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Advanced Applications of the Integration by Parts Formula I

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ABSTRACT
We have formulated our main delay stochastic differential equation in stratonovich form and we have also established some theorems and results which can be regarded as advanced applications of the integration by parts formula.
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I. INTRODUCTION, NOTATIONS AND DEFINITIONS

1.1 Introduction

In Chapter 1 of the Ph.D. thesis of Ahmed [15] we have proved the existence and uniqueness of a solution for certain types of delay (functional) stochastic differential equations (delay SDE's) with discontinuous initial data,see also [1], [9] and the web cite www.sfde.math.siu.edu. See the delay SDE (1.1) in the present work. In [18] we have established an integration by parts formula involving Mallivan derivatives of solutions to such type of delay (functional) SDE's. The integration by parts formula which we establish can be used to extend the formulas in [2] and [3] and to include delay SDE's as well as ordinary SDE's. In this work we also establish some other useful applications to delay SDE's. Generally speaking we can say that our work extends the first three chapters of the work by Norris to include delay SDE's as well as ordinary SDE's; see Theorems 2.3, 3.1 and 3.2 in [10] .

1.2 Notations and Definitions

The following notations and definitions will be used throughout this work: $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space; T is a positive real number; $\{\mathcal{F}_t\}_{t\in[0,T]}$ is an increasing family of sub-σ algebras of $\mathcal{F}$, each of which contains all null subsets of Ω; $\mathbb{N}$ is the set of natural numbers; $W = (W^1, \dots, W^r): [0,T] \times \Omega \to \mathbb{R}^r$ is a r-dimensional normalized Brownian motion. If X is a topological space, then $\mathcal{B}(X)$ denotes its Borel field. The symbol λ refers to the Lebesgue measure on $\mathbb{R}^d$, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, $d \in \mathbb{N}$.

Let G be a Banach space and let $\mathcal{A}$ be a sub-σ algebra of $\mathcal{F}$ containing all subsets of measure zero in $\mathcal{F}$, then $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbb{P}; G)$ denotes the space of all functions $f: \Omega \to G$ which are $\mathcal{A}$-$\mathcal{B}(G)$ measurable and are such that $\int_\Omega \|f\|_G^2 \, d\mathbb{P} < \infty$.

The symbol $L^2(\Omega, \mathcal{A}, \mathbb{P}; G)$ denotes the Banach space (with norm determined by $\|f\|_{L^2}^2 = \int_\Omega \|f(\omega)\|_G^2 \, d\mathbb{P}$) of all equivalence classes of functions $f: \Omega \to G$ which are $\mathcal{A}$-$\mathcal{B}(G)$ measurable and which are such that $\int_\Omega \|f\|_G^2 \, d\mathbb{P} < \infty$. The symbol $L(\mathbb{R}^m, \mathbb{R}^n)$ $(m, n \in \mathbb{N})$ denotes the space of all linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. The symbol J refers to the interval $[-1,0)$, and $\mathcal{H}(J)$ or $\mathcal{B}(J)$ refers to the Borel field on J.

If $X: [-1,T] \times \Omega \to \mathbb{R}^d$ is a process, then for each $t \in [0,T]$ and $\omega \in \Omega$ we define the map: $X_t: \Omega \to \mathcal{L}^2(J, \mathbb{R}^d)$ by $X_t(\omega)(s) = X(t+s, \omega)$ for all $s \in J$ and almost all ω. For each $0 \le t \le T$ we write $\|(X(t), X_t)\|^2 = \|X(t)\|^2 + \|X_t\|^2$. Let the function V belong to $\mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$, θ belong to $\mathcal{L}^2(J \times \Omega, \mathcal{H}(J) \otimes \mathcal{F}_0, \lambda \otimes \mathbb{P}; \mathbb{R}^d)$, and for $\ell = 1,2,\dots,r$ let f ,g^ℓ be functions from $[0,T] \times \Omega \times \mathbb{R}^d \times \mathcal{L}^2(J, \mathbb{R}^d)$ to $\mathbb{R}^d$. Then a process $X: [-1,T] \times \Omega \to \mathbb{R}^d$ is called a solution of the delay SDE with integral form

$$X(t) = \begin{cases} V + \int_0^t f(u, X(u), X_u) du + \sum_{\ell=1}^r \int_0^t g^\ell(u, X(u), X_u) \, dW^\ell(u), & 0 \le t \le T, \\ \theta(t), & t \in J, \end{cases} \tag{1.1}$$

If
(i) X is $\mathcal{B}([0,T]) \otimes \mathcal{F}$-$\mathcal{B}(\mathbb{R}^d)$ measurable;
(ii) For each$t \in [0,T]$, the process $X(t,\cdot)$ is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^d)$ measurable, and for each $t \in J$, the process $X(t,\cdot)$ is $\mathcal{F}_0$-$\mathcal{B}(\mathbb{R}^d)$ measurable;
(iii)$X \in \mathcal{L}^2([-1,T] \times \Omega, \mathcal{H} \times \mathcal{F}, \lambda \times \mathbb{P}; \mathbb{R}^d)$,
(IV) Xsatisfies the delay SDE(1.1).
The following conditions are sufficient for the existence of a unique solution to (1.1) (see [1] and [15]).
(i) $V \in \mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$.
(ii) $\theta \in \mathcal{L}^2(J \times \Omega, \mathcal{H} \otimes \mathcal{F}_0, \lambda \otimes \mathbb{P}, \mathbb{R}^d)$.
(iii) $f, g^\ell: [0,T] \times \Omega \times \mathbb{R}^d \times \mathcal{L}^2(J, \mathbb{R}^d) \to \mathbb{R}^d$ are such that
(a) $f$andg^ℓ are $\mathcal{B}([0,T]) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(L^2(J, \mathbb{R}^d))$-$\mathcal{B}(\mathbb{R}^d)$ measurable.
(b) For each $t \in [0,T]$, the stochastic variables $f(t,\cdot,\cdot,\cdot)$ and $g^\ell(t,\cdot,\cdot,\cdot)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{L}^2(J, \mathbb{R}^d))$-$\mathcal{B}(\mathbb{R}^d)$ measurable.
(c)There exists a constant K and a function $\zeta \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that

$$|f(t,\omega,s,h)| + \sum_{\ell=1}^r \left|g^\ell(t,\omega,s,h)\right| \le K(|s| + \|h\| + |\zeta(\omega)|) \text{ (1.2)} \qquad (1.2)$$

for almost all ω and for all $t \in [0,T]$; $s \in \mathbb{R}^d$ and h belongs to $\mathcal{L}^2(J, \mathbb{R}^d)$.
(d) There exists a constant K' such that, for almost all ω,

$$\begin{aligned} |f(t,\omega,s,h_1) - f(t,\omega,u,h_2)| + \sum_{\ell=1}^r \left|g^\ell(t,\omega,s,h_1) - g^\ell(t,\omega,u,h_2)\right| \\ \le K'(|s-u| + \|h_1 - h_2\|) \end{aligned} \qquad (1.3)$$

for all $t \in [0,T]$; for all $s, u \in \mathbb{R}^d$, and for all $h_1, h_2 \in \mathcal{L}^2(J, \mathbb{R}^d)$.

II. INTEGRATION BY PARTS FORMULA

In the beginning of this section we recall the following eight basic numbered equations and definitions, See (16) and (17). For$(X(0), X_0) = (x, \xi) \in \mathbb{R}^d \times L^2(J, \mathbb{R}^d)$, let$v \mapsto D^v X^{x,\xi}(t)$, be the Malliavin derivative of the solution process$X^{x,\xi}(t)$. We write $D^v X_t^{x,\xi}(\vartheta) = D^v X^{x,\xi}(t+\vartheta)$ $(t \in [0,T],\ \vartheta \in J = [-1,0))$ for its time delay. In the following definition we give a precise definition of the Malliavin derivative of a real-valued functional F of Brownian motion.

IDefinition: Let $F((W(s))_{0 \le s \le T})$ be a functional of r-dimensional Brownian motion, and let $v(t) = (v^1(t), \dots, v^r(t))^* = \begin{pmatrix} v^1(t) \\ \vdots \\ v^r(t) \end{pmatrix}$ be a deterministic vector-valued function in $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$. Then $D^v F((W(s))_{0 \le s \le T})$ is given by the limit:

$$\begin{aligned} & D^v F((W(s))_{0 \le s \le T}) \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(F\left(\left(W(s) + \varepsilon \int_0^s v(\sigma) d\sigma \right)_{0 \le s \le s} \right) - F((W(s))_{0 \le s \le t}) \right). \end{aligned} \qquad (2.1)$$

The mapping $v \mapsto D^v F((W(s))_{0 \le s \le T})$ is a linear map (functional) from the space $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)to\mathbb{R}$. Here $\mathbb{R}^r \otimes \mathbb{R}^d$ denotes the space of all $r \times d$-matrices (r rows, d columns).

Notice that, for $v(t) = (v^1(t), \dots, v^r(t))^\tau = \begin{pmatrix} v^1(t) \\ \vdots \\ v^r(t) \end{pmatrix}$ be a deterministic matrix-valued function in $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$, $U^v(t)$ can be considered as a $d \times d$-matrix where each entry is an $\mathbb{R}$-valued adapted stochastic process; U_t^v can be considered as a $d \times d$-matrix where each entry is an $L^2(J, \mathbb{R})$-valued adapted stochastic process. If $M = (m_{jk})_{1 \le j \le d,\, 1 \le k \le r}$is a real $d \times r$ matrix, then $M^\tau = (m_{kj})_{1 \le k \le r,\, 1 \le j \le d}$ denotes its transposed: it is $r \times d$ matrix with entriesm_{kj}.

The process $D^v X_t^{x,\xi}(\cdot)$ satisfies the following delay stochastic differential equation:

$$\begin{aligned} dD^v X_t(\vartheta) &= dD^v X(t+\vartheta) \\ &= \Big(\frac{\partial f}{\partial x}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) D^v X(t+\vartheta) \\ &\quad + \int_J \frac{\partial f}{\partial \xi}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})(\varphi) D^v X_{t+\vartheta}(\varphi)\, d\varphi\Big)\, dt \\ &\quad + \sum_{\ell=1}^r \frac{\partial g^\ell}{\partial x}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) D^v X(t+\vartheta) dW^\ell(t+\vartheta) \\ &\quad + \sum_{\ell=1}^r \int_J \frac{\partial g^\ell}{\partial \xi}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})(\varphi) D^v X_{t+\vartheta}(\varphi)\, d\varphi\, dW^\ell(t+\vartheta) \\ &\quad + \sum_{\ell=1}^r g^\ell(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) v^\ell(t+\vartheta, X(t+\vartheta), X_{t+\vartheta}) dt, \end{aligned} \tag{2.2}$$

Where ϑ belongs to J. If $t+\vartheta$ belongs to J we replace $t+\vartheta$ with 0 in (2.2). If $\vartheta = 0$ we obtain the delay stochastic differential equation for the process$D^v X(t)$:

$$\begin{aligned} &dD^v X(t) \\ &= \Big(\frac{\partial f}{\partial x}(t, X(t), X_t) D^v X(t) + \int_J \frac{\partial f}{\partial \xi}(t, X(t), X_t)(\vartheta) D^v X_t(\vartheta) d\vartheta\Big)\, dt \\ &+ \sum_{\ell=1}^r \Big(\frac{\partial g^\ell}{\partial x}(t, X(t), X_t) D^v X(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t, X(t), X_t)(\vartheta) D^v X_t(\vartheta) d\vartheta\Big)\, dW^\ell(t) \\ &+ \sum_{\ell=1}^r g^\ell(t, X(t), X_t) v^\ell(t, X(t), X_t) dt. \end{aligned} \tag{2.3}$$

We also write$U_{11}^{x,\xi}(t) = \frac{\partial}{\partial x} X^{x,\xi}(t)$, and$U_{12}^{x,\xi}(t) = \frac{\partial}{\partial \xi} X^{x,\xi}(t)$. In addition, we write $U_{21}^{x,\xi}(t) = \frac{\partial}{\partial x} X_t^{x,\xi} = U_{11,t}^{x,\xi}$ (the delay of$U_{11}^{x,\xi}(t)$), and$U_{22}^{x,\xi}(t) = \frac{\partial}{\partial \xi} X_t^{x,\xi} = U_{12,t}^{x,\xi}$, the delay of the process$U_{12}^{x,\xi}(t)$. The matrix $U_{11}^{x,\xi}(t)$ can be identified with an operator from $\mathbb{R}^d$ to itself, the matrix $U_{12}^{x,\xi}(t)$ can be considered as an linear mapping from $L^2(J, \mathbb{R}^d)$ to $\mathbb{R}^d$, the matrix $U_{21}^{x,\xi}(t)$ as a mapping from $\mathbb{R}^d$ to $L^2(J, \mathbb{R}^d)$, and, finally, $U_{22}^{x,\xi}(t)$ as a mapping from $L^2(J, \mathbb{R}^d)$ to itself. Notice that $U_{11}^{x,\xi}(t)$ can be considered as $d \times d$-matrix where each entry is an $\mathbb{R}$-valued adapted stochastic process; $U_{12}^{x,\xi}(t)$ can be considered as $d \times d$-matrix where each entry is an $L^2(J, \mathbb{R})$-valued adapted stochastic process. To be precise, write the solution process as a d-vector $X^{x,\xi}(t) = \left(X_1^{x,\xi}(t), \ldots, X_d^{x,\xi}(t)\right)$, and consider the mapping $(1 \le j, k \le d)$

$$\xi_k \to X_j^{x,(\xi_1,\ldots,\xi_{k-1},\xi_k,\xi_{k+1},\ldots,\xi_d)}(t), \tag{2.4}$$

Which is a mapping from $L^2(J, \mathbb{R})$to$\mathbb{R}$, and where each variableξ_ℓ, ,$l \neq k$, is a fixed function in $L^2(J, \mathbb{R})$. The derivative of the function in (2.4) can be considered as a continuous linear functional on$L^2(J, \mathbb{R})$. Therefore it can be represented as an inner-product with a function in $L^2(J, \mathbb{R})$, which is denoted by $\frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}$. Consequently, we write

$$\begin{aligned} \frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}(\eta) &= \lim_{h \to 0} \frac{X_j^{x,(\xi_1,\ldots,\xi_{k-1},\xi_k+h\eta,\xi_{k+1},\ldots,\xi_d)}(t) - X_j^{x,(\xi_1,\ldots,\xi_{k-1},\xi_k,\xi_{k+1},\ldots,\xi_d)}(t)}{h} \\ &= \int_J \eta(\varphi) \frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}(\varphi) d\varphi, \quad \eta \in L^2(J, \mathbb{R}). \end{aligned} \tag{2.5}$$

After giving a brief introduction to our work, we are now ready to continue the work that we have started in (16).

Here, and in the sequel, we write $f(t)$ and $g^\ell(t)$ instead of $f\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)$ and $g^\ell\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)$ respectively. For a concise formulation of the stochastic differential equation for the matrix-valued process $(U(t): t \ge)$ and its inverse we introduce the following *stochastic differentials*:

$$h_x(t) \quad = \frac{\partial f}{\partial x}(t)dt + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial x}(t) dW^\ell(t); \tag{2.6}$$

$$h_\xi(t) \quad = \frac{\partial f}{\partial \xi}(t)dt + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial \xi}(t)\, dW^\ell(t) \tag{2.7}$$

$$h_\xi(t,\vartheta) \quad = \frac{\partial f}{\partial \xi}(t,\vartheta)dt + \sum_{\ell=1}^{r} \frac{\partial g^\ell}{\partial \xi}(t,\vartheta) dW^\ell(t) \tag{2.8}$$

Advanced Applications of the Integration by Parts Formula:
Relevant SDE's are ($v(t)$ is a $r \times d$ matrix-valued adapted process: d columns, r rows)

$$dD^vX(t) \quad = h_x(t)D^vX(t) + \int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta + \sum_{\ell=1}^{r} g^\ell\,(t)v^\ell(t)^\tau\,dt; \tag{2.9}$$

$$dV^v(t) \quad = -V^v(t)h_x(t) - V^v(t)\int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}$$
$$+V^v(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell(t)}{\partial x} + \int_J \frac{\partial g^\ell(t,\vartheta)}{\partial \xi} D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}\right)^2 dt; \tag{2.10}$$

$$dU^v(t) \quad = h_x(t)U^v(t) + \int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}U^v(t). \tag{2.11}$$

2Theorem. Suppose the vector $v$is chosen in such a way that the Malliavin derivative $D^vX(t)$ is invertible. Put, for $1 \le \ell \le r$,

$$B^{v,0}(t) \quad = \sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)D^vX_t(\vartheta)(D^vX(t))^{-1}d\vartheta\right)^2 \tag{2.12}$$
$$-\int_J \frac{\partial f}{\partial \xi}(t,\vartheta)D^vX_t(\vartheta)\big(D^vX(t)\big)^{-1}d\vartheta - \frac{\partial f}{\partial x}(t);$$

$$B^{v,\ell}(t) \quad = \frac{\partial g^\ell}{\partial x}(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t,\vartheta)D^vX_t(\vartheta)\big(D^vX(t)\big)^{-1}d\vartheta. \tag{2.13}$$

Then

$$B^{v,0}(t) = \sum_{\ell=1}^{r}(B^{v,\ell}(t))^2 - \int_J \frac{\partial f}{\partial \zeta}(t,\vartheta)D^vX_t(\vartheta)\,(D^vX(t))^{-1}d\vartheta - \frac{\partial f}{\partial x}(t),$$

and

$$dV^v(t) = V^v(t)\,B^{v,0}(t)\,dt - V^v(t)\sum_{\ell=1}^{r} B^{v,\ell}(t)dW^\ell(t). \tag{2.14}$$

Suppose, in addition to the invertibilityof $D^vX(t)$, that the matrices $B^{v,\ell}(t)$, $\ell = 0,1,\dots,r$ are uniformly bounded. Under these hypotheses it follows that the inverse of the covariance matrix $\int_0^t V^v\,(s)\sum_{\ell=1}^{r} g^\ell\,(s)v^\ell(s)^\tau ds$ belongs to $L^p(\Omega,\mathcal{F},\mathbb{P})$ for all $1 \le p < \infty$ if and only if the inverse of the Malliavin derivative $D^vX(t)$ does so.

Proof. This assertion follows from the equality

$$(D^vX(t))^{-1} = \left(\int_0^t V^v(s)\sum_{\ell=1}^{r} g^\ell(s)v^\ell(s)^\tau ds\right)^{-1} V^v(t), \tag{2.15}$$

in conjunction with the equality:

$$\big(D^vX(t)\big)^{-1}U^v(t) = \left(\int_0^t V^v(s)\sum_{\ell=1}^{r} g^\ell(s)v^\ell(s)^\tau ds\right)^{-1} \tag{2.16}$$

From (2.14) together with the boundedness of the processes$B^{v,\ell}(t)$, $0 \le \ell \le r$, it follows that, for each $0 \le t \le T$ the stochastic variable $V^v(t)$ belongs to L^p for all $1 \le p < \infty$. Since

$$\begin{aligned} dU^v(t) &= h_x(t)U^v(t) + \int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}U^v(t) \\ &= \left(\frac{\partial f}{\partial x}(t) + \int_J \frac{\partial f}{\partial \xi}(t,\vartheta)D^vX_t(\vartheta)\big(D^vX(t)\big)^{-1}d\vartheta\right)U^v(t)\,dt \\ &\quad + \sum_{\ell=1}^{r} B^{v,\ell}(t)dW^\ell(t) \qquad (2.17) \end{aligned}$$

the same is true for the process $U^v(t)$. The assertion in the theorem then follows from (2.14),(2.15),(2.16),(2.17) and the arguments in Norris' paper.

3 Dfinition. In Stratonovich form our delay SDE (1.1) reads as follows ($\theta \in L^2(J,\mathbb{R}^d)$ is replaced with $\xi \in L^2(J,\mathbb{R}^d)$ and $X(0) = V$ is replaced with$x$):

$$dX(t) = g^0(t)dt + \sum_{\ell=1}^{r} g^\ell(t) \circ dW^\ell(t), \quad X(0) = x, \;\; X(\vartheta) = \xi(\vartheta), \text{for } \vartheta \in J,$$

where we write

$$g^0(t,x,\xi) = f(t,x,\xi) - \frac{1}{2}\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t,x,\xi)g^\ell(t,x,\xi).$$

Fix$(x,\xi) \in \mathbb{R}^d \times L^2(J,\mathbb{R}^d)$. Consider the span of the Lie brackets:

$$g^0(t),\, g^1(t), \dots, g^r(t); \quad [g^j(t), g^k(t)]_{j,k=0}^r; \quad \left[g^i(t),[g^j(t),g^k(t)]\right]_{i,j,k=0}^r;$$

and so on. If at the point (x,ξ) this span is all of $\mathbb{R}^d$, then we say that the coefficients $f(t)$, $g^1(t), \dots, g^r(t)$ satisfy Hörmander condition at the point (x,ξ).

4Theorem. Suppose that Hörmander's condition is satisfied at the point(x,ξ). Then there exists a $r \times d$ matrix $v = \begin{pmatrix} v^1 \\ \vdots \\ v^r \end{pmatrix}$ such that the Malliavin covariance matrix

$$\int_0^t V^v(s)\sum_{\ell=1}^{r} g^\ell(s)v^\ell(s)^\tau ds$$

is invertible with inverse in $L^p(\Omega,\mathcal{F},\mathbb{P})$ for all $1 \le p < \infty$. Here, for given v, the Malliavin derivative $D^vX(t)$, $V^v(t)$, and$U^v(t)$ satisfy the next stochastic differential equations respectively:

$$dD^vX(t) = h_x(t)D^vX(t) + \int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta + \sum_{\ell=1}^{r} g^\ell(t)v^\ell(t)^\tau\,dt; \qquad (2.18)$$

$$dV^v(t) = -V^v(t)h_x(t) - V^v(t)\int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1} \quad (2.19)$$

$$+V^v(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell(t)}{\partial x} + \int_J \frac{\partial g^\ell(t,\vartheta)}{\partial \xi}D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}\right)^2 dt; \qquad (2.20)$$

$$dU^v(t) = h_x(t)U^v(t) + \int_J h_\xi\,(t,\vartheta)D^vX_t(\vartheta)\,d\vartheta\big(D^vX(t)\big)^{-1}U^v(t). \qquad (2.21)$$

It turns out that, under appropriate initial value conditions, like $D^vX(0) = 0$, $V^v(0) = I$, and $U^v(0) = I$, the equalities

$$U^v(t)V^v(t) = I \text{ and } D^vX(t) = U^v(t)\int_0^t V^v(s)\sum_{\ell=1}^{r} g^\ell(s)v^\ell(s)^\tau ds \quad (2.22)$$

hold for $0 \le t \le T$.

Proof. For $\vartheta \in J$ we rewrite the expression $D^vX_t(\vartheta)D^vX(t)^{-1}$ as follows

$$D^v X_t(\vartheta) D^v X(t)^{-1} = D^v X(t+\vartheta) D^v X(t)^{-1}$$
$$= U^v(t+\vartheta)\int_0^{(t+\vartheta)\vee 0} V^v(s)\sum_{\ell=1}^{r} g^\ell(s) v^\ell(s)^\tau ds \left(\int_0^t V^v(s)\sum_{\ell=1}^{r} g^\ell(s) v^\ell(s)^\tau ds\right)^{-1} V^v(t)$$
$$= U^v(t+\vartheta) C^v(t+\vartheta) C^v(t)^{-1} V^v(t), \qquad (2.23)$$

where $C^v(t) = \int_0^t V^v(s)\sum_{\ell=1}^{r} g^\ell(s) v^\ell(s)^\tau ds$. Substituting (2.23) in the equations (2.20) and (2.21) and using (2.22) we obtain:

$$dV^v(t) = -V^v(t) h_x(t) - V^v(t)\int_J h_\xi(t,\vartheta) U^v(t+\vartheta) C^v(t+\vartheta) C^v(t)^{-1} V^v(t)\, d\vartheta$$
$$+ V^v(t)\sum_{\ell=1}^{r}\left(\frac{\partial g^\ell(t)}{\partial x} + \int_J \frac{\partial g^\ell(t,\vartheta)}{\partial \xi} U^v(t+\vartheta) C^v(t+\vartheta) C^v(t)^{-1} V^v(t)\, d\vartheta\right)^2 dt; \quad (2.24)$$
$$dU^v(t) = h_x(t) U^v(t) + \int_J h_\xi(t,\vartheta) U^v(t+\vartheta) C^v(t+\vartheta) C^v(t)^{-1}\, d\vartheta. \qquad (2.25)$$

Fix. $t > 0$. We need the following lemma about the family of matrices $U^v(s)C^v(s)$, $0 \le s \le t$.
5Lemma. Suppose that there exists a real constant C_t such that for every $0 \le s_1 < s_2 \le t$ and for every $x \in \mathbb{R}^n$ the following inequality holds:

$$-2\langle U^v(s_1)C^v(s_1)x, (U^v(s_2)C^v(s_2) - U^v(s_1)C^v(s_1)x\rangle$$
$$\le \frac{C_t^2}{C_t^2+1}\|U^v(s_1)C^v(s_1)x\|^2 + \|U^v(s_2)C^v(s_2) - U^v(s_1)C^v(s_1)x\|^2. \quad (2.26)$$

Then, for all $0 \le s_1 \le s_2 \le t$, the next inequality is valid:

$$\|U^v(s_1)C^v(s_1)C^v(s_2)^{-1}U^v(s_2)^{-1}\|^2 \le C_t^2 + 1. \qquad (2.27)$$

Here the space $\mathbb{R}^n$ is equipped with the Euclidean norm and $\|T\|$ stands for the corresponding operator norm of the matrix T considered as an operator on the Hilbert space $\mathbb{R}^n$.
Remark. The inequality in (2.26) says that for no $x \in \mathbb{R}^n$ the vector

$$\left(U^v(s_2)C^v(s_2) - U^v(s_1)C^v(s_1)\right)x$$

can be written as a negative scalar multiple of the vector $C^v(s_1)x$; in fact the angle between these two vectors is always strictly positive. More presicely, write

$$\langle (U^v(s_2)C^v(s_2) - U^v(s_1)C^v(s_1))x, U^v(s_1)C^v(s_1)x\rangle$$
$$= \|U^v(s_2)C^v(s_2) - U^v(s_1)C^v(s_1)x\|\|U^v(s_1)C^v(s_1)x\|\cos(\omega(s_1,s_2,x)) \quad (2.28)$$

Then (2.26) is satisfied, provided

$$-\cos\left(\omega(s_1,s_2,x)\right) \le \frac{C_t^2}{C_t^2+1}.$$

Proof. Fix $0 \le s_1 < s_2 \le t$. Then we have by inequality (2.26)

$$(C_t^2+1)\langle U^v(s_2)C^v(s_2)x, U^v(s_2)C^v(s_2)x\rangle - \langle U^v(s_1)C^v(s_1)x, U^v(s_1)v(s_1)x\rangle$$
$$= (C_t^2+1)\{\|U^v(s_2)C^v(s_2)x - U^v(s_1)C^v(s_1)x\|^2 + \|U^v(s_1)C^v(s_1)x\|^2$$
$$+2\Re\langle U^v(s_2)C^v(s_2)x - U^v(s_1)C^v(s_1)x, U^v(s_1)C^v(s_1)x\rangle \} - \|C^v(s_1)x\|^2$$
$$= (C_t^2+1)\left\{\|U^v(s_2)C^v(s_2)x - U^v(s_1)C^v(s_1)x\|^2 + \frac{C_t^2}{C_t^2+1}\|U^v(s_1)C^v(s_1)x\|^2\right.$$
$$\left.+2\Re\langle U^v(s_2)C^v(s_2)x - U^v(s_1)C^v(s_1)x, U^v(s_1)C^v(s_1)x\rangle \right\} \ge 0.$$
(2.29)
From (2.29) we obtain the operator inequality in Hilbert space:

$C^v(s_1)^*U^v(s_1)^*U^v(s_1)C^v(s_1) \leq (C_t^2+1)C^v(s_2)^*U^v(s_2)^*U^v(s_2)C^v(s_2)$ (2.30)

Then we have

$$\begin{gathered}\|U^v(s_1)C^v(s_1)C^v(s_2)^{-1}U^v(s_2)^{-1}\|^2\\ = \|(U^v(s_2)^*)^{-1}(C^v(s_2)^*)^{-1}C^v(s_1)^*U^v(s_1)^*U^v(s_1)C^v(s_1)C^v(s_2)^{-1}U^v(s_2)^{-1}\|\\ \leq (C_t^2+1)\|(C^v(s_2)^*U^v(s_2)^*)^{-1}C^v(s_2)^*U^v(s_2)^*U^v(s_2)C^v(s_2)C^v(s_2)^{-1}U^v(s_2)^{-1}\|\\ = C_t^2+1.\end{gathered} \tag{2.31}$$

The inequality in (2.31) implies inequality (2.27) in Lemma5.

Remarks:

1. All the results which we have established in this work can be extended by replacing the Brownian motion W by another process$Z: [0,a]\times \Omega \to \mathbf{R^d}$, $(d \in \mathbf{N})$ which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant K the inequalities $|Z(t)-Z(s)\mathcal{F}_s| \leq K(t-s)$ and $\mathbf{E}(|Z(t)-Z(s)|^2\mathcal{F}_s \leq K(t-s)$for $0 \leq s \leq t \leq a$. Observe that the above properties of Z which we have just mentioned are the only properties of W which we have used (in case of Brownian motion) to prove the results which we have obtained in this work.See [1], [15], [16], [17], [18], [19] and [20].

2. All the lemmas and theorems in this work hold for any delay interval $J' = [-r,0)$ $(r \geq 0)$ inplace of $J = [-1,0)$. See [1], [15], [16], [17], [18], [19] and [20].

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