} \cdot \right.$

s ; 0≤s<dℝ3ℬ2,𝒫∗

are obtained. By (3.12) – (3.14) we have that $\underline{\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[)}$, $\underline{\mathcal{L}([\mathcal{P}_*, \mathcal{B}_1[)}$ and $\underline{\mathcal{L}([\mathcal{P}_*, \mathcal{B}_2[)}$are (open and half open) linear passages in$\overline{\underline{\mathcal{B}_1}, \underline{\mathcal{B}_2}}$, $\overline{\underline{\mathcal{B}_1}, \underline{\mathcal{P}_*}}$ and $\overline{\underline{\mathcal{B}_2}, \underline{\mathcal{P}_*}}$, respectively. Considering that $\overline{\underline{\mathcal{B}_2}, \underline{\mathcal{P}_*}} = \overline{\underline{\mathcal{P}_*}, \underline{\mathcal{B}_2}}$we have

(3.15) $\overline{\underline{\mathcal{B}_1}, \underline{\mathcal{P}_*}} \cup \overline{\underline{\mathcal{B}_2}, \underline{\mathcal{P}_*}} = \overline{\underline{\mathcal{B}_1}, \underline{\mathcal{B}_2}}$

First, we investigate the connection betveen $\underline{\mathcal{L}([\mathcal{P}_*, \mathcal{B}_1[)}$ and $\underline{\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[)}$. As $\underline{\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[)}$ is an open passage, so, $\underline{\mathcal{B}_1} \ne \underline{\mathcal{P}_*}$. Let us denote the parameter $t_* \leftrightarrow \underline{\mathcal{P}_*}$, that is $\underline{\mathcal{P}_*} = \frac{\underline{\mathcal{B}_1 + \mathcal{B}_2}}{2} + \frac{\underline{\mathcal{B}_1 - \mathcal{B}_2}}{d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1}, \underline{\mathcal{B}_2}\right)} \cdot t_*$, and consider the parameter interval $t_* \le t < d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1}, \underline{\mathcal{B}_2}\right)$, The (3.12) gives the mapping $d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1}, \underline{\mathcal{B}_2}\right) \leftrightarrow \underline{\mathcal{B}_1}$, so, the mentioned parameter interval determines the (half open) passage $\overline{\underline{\mathcal{B}_1}, \underline{\mathcal{P}_*}}$. On the other hand, by (3.13) we can see the mappings $0 \leftrightarrow \underline{\mathcal{P}_*}$ and$d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1}, \underline{\mathcal{P}_*}\right) \leftrightarrow \underline{\mathcal{B}_1}$. So, the parameter interval $0 \le r < d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1}, \underline{\mathcal{P}_*}\right)$ derermines determines the (half open) passage $\overline{\underline{\mathcal{B}_1}, \underline{\mathcal{P}_*}}$,again.

Sscond, we investigate the connection betveen $\underline{\mathcal{L}([\mathcal{P}_*, \mathcal{B}_2[)}$ and $\underline{\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[)}$. As $\underline{\mathcal{L}(]\mathcal{B}_1, \mathcal{B}_2[)}$ is an open passage, so, $\underline{\mathcal{B}_2} \ne \underline{\mathcal{P}_*}$. The (3.12) gives the mapping $-d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1}, \underline{\mathcal{B}_2}\right) \leftrightarrow \underline{\mathcal{B}_2}$. So, the parameter

interval $-d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1},\underline{\mathcal{B}_2}\right) < t \le t_*$ determines the (half open) passage $\underline{\mathcal{B}_2},\underline{\mathcal{P}_*}$. On the other hand, by (3.14) we can see the mappings $0 \leftrightarrow \underline{\mathcal{P}_*}$ and $d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_2},\underline{\mathcal{P}_*}\right) \leftrightarrow \underline{\mathcal{B}_2}$. So, the parameter interval $0 \le s < d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1},\underline{\mathcal{P}_*}\right)$ derermines determines the (half open) passage$\overline{\mathcal{P}_*,\mathcal{B}_2}$.
Collecting our results by (3.15)

(3.16) $\underline{\mathcal{L}([\mathcal{P}_*,\mathcal{B}_1[)} \cup \underline{\mathcal{L}([\mathcal{P}_*,\mathcal{B}_2[)} = \underline{\mathcal{L}(]\mathcal{B}_1,\mathcal{B}_2[)}$

is obtained. Hence, having the trivial identity $(\widetilde{\mathbb{S}_1 \cup \mathbb{S}_2}) = \widetilde{\mathbb{S}_1} \cup \widetilde{\mathbb{S}_2}$; $\mathbb{S}_1,\mathbb{S}_2 \subset \mathbb{R}^3$ and using the right hand side of (0.9) by the explosion (3.16) yields (3.6). ■

Definition 3.17. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be different points of our universe and $\mathcal{B}$ a point on the border of our universe such that they are non super – collinear points of the Multiverse. The half extra - lines
(3.18)
$\mathcal{L}([\mathcal{P}_1,\mathcal{B}[) =$
$\{\mathcal{P} \in \mathbb{R}^3 | \mathcal{P} =$
$\mathcal{P}1\oplus\varrho\odot\mathcal{B}\ominus\mathcal{P}1\oslash d\mathbb{R}3\mathcal{P}1,\mathcal{B};0\le\varrho<d\mathbb{R}3\mathcal{P}1,\mathcal{B}$

and
(3.19)
$\mathcal{L}([\mathcal{P}_2,\mathcal{B}[) =$
$\{\mathcal{P} \in \mathbb{R}^3 | \mathcal{P} =$
$\mathcal{P}1\oplus\sigma\odot\mathcal{B}\ominus\mathcal{P}2\oslash d\mathbb{R}3\mathcal{P}2,\mathcal{B};0\le\sigma<d\mathbb{R}3\mathcal{P}2,\mathcal{B}$

are called extra – parallel half extra – lines.■
Definition 3.20. If two extra – lines have a joint border point, then they are called extra – parallel extra – lines. If an extra – line and a half extra – line have a joint border point then the half extra- line iscalled to be extra – prallel with respect the extra - line■
Remark 3.21. If both border points of two extra – lines are joint points then the two extra – lines are equal. (See, Theorem 2.9, (iii).) .

Theorem 3.22. Let be $\mathcal{P}_0 \in \mathbb{R}^3$ and $\mathcal{L}(]\mathcal{B}_1,\mathcal{B}_2[)$ an extra – line such that $\mathcal{P}_0$, $\mathcal{B}_1$ *and* $\mathcal{B}_2$ are not super – collinear points. The half extra – lines
(3.23)$\mathcal{L}([\mathcal{P}_0,\mathcal{B}_1[) =$
$\{\mathcal{P} \in \mathbb{R}^3 | \mathcal{P} =$
$\mathcal{P}0\oplus\varrho\odot\mathcal{B}1\ominus\mathcal{P}0\oslash d\mathbb{R}3\mathcal{B}1,\mathcal{P}0,\ 0\le\varrho<d\mathbb{R}3\mathcal{B}1,\mathcal{P}0$

and
(3.24)$\mathcal{L}([\mathcal{P}_0,\mathcal{B}_2[) =$
$\{\mathcal{P} \in \mathbb{R}^3 | \mathcal{P} =$
$\mathcal{P}_0\overline{\oplus}\left(\sigma\overline{\odot}\left((\mathcal{B}_2 \overline{\ominus} \mathcal{P}_0)\ \overline{\oslash}\ d_{\widetilde{\mathbb{R}^3}}(\mathcal{B}_2,\mathcal{P}_0)\right)\right)\}, 0 \le \sigma <$
$d_{\widetilde{\mathbb{R}^3}}(\mathcal{B}_2,\mathcal{P}_0)$
are extra parallel half extra – lines with respect to the extra line $\mathcal{L}(]\mathcal{B}_1,\mathcal{B}_2[)$such that
$\mathcal{L}([\mathcal{P}_0,\mathcal{B}_1[) \cup \mathcal{L}([\mathcal{P}_0,\mathcal{B}_2[)$
is not an extra – line.
Proof. Denoting $\underline{\tau} = t$, $\underline{\varrho} = r$ *and* $\underline{\sigma} = s$ and compessing the sets under (3.7) . (3.23) and (3.24) together their parameter – domains

(3.25) $\underline{\mathcal{L}(]\mathcal{B}_1,\mathcal{B}_2[)} = \{\underline{\mathcal{P}} \in \underline{\mathbb{R}^3} | \underline{\mathcal{P}} = \frac{\mathcal{B}_1+\mathcal{B}_2}{2} +$
$\mathcal{B}1-\mathcal{B}2d\mathbb{R}3\mathcal{B}1,\mathcal{B}2\cdot t$;
$-d\mathbb{R}3\mathcal{B}1,\mathcal{B}22<t<d\mathbb{R}3\mathcal{B}1,\mathcal{B}22,$

(3.26) $\underline{\mathcal{L}([\mathcal{P}_0,\mathcal{B}_1[)} = \{\underline{\mathcal{P}} \in \underline{\mathbb{R}^3} | \underline{\mathcal{P}} = \underline{\mathcal{P}_0} +$
$\mathcal{B}1-\mathcal{P}0d\mathbb{R}3\mathcal{B}1,\mathcal{P}0\cdot r$; $0\le r<d\mathbb{R}3\mathcal{B}1,\mathcal{P}0$

and

(3.27) $\underline{\mathcal{L}([\mathcal{P}_0,\mathcal{B}_2[)} = \{\underline{\mathcal{P}} \in \underline{\mathbb{R}^3} | \underline{\mathcal{P}} = \underline{\mathcal{P}_0} +$
$\mathcal{B}2-\mathcal{P}0d\mathbb{R}3\mathcal{B}2,\mathcal{P}0\cdot s$; $0\le s<d\mathbb{R}3\mathcal{B}2,\mathcal{P}0$

are obtained. By (3.25) – (3.27) we have that $\underline{\mathcal{L}(]\mathcal{B}_1,\mathcal{B}_2[)}$, $\underline{\mathcal{L}([\mathcal{P}_0,\mathcal{B}_1[)}$ and $\underline{\mathcal{L}([\mathcal{P}_0,\mathcal{B}_2[)}$are (open and half open) linear passages in $\overline{\underline{\mathcal{B}_1},\underline{\mathcal{B}_2}}$,$\overline{\underline{\mathcal{P}_0},\underline{\mathcal{B}_1}}$ and$\overline{\underline{\mathcal{P}_0},\underline{\mathcal{B}_2}}$, respectively. Considering that $\mathcal{P}_0$, $\mathcal{B}_1$ *and* $\mathcal{B}_2$ are not super – collinear points, $\underline{\mathcal{P}_0} \notin \overline{\underline{\mathcal{B}_1},\underline{\mathcal{B}_2}}$. So, $\underline{\mathcal{L}([\mathcal{P}_0,\mathcal{B}_1[)} \cup \underline{\mathcal{L}([\mathcal{P}_0,\mathcal{B}_1[)}$ is a breaked linear passage. $(d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1},\underline{\mathcal{B}_2}\right) < d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_1},\underline{\mathcal{P}_0}\right) + d_{\mathbb{R}^3}\left(\underline{\mathcal{B}_2},\underline{\mathcal{P}_0}\right))$ Hence, $\mathcal{L}([\mathcal{P}_0,\mathcal{B}_1[) \cup \mathcal{L}([\mathcal{P}_0,\mathcal{B}_2[)$ is not an extra – line. Moreover, by Definitions 3.20, $\mathcal{L}([\mathcal{P}_0,\mathcal{B}_1[)$ *and* $\mathcal{L}([\mathcal{P}_0,\mathcal{B}_2[)$are extra parallel half extra – lines with respect to the extra line $\mathcal{L}(]\mathcal{B}_1,\mathcal{B}_2[)$.■
Exercise 3.28. Let usprove that the set

(3.29) $\mathcal{L} = \{(u, v, w) \in$

$\mathbb{R}3u$=$-\tanh-114+914 \cdot tv$=$\tanh-114-114 \cdot tw$=$\tanh-112-27 \cdot t$, , where $-78<t<78$

forms an extra – line.

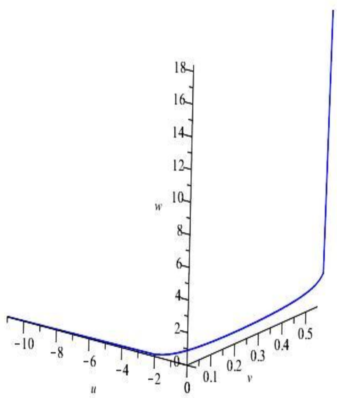

Fig. 3.29*

Moreover, let us show there exist twohalf extra - lines setting out from the origo which are extra – parallel half extra – lines with respect to $\mathcal{L}$, such that their union does not give an extra – line.

Solution. First we prove that the set $\mathcal{L}$ forms an extra – line. Denoting $\underline{u} = x, \underline{v} = y, \underline{w} = z$ and comprssing the set $\mathcal{L}$

(3.30)

$$\underline{\mathcal{L}} = \left\{ (x,y,z) \in \mathbb{R}^3 | \begin{matrix} x=-\frac{1}{4}-\sqrt{\frac{9}{14}}\cdot t \\ y=\frac{1}{4}-\sqrt{\frac{1}{14}}\cdot t \\ z=\frac{1}{2}-\sqrt{\frac{2}{7}}\cdot t \end{matrix}, -\sqrt{\frac{7}{8}} < t < \sqrt{\frac{7}{8}} \right\},$$

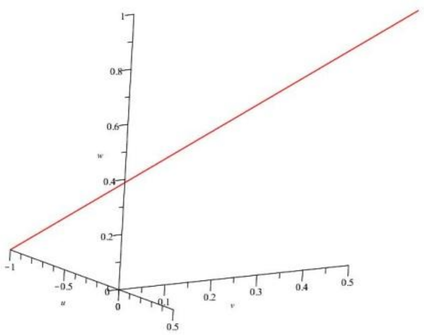

Fig. 3.30*

is obtained. We can see, that $\underline{\mathcal{L}}$is an open linear passage with the endpoints $\left(\frac{1}{2}, \frac{1}{2}, 1\right)$ and $(-1,0,0)$ situated on the border of compressed universe $\underline{\mathbb{R}^3}$.

Hence, $\mathcal{L}$ is an extra – line with its border – points $\mathcal{B}_1 = \left(\widetilde{\left(\frac{1}{2}\right)}, \widetilde{\left(\frac{1}{2}\right)}, \breve{1}\right)$ and $\mathcal{B}_1 = \left(\widetilde{(-1)}, 0, 0\right)$.

Second, having that $\mathcal{P}_0 = \mathcal{O}$ we apply Theorem 3.22, by (3.23) and (3.24) we get the half extra - lines

(3.31)$\mathcal{L}([\mathcal{O}, \mathcal{B}_1[) =$

$\left\{\mathcal{P} \in \mathbb{R}^3] \mathcal{P} = \varrho \overline{\odot} \left(\mathcal{B}_1 \overline{\oslash} \widetilde{\left(\frac{\sqrt{6}}{2}\right)}\right)\right\}, 0 \le \varrho < \widetilde{\left(\frac{\sqrt{6}}{2}\right)}$

and

(3.32)$\mathcal{L}([\mathcal{P}_0, \mathcal{B}_2[) = \{\mathcal{P} \in \mathbb{R}^3] \mathcal{P} = \sigma \overline{\odot}\, \mathcal{B}_2\}, 0 \le \sigma < \breve{1}$.

Exchanging the vector equations for scalar equation systems

(3.33) $\mathcal{L}([\mathcal{O}, \mathcal{B}_1[) = \{(u,v,w) \in$

ℝ3u=tanh−*116·* *ϱv*=tanh−*116·* *ϱw*=tanh−*126·* *ϱ*, , where *0≤ϱ<62*

and

(3.34) $\mathcal{L}([\mathcal{O}, \mathcal{B}_2[) = \{(u,v,w) \in$

ℝ3u=−σv=0w=0, , where *0≤σ<∞*

are obtained.

The half extra – line $\mathcal{L}([\mathcal{O}, \mathcal{B}_1[)$ is a part of the extra – line $\mathcal{L}_{\mathcal{O},\mathcal{E}}$ (see (1.8) and Fig. 1.9) and the half extra – line $\mathcal{L}([\mathcal{O}, \mathcal{B}_2[)$ is a part of the „u – axis".

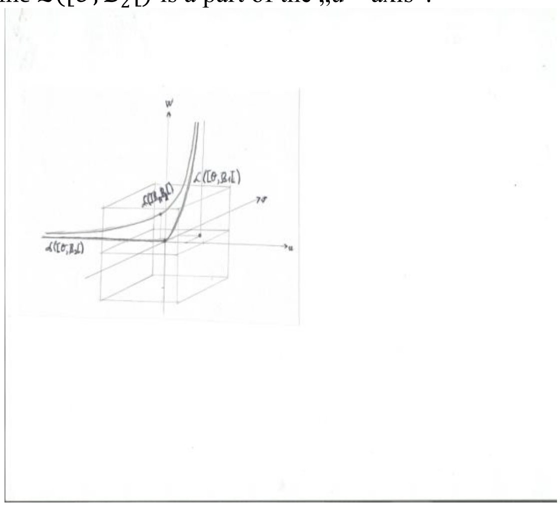

Fig. 3.35

By Fig. 3.35 we can see that $\mathcal{L}([\mathcal{O}, \mathcal{B}_1[) \cup \mathcal{L}([\mathcal{O}, \mathcal{B}_2[)$ is not an extra – line.

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