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Elzaki Transform Homotopy Perturbation Method for Solving Fourth Order Parabolic PDE with Variable Coefficients

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ABSTRACT
In this paper, we apply a new method called ELzaki transform homotopy perturbation method (ETHPM) to solve one dimensional fourth order parabolic linear partial differential equations with variable coefficients. This method is a combination of the new integral transform "ELzaki transform" and the homotopy perturbation method. Some cases of one dimensional fourth order parabolic linear partial differential equations are solved to illustrate ability and reliability of mixture of ELzaki transform and homotopy perturbation method. We have compared the obtained analytical solution with the available Laplace decomposition solution and homotopy perturbation method solution which is found to be exactly same. The results reveal that the combination of ELzaki transform and homotopy perturbation method is quite capable, practically well appropriate for use in such problems.
Keywords – *ELzaki transform, homotopy perturbation method, linear partial differential equation.*

I. INTRODUCTION

Many problems of physical interest are described by linear partial differential equations with initial and boundary conditions. One of them is fourth order parabolic partial differential equations with variable coefficients; these equations arise in the transverse vibration problems[1]. In recent years, many research workers have paid attention to find the solution of these equations by using various methods. Among these are the variational iteration method [Biazar and Ghazvini (2007)], Adomian decomposition method [Wazwaz (2001) and Biazar et al (2007)], homotopy perturbation method [Mehdi Dehghan and Jalil Manafian (2008)], homotopy analysis method [Najeeb Alam Khan, Asmat Ara, Muhammad Afzal and Azam Khan (2010)] and Laplace decomposition algorithm [Majid Khan, Muhammad Asif Gondal and Yasir Khan (2011)]. In this paper we use coupling of new integral transform "ELzaki transform" and homotopy perturbation method. This method is a useful technique for solving linear and nonlinear differential equations. The main aim of this paper is to consider the effectiveness of the ELzaki transform homotopy perturbation method in solving higher order linear partial differential equations with variable coefficients. This method provides the solution in a rapid convergent series which leads the solution in a closed form.

II. ELZAKI TRANSFORM HOMOTOPY PERTURBATION METHOD [1, 2, 3, 4]

Consider a one dimensional linear nonhomogeneous fourth order parabolic partial differential equation with variable coefficients of the form

$$\frac{\partial^2 u}{\partial t^2} + \psi(x)\frac{\partial^4 u}{\partial x^4} = \phi(x,t), \quad (1)$$

where $\psi(x)$ is a variable coefficient, with the following initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = h(x). \quad (2)$$

and boundary conditions are

$$u(a,t) = \beta_1(t), u(b,t) = \beta_2(t),$$
$$\frac{\partial^2 u}{\partial x^4}(a,t) = \beta_3(t), \frac{\partial^2 u}{\partial x^4}(b,t) = \beta_4(t), \quad (3)$$

Apply ELzaki transform on both sides of Eq. (1)

$$E\left[\frac{\partial^2 u}{\partial t^2} + \psi(x)\frac{\partial^4 u}{\partial x^4}\right] = E[\phi(x,t)], \quad (4)$$

But, ELzaki transform of partial derivative is given by [3]

$$E\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right] = \frac{1}{v^2}E[f(x,t)] - f(x,0) - v\frac{\partial f}{\partial t}(x,0),$$

Using this property, Eq. (3) can be written as

$$\frac{1}{v^2}E[u(x,t)] - u(x,0) - v\frac{\partial u}{\partial t}(x,0) = E[\phi(x,t)] - E\left[\psi(x)\frac{\partial^4 u}{\partial x^4}\right], \quad (5)$$

Put the values of initial conditions in Eq. (5), we get

$$\frac{1}{v^2}E[u(x,t)] - f(x) - vh(x) = E[\phi(x,t)] - E\left[\psi(x)\frac{\partial^4 u}{\partial x^4}\right],$$

By simple calculations, we have

$$E[u(x,t)] = g(x,v) - v^2 E\left[\psi(x)\frac{\partial^4 u}{\partial x^4}\right], \quad (6)$$

where $g(x,v) = v^2 f(x) + v^3 h(x) + E[\phi(x,t)]$.

Applying ELzaki inverse on both sides of Eq. (6), we get

$$u(x,t) = K(x,t) - E^{-1}\left\{v^2 E\left[\psi(x)\frac{\partial^4 u}{\partial x^4}\right]\right\}, \qquad (7)$$

where $K(x,t) = E^{-1}\{g(x,v)\}$, represents the term arising from the source term and the prescribed initial conditions.

Now, we apply the homotopy perturbation method.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \qquad (8)$$

By substituting Eq. (8) into Eq. (7), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = K(x,t)$$
$$-pE^{-1}\left\{v^2 E\left[\psi(x)\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xxxx}\right]\right\}, \qquad (9)$$

This is the coupling of the ELzaki transform and the homotopy perturbation method. Comparing the coefficient of like powers of p, the following approximations are obtained.

$p^0: u_0(x,t) = K(x,t)$,
$p^1: u_1(x,t) = -E^{-1}\{v^2 E[\psi(x)u_{0xxxx}(x,t)]\}$,
$p^2: u_2(x,t) = -E^{-1}\{v^2 E[\psi(x)u_{1xxxx}(x,t)]\}$,
$p^3: u_3(x,t) = -E^{-1}\{v^2 E[\psi(x)u_{2xxxx}(x,t)]\}$,
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In general recursive relation is given by,

$p^m: u_m(x,t) = -E^{-1}\{v^2 E[\psi(x)u_{(m-1)xxxx}(x,t)]\}$,

Then the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots \dots \dots \qquad (10)$$

III. APPLICATION

To demonstrate the effectiveness of the method we have solved homogeneous and nonhomogeneous one dimensional fourth order linear partial differential equations with initial and boundary conditions.

Example 1. Consider fourth order homogeneous partial differential equation as [1, 5]

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1, t > 0 \qquad (11)$$

with the following initial conditions

$$u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \qquad (12)$$

and boundary conditions

$$u(0.5,t) = \left(\frac{1+(0.5)^5}{120}\right)\sin t, u(1,t) = \frac{121}{120}\sin t,$$
$$\frac{\partial^2 u}{\partial x^2}(0.5,t) = 0.02084\sin t, \frac{\partial^2 u}{\partial x^2}(1,t) = \frac{1}{6}\sin t. \qquad (13)$$

Applying ELzaki transform to Eq. (11), we get

$$E\left[\frac{\partial^2 u}{\partial t^2}\right] + E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}\right] = 0,$$

$$\frac{1}{v^2}E[u(x,t)] - u(x,0) - v\frac{\partial u}{\partial t}(x,0) + E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}\right] = 0, \qquad (14)$$

Using initial conditions from Eq. (12), we get

$$E[u(x,t)] = v^3\left(1 + \frac{x^5}{120}\right) - v^2 E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}\right], \qquad (15)$$

Now taking ELzaki inverse on both sides of above Eq. (15), we have

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)t - E^{-1}\left\{v^2 E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}\right]\right\}, \qquad (16)$$

Now, we apply the homotopy perturbation method.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t),$$

Putting this value of $u(x,t)$ into Eq. (16), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \left(1 + \frac{x^5}{120}\right)t$$
$$-pE^{-1}\left\{v^2 E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xxxx}\right]\right\} \qquad (17)$$

Comparing the coefficient of like powers of p, in Eq. (17) the following approximations are obtained.

$$p^0: u_0(x,t) = K(x,t) = \left(1 + \frac{x^5}{120}\right)t,$$
$$p^1: u_1(x,t) = -E^{-1}\left\{v^2 E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{0xxxx}(x,t)\right]\right\} = -\left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!},$$
$$p^2: u_2(x,t) = -E^{-1}\left\{v^2 E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{1xxxx}(x,t)\right]\right\} = \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!},$$
$$p^3: u_3(x,t) = -E^{-1}\left\{v^2 E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{2xxxx}(x,t)\right]\right\} = -\left(1 + \frac{x^5}{120}\right)\frac{t^7}{7!},$$
$$\vdots$$
$$p^m: u_m(x,t) = -E^{-1}\left\{v^2 E\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{(m-1)xxxx}\right]\right\} = (-1)^m\left(1 + \frac{x^5}{120}\right)\frac{t^{2m+1}}{(2m+1)!},$$
$$m = 1,2,3, \dots \dots \dots .$$

And so on in the same manner the rest of the components of iteration formula can be obtained and thus solution can be written in closed form as

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)t - \left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!} + \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!} - \left(1 + \frac{x^5}{120}\right)\frac{t^7}{7!} + \cdots \dots \dots \dots ..$$

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots..\right)$$

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\text{sint}, \quad (18)$$

which is an exact solution of Eq. (11) and can be verified through substitution.

Example 2. Consider fourth order homogeneous partial differential equation as [5]

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{x}{\text{sinx}}-1\right)\frac{\partial^4 u}{\partial x^4}=0, \quad 0<x<1, t>0 \quad (19)$$

with the following initial conditions

$$u(x,0)=x-\text{sinx}, \frac{\partial u}{\partial t}(x,0)=-x+\text{sinx}, \quad (20)$$

and boundary conditions

$$u(0,t)=0, u(1,t)=e^{-t}(1-\text{sin}1),$$

$$\frac{\partial^2 u}{\partial x^2}(0,t)=0, \frac{\partial^2 u}{\partial x^2}(1,t)=e^{-t}\text{sin}1. \quad (21)$$

Applying ELzaki transform to Eq. (19), we get

$$E\left[\frac{\partial^2 u}{\partial t^2}\right]+E\left[\left(\frac{x}{\text{sinx}}-1\right)\frac{\partial^4 u}{\partial x^4}\right]=0,$$

$$\frac{1}{v^2}E[u(x,t)]-u(x,0)-v\frac{\partial u}{\partial t}(x,0) + E\left[\left(\frac{x}{\text{sinx}}-1\right)\frac{\partial^4 u}{\partial x^4}\right]=0, \quad (22)$$

Using initial conditions from Eq. (20), we get

$$E[u(x,t)]=v^2(x-\text{sinx})+v^3(-x+\text{sinx}) - v^2 E\left[\left(\frac{x}{\text{sinx}}-1\right)\frac{\partial^4 u}{\partial x^4}\right], \quad (23)$$

Now taking ELzaki inverse on both sides of above Eq. (23), we have

$$u(x,t)=(x-\text{sinx})+(-x+\text{sinx})t -E^{-1}\left\{v^2 E\left[\left(\frac{x}{\text{sinx}}-1\right)\frac{\partial^4 u}{\partial x^4}\right]\right\}, \quad (24)$$

Now, we apply the homotopy perturbation method.

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t),$$

Putting this value of $u(x,t)$ into Eq. (24), we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=(x-\text{sinx})+(-x+\text{sinx})t -pE^{-1}\left\{v^2 E\left[\left(\frac{x}{\text{sinx}}-1\right)\left(\sum_{n=0}^{\infty}p^n u_n(x,t)\right)_{xxxx}\right]\right\}, \quad (25)$$

Comparing the coefficient of like powers of p , in Eq. (25) the following approximations are obtained.

$$p^0: u_0(x,t)=K(x,t)=(x-\text{sinx})+(-x+\text{sinx})t$$

$$p^1: u_1(x,t)=-E^{-1}\left\{v^2 E\left[\left(\frac{x}{\text{sinx}}-1\right)u_{0xxxx}(x,t)\right]\right\} = (x-\text{sinx})\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right),$$

$$p^2: u_2(x,t)=-E^{-1}\left\{v^2 E\left[\left(\frac{x}{\text{sinx}}-1\right)u_{1xxxx}(x,t)\right]\right\} = (x-\text{sinx})\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right),$$

$$p^3: u_3(x,t)=-E^{-1}\left\{v^2 E\left[\left(\frac{x}{\text{sinx}}-1\right)u_{2xxxx}(x,t)\right]\right\} = (x-\text{sinx})\left(\frac{t^6}{6!}-\frac{t^7}{7!}\right),$$

$$\vdots$$

$$p^m: u_m(x,t)=-E^{-1}\left\{v^2 E\left[\left(\frac{x}{\text{sinx}}-1\right)u_{(m-1)xxxx}\right]\right\} = (x-\text{sinx})\left(\frac{t^{2m}}{2m!}-\frac{t^{2m+1}}{(2m+1)!}\right),$$

$$m=1,2,3,4\ldots\ldots..$$

And so on in the same manner the rest of the components of iteration formula can be obtained and thus solution can be written in closed form as

$$u(x,t)=(x-\text{sinx})+(-x+\text{sinx})t +(x-\text{sinx})\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right)+(x-\text{sinx})\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right) +(x-\text{sinx})\left(\frac{t^6}{6!}-\frac{t^7}{7!}\right)\ldots\ldots\ldots\ldots..$$

$$u(x,t)=(x-\text{sinx})\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+..\right),$$

$$u(x,t)=(x-\text{sinx})e^{-t}, \quad (26)$$

which is an exact solution of Eq. (19) and can be verified through substitution.

Example 3. Consider fourth order nonhomogeneous partial differential equation as [1, 5]

$$\frac{\partial^2 u}{\partial t^2}+(1+x)\frac{\partial^4 u}{\partial x^4}=\left(x^4+x^3-\frac{6}{7!}x^7\right)\text{cost}, \quad 0<x<1, t>0 \quad (27)$$

with the following initial conditions

$$u(x,0)=\frac{6}{7!}x^7, \frac{\partial u}{\partial t}(x,0)=0. \quad (28)$$

and boundary conditions

$$u(0,t)=0, u(1,t)=\frac{6}{7!}\text{cost},$$

$$\frac{\partial^2 u}{\partial x^2}(0,t)=0, \frac{\partial^2 u}{\partial x^2}(1,t)=\frac{1}{20}\text{cost}. \quad (29)$$

Applying ELzaki transform to Eq. (27), we get

$$E\left[\frac{\partial^2 u}{\partial t^2}\right]+E\left[(1+x)\frac{\partial^4 u}{\partial x^4}\right] = E\left[\left(x^4+x^3-\frac{6}{7!}x^7\right)\text{cost}\right],$$

$$\frac{1}{v^2}E[u(x,t)]-u(x,0)-v\frac{\partial u}{\partial t}(x,0) =\left(x^4+x^3-\frac{6}{7!}x^7\right)\frac{v^2}{1+v^2}-E\left[(1+x)\frac{\partial^4 u}{\partial x^4}\right],$$

Using initial conditions from Eq. (28), we get

$$E[u(x,t)]=v^2\frac{6}{7!}x^7+\left(x^4+x^3-\frac{6}{7!}x^7\right)\frac{v^4}{1+v^2} - v^2 E\left[(1+x)\frac{\partial^4 u}{\partial x^4}\right], \quad (30)$$

Now taking ELzaki inverse on both sides of above Eq. (30), we have

$$u(x,t)=\frac{6}{7!}x^7+\left(x^4+x^3-\frac{6}{7!}x^7\right)(1-\text{cost}) -E^{-1}\left\{v^2 E\left[(1+x)\frac{\partial^4 u}{\partial x^4}\right]\right\}, \quad (31)$$

Now, we apply the homotopy perturbation method.

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t),$$

Putting this value of $u(x,t)$ into Eq. (31), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \frac{6}{7!}x^7 + \left(x^4 + x^3 - \frac{6}{7!}x^7\right)(1 - \cos t) - pE^{-1}\left\{v^2 E\left[(1+x)\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xxxx}\right]\right\} \quad (32)$$

Here, we assume that

$$K(x,t) = \frac{6}{7!}x^7 + \left(x^4 + x^3 - \frac{6}{7!}x^7\right)(1 - \cos t)$$

can be divided into the sum of two parts namely $K_0(x,t)$ and $K_1(x,t)$, therefore we get [6]

$$K(x,t) = K_0(x,t) + K_1(x,t)$$

Under this assumption, we propose a slight variation only in the components u_0, u_1. The variation we propose is that only the part $K_0(x,t)$ be assigned to the u_0, whereas the remaining part $K_1(x,t)$, be combined with the other terms to define u_1.

$$K_0(x,t) = \frac{6}{7!}x^7 \cos t\,;\; K_1(x,t) = (x^4 + x^3)(1 - \cos t)$$

In view of these, we formulate the modified recursive algorithm as follows:

$$p^0: u_0(x,t) = \frac{6}{7!}x^7 \cos t,$$

$$p^1: u_1(x,t) = (x^4 + x^3)(1 - \cos t) - E^{-1}\{v^2 E[(1+x)u_{0xxxx}(x,t)]\} = 0,$$

$$p^2: u_2(x,t) = -E^{-1}\{v^2 E[(1+x)u_{1xxxx}(x,t)]\} = 0,$$

And so on in the same manner the rest of the components of iteration formula can be obtained

$$u_m(x,t) = 0, \text{for } m \geq 1.$$

Thus solution can be written in closed form as

$$u(x,t) = \frac{6}{7!}x^7 \cos t + 0 + 0 + 0 + 0 + \cdots \ldots$$

$$u(x,t) = \frac{6}{7!}x^7 \cos t. \quad (33)$$

which is an exact solution of Eq. (27) and can be verified through substitution.

IV. CONCLUSION

The main goal of this paper is to show the applicability of the mixture of new integral transform "ELzaki transform" with the homotopy perturbation method to solve one dimensional fourth order homogeneous and nonhomogeneous linear partial differential equations with variable coefficients. This combination of two methods successfully worked to give very reliable and exact solutions to the equation. This method provides an analytical approximation in a rapidly convergent sequence with in exclusive manner computed terms. Its rapid convergence shows that the method is trustworthy and introduces a significant improvement in solving linear partial differential equations over existing methods.

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