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Proof of Goldbach's binary conjecture

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ABSTRACT
The article analyzes the problem of representation of even number as a sum of two odd summands. On this basis the proof of Goldbach's binary conjecture is given.
***Keywords and phrases*:** Number theory; natural numbers; representation of even number as a sum of two odd summands; Goldbach's binary hypothesis.

I. INTRODUCTION. GENERAL RELATIONS

Every natural number admits a trivial representation as a sum of units. Combining (grouping) units in different ways we obtain all the representations [1]. Consider the even numbers and their representations as a sum of two odd summands. For an arbitrary even integer p the trivial representation has the form $p = 1 + 1 + ... + 1$ (p units). Let for definiteness p is divisible by 4, then its representation as a sum of two odd numbers can be written as a chain of equalities

$$p= [(1+ (p-1)] = [(3+ (p-3)] = ...= [(p / 2-1, p / 2+1)] \quad (1),$$

where $p/2$ is the center of representation (even number). The terms in square brackets we call the conjugate numbers. The first term is to the left of the center and the second term is to the right of the center. If p is not divisible by 4, its representation has the form

$$p= [(1+ (p-1)] = [(3+ (p-3)] = ...= [(p / 2-2, p / 2+2)] = [p / 2, + p / 2], \quad (2)$$

where $p/2$ is the center of representation (odd number). For number $p + 2$ the representation is written in the form

$$p + 2= [(1+ (p+1)] = [(3+ (p-1)] = ...= [(p / 2-1) + p / 2+3)] = [(p / 2+1) + p / 2+1)], \quad (3)$$

where $(p / 2+1)$ is the center of representation (odd number). Acceptable combinations of ends in conjugate numbers and their sequence are completely determined by the end of even number. The number of pairs depends on the parity of center. There are two cases for each end: the center is an even number or it is an odd number. For example, if p ends in 0 and it is divisible by 4 (center – an even number), acceptable combinations of ends are (towards the center) 1-9; 3-7; 5-5; 7-3; 9-1 and then the ends are repeated (first end corresponds with numbers to the left of the center, and the second end corresponds with numbers to the right of the center). The number of pairs of conjugate numbers is equal to $p/4$. If we consider only prime conjugate numbers, the combination 5-5 can be eliminated. If p ends in 0, but it is not divisible by 4 (center – an odd number), acceptable combinations of ends are the same, but the number of pairs equals $p/4 + 1$. For number $p + 2$ ending in 2 permissible combinations of ends are (towards the center) 1-1; 3-9; 5(number)-7; 7-5; 9-3 and then the ends are repeated. The number of pairs of conjugate numbers is equal to $p/4 + 1$. If we take into account only prime conjugate numbers, the combination 7-5 can be eliminated. Consider what it will be, when p increases. With increasing of even number p the number of conjugate pairs and the number of primes included in the representation of even number increase. From (1) it can be seen that initial prime numbers 3, 5, 7, 11, 13, 17, 19, 23 etc. are included in all representations for sufficiently large p and their number increases with increasing p. There is an asymmetry in the distribution of prime numbers to the left and to the right of the center which depends on two processes. The number of primes to the left of the center rises (not decreases), as they transfer from the right to the left. Therefore the number of primes to the right of the center can decrease (does not increase), but not much. It is compensated by appearance of new primes and does not vanish when p increases. With increasing p between these two processes is the dynamic equilibrium that determines an asymmetry of the distribution of prime numbers and depends on order of magnitude of number p. Prime numbers regularly occur both to the left and to the right of the center. The appearance of pairs of prime conjugate numbers depends mainly on the total number of primes less than p and to a certain extent on the irregularity of their distribution to the left and to the right of the center. Since prime numbers are infinitely many, pairs of prime conjugate numbers appear regularly and can not completely vanish with increasing p. Otherwise after certain p would not be prime numbers which is impossible (*vide infra*).

II. WAY OF PROOF

We prove the binary conjecture (hypothesis) by induction. We are going to prove the statement: every even integer not less 6 admits a representation as a sum of two primes. This hypothesis can be verified for any finite even number; for example, 6 = (1 + 1 + 1) + (1 + 1 + 1) = 3 + 3; 8 = (1 + 1 + 1) + (1 + 1 + 1 +1 + 1) = 3 + 5 etc. Suppose that for numbers $n \le p$ the binary hypothesis is correct while for $n > p$ it is not true. Consider the set of even integers of the form 10*m*, where *m*=1, 2, 3,..... Let *p* is such a number; let for definiteness *p* is divisible by 4; thence p +10 is not divisible by 4 and *p*+20 is divisible by 4 etc. The representation of number *p*+10 in brief notation has the form

p+10= (1, p+9) = (3, p+7)= ...= (p / 2+3, p / 2+7)= (p / 2+5, p / 2+5) , (4)

where in the round brackets are the pairs of conjugate numbers. Under what conditions the binary hypothesis will not be true for certain even number? This is possible, if there is one of two conditions (A) or (B).

(A) There are no primes to the right of the center in the representation of even number.

(B) There are primes to the right of the center, but they form conjugate pairs only with composite numbers.

We will show that these conditions do not occur. Consider number p +10. Compare representations of numbers p and p+10. In our case the center in the representation of p is equal to p/2 (even number); the center in the representation of p+10 is p / 2+5 (odd number). The number of conjugate pairs in the representation of p is equal to p/4 and in the representation of p+10 it is equal to p / 4+3. The number of primes is not reduced after transition from p to p+10. To the left of the center in the representation of p+10 there are three new numbers p / 2+1, p / 2+3 and p / 2+5 . Numbers p / 2+1 and p / 2+3 appear because of transition from the right of the center in the representation of p to the left of the center in the representation of p+10. Number p / 2+5 appears due to change in the parity of center. The other numbers are the same as in the representation of p. Find out which of new numbers are primes. Number p/2+5 is composite as it is divisible by 5. Numbers p/2+1and p/2+3 can be prime or composite (*vide infra*). In the representation of p+10 to the right of the center there are five new numbers p+9, p+7, p+5, p+3 and p+1. The other numbers are the same as in the representation of p; they change position (increase) by 10 relative to the numbers in the representation of p. Find out which of these new numbers are primes. Number p+5 is composite as it is divisible by 5. Numbers p+7 and p+3 can not be primes. Otherwise the binary hypothesis will be true for all numbers p +10m in representation of which pairs (10m - 7, p+7) and (10m - 3, p+3) are the pairs of prime conjugate numbers; in particular the binary hypothesis will be true for p+10 , which contradicts our assumption (*vide supra*). Numbers p+1 and p+9 can not be primes. Otherwise the binary hypothesis will be true for all numbers p +10m in representation of which pairs (10m -1, p+1) and (10m - 9, p+9) are the pairs of prime conjugate numbers; in particular the binary hypothesis will be true for p+20, which contradicts our assumption. We must consider special case, when numbers p / 2+1 and/or p / 2+3 are unique (sole) prime numbers to the right of the center in the representation of p (*vide supra*). It follows from the foregoing analysis which is valid for arbitrary two adjacent even numbers ending in 0 (with identical ends) that in this case numbers p / 2+3 and/or p / 2+1 will be unique (sole) prime numbers to the right of the center in the representation of all even numbers on the interval [p / 2+10, p]. As the binary hypothesis is true by assumption for all even numbers $n \le p$, numbers p / 2+3 and/or p / 2+1 should form pairs of prime conjugate numbers in the representation of all even numbers on the interval [p / 2+10, p]. Let q is an arbitrary even number from this interval (p is sufficiently large). In the representation of q there should be pairs of prime conjugate numbers of the form (α, p/2 + 3) and (β, p/2 +1), where α and β are primes; $\alpha + p / 2 + 3 = \beta + p / 2 + 1 = q$; $\beta - \alpha = 2$. Put $q = p/2+120$; then α =117, β =119. But 117 and 119 are not primes. We have the contradiction. Number p/2+120 is the smallest number of that kind. There are other numbers of this kind; for example q = p / 2+190 (α =187, β =189); q = p 2+210 (α = 207, β = 209); q = p 2+220 (α = 217, β = 219) etc. Here α and β are not primes; thence the binary hypothesis is not true for numbers p / 2+120, p / 2+190, p / 2+210, p / 2+220, which contradicts our assumption. Hence numbers p / 2+3 and p / 2+1 are not unique (sole) prime numbers to the right of the center in the representation of p. So to the right of the center in the representation of p there are other prime numbers; thence there are primes also to the right of the center in the representation of p+10 and the same as in the representation of p. Therefore it follows from our analysis which is valid *mutatis mutandis* for arbitrary two adjacent even numbers ending in 0 (with identical end) that condition (A) does not take place for p+10 and numbers follow it (*vide infra*). Consider condition (B). Suppose it occurs. We will increase number p

sequentially by 10. After [p/10] steps (here [p/10] is the integer part of number p/10) it will be complete substitution (displacement) of numbers situated to the right of the center in the representation of p. All of them are found on the left in the representation of p +10[p/10]. Consequently in accordance with foregoing analysis which is valid for arbitrary two adjacent even numbers ending in 0 (with identical ends) there are no primes to the right of the center in the representation of number p +10[p/10] and numbers follow it. On the other hand the number of primes should increase with increasing p. We have the contradiction. Therefore the condition (B) does not occur and pairs of prime conjugate numbers are regularly formed, when we increase the number p. So we have proved the binary hypothesis for all numbers ending in 0. Generally speaking two conclusions are possible in accordance with formal logic: the binary hypothesis is true for all even numbers more than p or it is true only for some numbers more than p. But the second conclusion can not be accepted because of insufficient reason, as it follows from our foregoing analysis. For complete proof of the binary hypothesis we should consider even numbers with other ends 2, 4, 6 and 8. Way of proof *mutatis mutandis* and conclusions for even numbers with other ends remain the same as in the case of numbers ending in 0. For example, consider numbers ending in 2. Let p is such a number; let for definiteness p is divisible by 4; thence p +10 is not divisible by 4 and p + 20 is divisible by 4 etc., as above for numbers ending in 0. It is necessary to take account of two factors. Firstly combinations of acceptable ends and their sequence change. Secondly even numbers with other ends are not divisible by 10; p/10 is not an integer and there is a remainder. Consider the first factor. For even numbers ending in 2 the pairs of ends and their sequence (towards the center) have the form 1-1, 3-9 etc. (*vide supra*). For even numbers ending in 4 the pairs of ends and their sequence have the form 1-3, 3-1, 5(number)-9, 7-7, 9-5, 1-3 etc. If we consider only prime numbers, the pair 9-5 can be eliminated. For even numbers ending in 6 the pairs of ends and their sequence have the form 1-5 (can be eliminated), 3-3, 5(number)-1, 7-9, 9-7, 1-5 etc. For even numbers ending in 8 the pairs of ends and their sequence have the form 1-7, 3-5 (can be excluded), 5(number)-3, 7-1, 9-9, 1-7 etc. Consider the second factor. For numbers ending in 2 the remainder is 2 etc., for numbers ending in 8 the remainder is 8. In this case numbers situated to the right of the center in the representation of p are not displaced completely after [p/10] steps; one or two of them remain to the right of the center in the representation of p +10[p/10]. But they can not be prime numbers, as it follows from the foregoing analysis. Indeed. If p ends in 2 or 4, one number p -1 remains to the right of the center in the representation of p +10[p/10]. Number p -1 can not be prime number. Otherwise the binary hypothesis will be true for all numbers p +10m in representation of which pair (10m+1, p-1) is the pair of prime conjugate numbers; in particular the binary hypothesis will be true for p+10 , which contradicts our assumption (*vide supra*). If p ends in 6, two numbers p-1 and p-3 remain to the right of the center in the representation of p +10[p/10]. Number p-1 is composite, as it is divisible by 5. Number p-3 can not be prime number. Otherwise the binary hypothesis will be true for all numbers p +10m in representation of which pair (10m + 3, p - 3) is the pair of prime conjugate numbers; in particular the binary hypothesis will be true for p+10, which contradicts our assumption (*vide supra*). If p ends in 8, two numbers p-1 and p-3 remain to the right of the center in the representation of p +10[p/10]. Number p-3 is composite, as it is divisible by 5. Number p-1 can not be prime, as it is contrary to our assumption of the binary hypothesis (*vide supra*). Therefore values [p/10] and p +10[p/10], as well as, the conclusions obtained above for numbers ending in 0 remain correct for numbers with other ends. Thus we have proved the binary hypothesis for all even numbers.

III. CONCLUSION

From validity of the binary hypothesis we immediately get two inferences. *Inference1*. There are always prime numbers on the interval [n, 2n], where n is arbitrary natural number. It follows from this that there are also prime numbers between 2^k and 2^{k+1}, where k is a natural number. *Inference2*. The hypothesis of Legendre is correct, i.e. there is always a prime number between n^2 and $(n+1)^2$, where n is arbitrary natural number. The hypothesis of Legendre can be easily proved by induction on the basis of the binary hypothesis [2].

REFERENCES

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