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Adomian Decomposition Method for Solving the Nonlinear Heat Equation

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ABSTRACT
This paper studies the application of the Adomian Decomposition Method to find the exact and approximate solutions of the heat equation with power nonlinearity. First, the relevant literature is studied in understanding the importance and extent of applicability of the method in the applied science. The literature review has been incorporated in the introduction of the paper. The rest of the paper is divided in three further sections. The first part Adomian Decomposition Method provides a step-by-step guide of applying the method on any heat equation with nonlinearity. The second section is labeled as Applications. It considers two examples from the previous works of Pumak (2005) and Hetmaniok et al. (2010) to find the exact and approximate solutions of the equations respectively.

I. INTRODUCTION

Contemporary mathematics faces major challenges in solving problems of various differential equations such as Cauchy problems. Various methods such as differential transform method (Smarda&Dibl´ık, 2013), Taylor collection method (Sezer&Akyuz-Dascioglu, 2006), variational iterative method (Chen & Wang, 2010), homotopy perturbation method (Shakeri&Dehghan, 1994) and homtopy analysis methods (Wang, 2010) have been considered by the researchers over the past two decades. Adomian's decomposition method is of great interest to applied sciences today. It is necessary for the solution of many different problems such as partial differential equations (Adomian, 1994), algebraic equations (Al-Hayani&Casasus, 2005), and boundary value problems (Wazwaz, 2005). It is a great method to accurately compute a rapidly convergent series solution. The main advantage of the method lies in its direct applicability to homogeneous or inhomogeneous, linear or nonlinear, integral and differential equations, and with variable or constant coefficients. Also, it maintains the high accuracy of the numerical solutions and reduces the great computational work simultaneously. The most recent application of the Adomian Decomposition Method lies in solving the heat equations with power nonlinearity (Akram& Pasha, 2005).

II. ADOMIAN DECOMPOSITION METHOD

George Adomian proposed the Decomposition Method in the 1980s to solve nonlinear function equations (Cherruault, 1995). This method has been read and applied ever since (Babolian&Biazar, 2002; Lesnic, 2002). This method is based on separating the linear and nonlinear portions of any given equation and then inverting the linear operator of the equation and applying the inverted operator to the equation. The nonlinear portion of any given equation is decomposed in various Adomian polynomials which are used to determine the recursive relationship among the terms of the series. Adomian Decomposition Method produces the convergent series solution (Hashim, 2006). Many researchers have been investing the issue of convergence (Cherruault et al., 1995; Babolian&Biazar, 2002; Lesnic, 2002). The definition from which the order of convergence for the method could be determined was provided by Babolian&Biazar (2002). It has also been found that the series produced by the decomposition method is absolutely convergent as well as uniformly convergent (Cherruault et al., 1995) where a high degree of convergence is desirable. It is for this reason that the series converges more rapidly. The researchers have also pointed out various advantages and disadvantages of the decomposition method. The method significantly reduces the computational work (Bulut, 2004). The method can widely handle quite general nonlinearities (Wang, 2004). It enables the researchers to develop an analytical, reliable solution of the nonlinear problems without linearization. Alongside, the method also suffers from certain disadvantages as well. First of all, the major concern lies in the region and rate of convergence of the series produced by the Adomian Decomposition Method. Also, it has very slow convergence rate in the wider regions (Jiao et al., 2002). However, a further investigation in this particular area is desirable.Nonetheless, the

Adomian Decomposition Method is one of the finest solutions to the nonlinearities today. The advantages of the method provide a strong base of wide-ranging applicability of the decomposition solution in fields like engineering, chemistry, biology, and physics (Adomian, 1995, 1996; Babolian&Biazar, 2002; Biazar, 2006). In fact, the method has proven to be highly applicable to such diverse areas as nonlinear optics, particle transport, mass and/or heat transfer, chaos theory and the fermentations theory (Hashim et al., 2006). The most recent application of the decomposition method lies in finding the solution of the nonlinear heat equations which are one of the most important phenomena in engineering, physics, and mathematics. Here is an example of how the decomposition method can be used to solve a simple heat equation with a power nonlinearity:

$$u_t(x,t) = u_{xx} + u^m \qquad (1)$$

subject to the initial condition

$$u(x,0) = f(x) \qquad (2)$$

Approximating by an operator

$$L_t u(x,t) = L_{xx} u + u^m \ (3)$$

Taking the inverse operator of the operator L_t exists and it defined as

$$L_t^{-1}(.) = \int_0^t (.)dt \ (4)$$

Thus, applying the inverse operator L_t^{-1} to equation 1 yields

$$L_t^{-1} L_t u(x,t) = L_t^{-1} L_{xx} u + \varepsilon L_t^{-1} u^m \ (5)$$

$$u(x,t) = u(x,0) + L_t^{-1} L_{xx} u + \varepsilon L_t^{-1} u^m. \qquad (6)$$

Presenting the Adomian Decomposition Method solution as under

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \ (7)$$

Decomposing the nonlinear portion of equation 1 in Adomian polynomials

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\right]_{\lambda=0} \quad , n \geq 0 \qquad (8)$$

thus,

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n \qquad (9)$$

Substituting equation 7 in equation 9

$$u(x,t) = u(x,0) + L_t^{-1} L_{xx} \sum_{n=0}^{\infty} u_n(x,t) + L_t^{-1} \sum_{n=0}^{\infty} A_n \ . \qquad (10)$$

Recurrent relation from equation 10 can be defined as

$$u_0(x,0) = f(x)$$

$$u_{n+1}(x,t) = L_t^{-1} L_{xx} u_n(x,t) + \varepsilon L_t^{-1} A_n \text{ for } n = 0,1,2, \ldots \ldots$$

form which

$$u_1(x,t) = L_t^{-1} L_{xx} u_0 + L_t^{-1} A_0$$

$$u_2(x,t) = L_t^{-1} L_{xx} u_1 + L_t^{-1} A_1$$

$$u_3(x,t) = L_t^{-1} L_{xx} u_2 + L_t^{-1} A_2$$

$$\vdots$$

$$u_n(x,t) = L_t^{-1} L_{xx} u_{n-1} + L_t^{-1} A_{n-1}$$

We can estimate the approximate solution ϕ_γ by using the γ- term approximation . That is ,

$$\phi_\gamma = \sum_{n=0}^{\gamma-1} u_n(x,t) \qquad (11)$$

Equation 7 and equation 11 make it clear that

$$u(x,t) = \lim_{\gamma \to \infty} \phi_\gamma(x,t)$$

Finally, we want to apply the Adomian Decomposition Method to two examples and compare the exact solution and approximate solution. The importance of arriving at approximate (Dehghan, 2004; Grzymkowski&Słota, 2005; Hetmaniok et al., 2010) or exact (Lesnic, 2002; Pamuk&Pamuk, 2014) solution is explained by the efficiency of these problems. Pumak (2005) utilised the Adomian Decomposition Method to find the exact location of a heat equation with both linear and nonlinear powers in his paper titled “An Application for Linear and Nonlinear Heat Equations by Adomian’s Decomposition Method.” For the purpose of this paper, I am considering the nonlinearity only. Here is the heat equation Pumak considered in the paper;

$$u_t(x,t) = u_{xx} - 2u^3$$

Applications

Consider the nonlinear heat equation and subject to initial condition

$$u_t(x,t) = u_{xx} - 2u^3 \ (12)$$

$$u(x,0) = \frac{1+2x}{x^2+x+1} \ (13)$$

Where for the exact solution of equation 12 as

$$u(x,t) = \frac{1+2x}{x^2+x+6t+1} \qquad (14)$$

In an operator form, equation 12 becomes,

$$u(x,t) = u(x,0) + L_t^{-1} L_{xx} u - 2L_t^{-1} u^3 \ (15)$$

the Adomian Polynomials are

$$A_0 = -2u_0^3$$

$$A_1 = -6u_0^2 u_1$$

$$A_2 = -6(u_0 u_1^2 + u_0^2 u_2)$$

And so on. Thus,

$$u_0 = \frac{1+2x}{x^2+x+1}$$

$$u_1 = L_t^{-1} L_{xx}(u_0) - 2L_t^{-1}(u_0^3) = \frac{-6(1+2x)}{(x^2+x+1)^2} t$$

$$u_2 = L_t^{-1} L_{xx}(u_1) - 6L_t^{-1}(u_0^2 u_1) = \frac{36(1+2x)}{(x^2+x+1)^3} t^2$$

$$u_3 = L_t^{-1}L_{xx}(u_2) - 6L_t^{-1}(u_0^2u_2 + u_1^2u_0) = \frac{-216(1+2x)}{(x^2+x+1)^4}t^3$$

$\vdots$

and so on.

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$u(x,t) = \frac{1+2x}{x^2+x+1} - \frac{6(1+2x)}{(x^2+x+1)^2}t + \frac{36(1+2x)}{(x^2+x+1)^3}t^2 - \frac{216(1+2x)}{(x^2+x+1)^4}t^3 + \cdots \quad (16)$$

So the equation 16 is almost equal

$$u(x,t) \approx \frac{1+2x}{x^2+x+6t+1}$$

It is the solution of the heat equation 12 by using ADM.

III. CONCLUSION

This paper proposes the application of the Adomian Decomposition Method for solving the nonlinear heat equations. It is a semi analytical method and it is the best method to solve the ordinary and partial equation with both linear and nonlinear powers. Examples from the previous works are used. It is concluded that Adomian Decomposition Method provides both exact and approximate numerical solutions for the heat equations with nonlinear power in comparison to other solutions such as antireductiori method (Fushchych& Zhdanov, 1994) and Lie symmetry reduction method (Euler & Euler, 1997).

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