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On the Estimation of $P\ (Y < X < Z)$for Weibull Distribution in the Presence of k Outliers

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ABSTRACT
This paper deals with the estimation problem of reliability $\boldsymbol{R = P(Y < X < Z)}$, where X, Y and Z are independently Weibull distribution with common known shape parameter but with different scale parameters, in presence of k outliers in strength X. The moment, maximum likelihood and mixture estimators of R are derived. An extensive computer simulation is used to compare the performance of the proposed estimators using MathCAD (14). Simulation study showed that the mixture estimators are better and easier than the maximum likelihood and moment estimators.
Keywords-Maximum likelihood estimator, mixture estimator, moment estimator, outliers and Weibull distribution.

I. Introduction

In the context of reliability the stress-strength model describes the life of a component which has a random strength X and is subjected to random stress Y. This problem arises in the classical stress–strength reliability where one is interested in estimating the proportion of the times the random strength X of a component exceeds the random stress Y to which the component is subjected. If $X \leq Y$, then either the component fails or the system that uses the component may malfunction, and there is no failure when $Y<X$. The stress-strength models of the types $P(Y<X)$, $P(Y<X<Z)$ have extensive applications in various subareas of engineering, psychology, genetics, clinical trials and so on. [See,Kotz[1]].

The germ of this idea was introduced by Birnbaum [2] and developed by Birnbaum and McCarty [3]. An important particular case is estimation of $R= P(Y<X<Z)$ which represents the situation where the strength X should not only be greater than stress Y but also be smaller than stress Z. For example, many devices cannot function at high temperatures; neither can do at very low ones. Similarly, person's blood pressure has two limits systolic and diastolic and his/her blood pressure should lie within these limits.

Chandra and Owen [4] constructed maximum likelihood estimators (MLEs) and uniform minimum unbiased estimators (UMVUEs) for $R= P(Y<X< Z)$. Singh [5] presented the minimum variance unbiased, maximum likelihood and empirical estimators of $R= P(Y<X<Z)$, where X, Y and Z are mutually independent random variables and follow the normal distribution. Dutta and Sriwastav[6] dealt with the estimation of R when X, Y and Z are exponentially distributed. Ivshin[7] investigated the MLE and UMVUE of R when X, Y and Z are either uniform or exponential random variables with unknown location parameters.

Wang et al. [8] make statistical inference for $P(X<Y<Z)$ via two methods, the nonparametric normal approximation and the jackknife empirical likelihood, since the usual empirical likelihood method for U-statistics is too complicated to apply. The results of the simulation studies indicate that these two methods work promisingly compared to other existing methods. Some classical and real data sets were analyzed using these two proposed methods.

The main aim of this article is to focus on the estimation of $R= P(Y<X<Z)$, under the assumption that, X, Y and Z are independent. The stresses Y and Z have Weibull distribution with common known shape parameter β and scale parameters α and θ respectively. While the strength X has Weibull distribution with known shape parameter β and scale parameter γ in presence of k outliers. Maximum likelihood estimator, moment estimator (ME) and mixture estimator (Mix) are obtained. Monte Carlo simulation is performed for comparing different methods of estimation.

The rest of the paper is organized as follows. In Section 2, the estimation of $R= P(Y<X< Z)$ will be derived. The moment estimator of R derived in Section 3. Section 4 discussed MLE of R. The mixture estimator of R is obtained in Section 5. Monte Carlo simulation results are laid out in Section 6. Finally, conclusionsare presented in Section 7.

II. Estimation of $R = P(Y < X < Z)$

This Section deals with estimate the reliability of $R$where Z, Y and X have Weibull distribution in the presence of k outliers in the strength X, such that X, Y and Z are independent.

Let $X = (X_1, X_2, \ldots\ldots, X_{n_1})$ is the strength of n_1 independent observations such that k of them are distributed Weibully with scale parameter δ, shape parameter $\beta$and has the following probability density function (*pdf*),

$$f_1(x;\ \delta,\beta) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} e^{-\left(\frac{x}{\delta}\right)^{\beta}}\ x > 0, \beta > 0\ and\ \delta > 0, \qquad (1)$$

while the remaining $(n_1 - k)$random variables are distributed Weibully with scale parameterγ, known shape parameter $\beta$and has the following *pdf*,

$$f_2(x;\ \gamma,\beta) = \frac{\beta}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1} e^{-\left(\frac{x}{\gamma}\right)^{\beta}}\ x > 0,\ \beta,\gamma > 0 \qquad (2)$$

According to Dixit [9] and Dixit and Nasiri [10], the joint distribution of$(X_1, X_2, \ldots\ldots, X_{n_1})$in the presence of k outliers, can be expressed as

$$f_X(x;\ \gamma,\delta,\beta) = \frac{k}{n}\frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} e^{-\left(\frac{x}{\delta}\right)^{\beta}} + \frac{(n-k)}{n}\frac{\beta}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1} e^{-\left(\frac{x}{\gamma}\right)^{\beta}}\quad x,\beta,\gamma,\delta > 0 \qquad (3)$$

The corresponding cumulative distribution function (*cdf*)

$$F_X(x;\ \gamma,\delta,\beta) = b\left(1 - e^{-\left(\frac{x}{\delta}\right)^{\beta}}\right) + \bar{b}\left(1 - e^{-\left(\frac{x}{\gamma}\right)^{\beta}}\right), \qquad (4)$$

where

$$\frac{k}{n} = b, \qquad \frac{(n-k)}{n} = \bar{b}, \quad \text{and} \qquad b + \bar{b} = 1$$

Let $Y = (Y_1, Y_2, \ldots\ldots, Y_{n_2})$ is the stress of n_2 independent observations of Weibull distribution with scale parameterα, known shape parameter $\beta$and has the following *pdf*,

$$g_Y(y;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{\beta-1} e^{-\left(\frac{y}{\alpha}\right)^{\beta}}\ y,\alpha,\beta, > 0 \qquad (5)$$

In addition, let $Z = (Z_1, Z_2, \ldots\ldots, Z_{n_3})$ is the stress of n_3 independent observations of random Z with known shape parameter β and scale parameterθ,and has the following *pdf*,

$$h_Z(z;\ \theta,\beta) = \frac{\beta}{\theta}\left(\frac{z}{\theta}\right)^{\beta-1} e^{-\left(\frac{z}{\theta}\right)^{\beta}} \qquad (6)$$

According to Singh [5], the reliability $R = P(Y < X < Z)$ takes the following form

$$P(Y < X < Z) = \int_{-\infty}^{\infty} G_Y(x)\overline{H}_Z(x)\, dF_X(x),$$

where $H_Z(x)$ is the *cdf* of Z at x, $G_Y(x)$ is the *cdf* of Y at x and $\overline{H}_Z(x)$ is the survival function of Z at x.

Then, the reliability R in the presence of k outliers is given by

$$R = \int_0^{\infty}\left[1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right] e^{-\left(\frac{x}{\theta}\right)^{\beta}}\left[\frac{b\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} e^{-\left(\frac{x}{\delta}\right)^{\beta}} + \frac{\bar{b}\beta}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1} e^{-\left(\frac{x}{\gamma}\right)^{\beta}}\right] dx$$

$$R = \frac{b(\delta\theta^2)^{\beta}}{[\delta^{\beta} + \theta^{\beta}][(\theta\alpha)^{\beta} + (\delta\alpha)^{\beta} + (\delta\theta)^{\beta}]} + \frac{\bar{b}(\gamma\theta^2)^{\beta}}{[\gamma^{\beta} + \theta^{\beta}][(\theta\alpha)^{\beta} + (\gamma\alpha)^{\beta} + (\gamma\theta)^{\beta}]} \qquad (7)$$

Now to compute the estimate of R, the estimate of the parameters δ, α, θ and γ must be obtained firstly. Three well known methods, namely, maximum likelihood, moments and mixture will be used to obtain the estimate of the parameters and will be discussed in details the following Sections.

III. Moment Estimator of *R*

In this Section, the moment estimator of R denoted byR_{ME}will be obtained. The moment estimators $\hat{\theta}_1, \hat{\alpha}_1, \hat{\gamma}_1and\hat{\delta}_1$ of the unknown parameters θ, α, γ and δ will be obtained by equating the population moments with the corresponding sample moments.

The population means ofrandom stresses Y and Z are given by

$$\acute{\mu}_y = E(y) = \alpha\Gamma\left(\frac{1}{\beta} + 1\right),$$

$$\acute{\mu}_z = E(z) = \theta\Gamma\left(\frac{1}{\beta} + 1\right)$$

In addition the first and second population moments of strength X are given by,

$$\acute{\mu}_x = \left(b\delta + \bar{b}\gamma\right)\Gamma\left(\tfrac{1}{\beta} + 1\right),$$

$$\grave{\mu}^2{}_x = \left(b\delta^2 + \bar{b}\gamma^2\right)\Gamma\left(\tfrac{2}{\beta} + 1\right).$$

Suppose that $Y = (Y_1, Y_2, \ldots\ldots, Y_{n_2})$be a random sample of size n_2 and $Z = (Z_1, Z_2, \ldots\ldots, Z_{n_3})$ be a random sample of size $n_3$drawn from Weibull distribution with known shape parameter β and scale parameterα and θ respectively. Then the means of these samples are given by

$$\bar{y} = \frac{1}{n_2}\sum_{i=1}^{n_2} y_i, \qquad \bar{z} = \frac{1}{n_3}\sum_{i=1}^{n_3} z_i$$

Let $X_1, X_2, \ldots\ldots, X_{n_1}$ be a random sample of size n_1 drawn from Weibull distribution with known shape parameter β and scale parameterδ, then the first and second sample moments are given by

$$m_1' = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i, \qquad m_2' = \frac{1}{n_1}\sum_{i=1}^{n_1} {x_i}^2$$

By equating the samples moments with the corresponding population moments, then

$$\bar{y} = \alpha\Gamma\left(\frac{1}{\beta}+1\right) \quad , \qquad \bar{z} = \theta\Gamma\left(\frac{1}{\beta}+1\right) \quad (8)$$

$$\bar{x} = \left(b\delta + \bar{b}\gamma\right)\Gamma\left(\frac{1}{\beta}+1\right) \quad (9)$$

$$m_2' = \left(b\delta^2 + \bar{b}\gamma^2\right)\Gamma\left(\frac{2}{\beta}+1\right) \quad (10)$$

The moment estimator of α and θ denoted by $\hat{\alpha}_1$ and $\hat{\theta}_1$ can be obtained from (8), respectively as,

$$\hat{\alpha}_1 = \frac{\bar{y}}{\Gamma\left(\frac{1}{\beta}+1\right)} \quad , \qquad \hat{\theta}_1 = \frac{\bar{z}}{\Gamma\left(\frac{1}{\beta}+1\right)} \quad (11)$$

The moment estimator of δ denoted by $\hat{\delta}_1$ can be obtained from (9) as follow

$$\hat{\delta}_1 = \frac{\bar{x}}{b\Gamma\left(\frac{1}{\beta}+1\right)} - \frac{\bar{b}\hat{\gamma}_1}{b} \qquad (12)$$

Thus it must be first obtain the moment estimator of γ denoted by $\hat{\gamma}_1$, by substitute (12) in (10), therefore

$$\bar{b}\left(\frac{\bar{b}}{b}+1\right)\hat{\gamma}_1{}^2 - 2\frac{\bar{b}\bar{x}}{b\,\Gamma\left(\frac{1}{\beta}+1\right)}\hat{\gamma}_1 + \left[\frac{\bar{x}^2}{b\left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2} - \frac{m_2'}{\Gamma\left(\frac{2}{\beta}+1\right)}\right] = 0$$

Let

$\xi_1 = \bar{b}\left(\frac{\bar{b}}{b}+1\right)$, $\xi_2 = -2\frac{\bar{b}\bar{x}}{b\,\Gamma\frac{1}{\beta}+1}$ and $\xi_3 = \frac{\bar{x}^2}{b\left(\Gamma\left(\frac{1}{\beta}+1\right)\right)^2} - \frac{m_2'}{\Gamma\left(\frac{2}{\beta}+1\right)}$

$$\xi_1\hat{\gamma}_1{}^2 + \xi_2\hat{\gamma}_1 + \xi_3 = 0 ,$$

if $\Delta = \xi_2{}^2 - 4\xi_1\xi_3$ is non-negative then the roots are real. Therefore the moment estimator of γ_1 denoted by $\hat{\gamma}_1$, takes the following form,

$$\hat{\gamma}_1 = \frac{-\xi_2 + \sqrt{\xi_2{}^2 - 4\xi_1\xi_3}}{2\xi_1} \qquad (13)$$

Hence the moment estimator $\hat{\delta}_1$ can be obtained by substitute (13) in (12).

Finally, the moment estimator of R, denoted by R_{ME} is obtained by substitute $\hat{\theta}_1, \hat{\alpha}_1, \hat{\gamma}_1$ and $\hat{\delta}_1$ in (7), therefore R_{ME} takes the following form,

$$R_{ME} = \frac{b\left(\hat{\delta}_1\hat{\theta}_1{}^2\right)^\beta}{\left[\hat{\delta}_1{}^\beta + \hat{\theta}_1{}^\beta\right]\left[\left(\hat{\theta}_1\hat{\alpha}_1\right)^\beta + \left(\hat{\delta}_1\hat{\alpha}_1\right)^\beta + \left(\hat{\delta}_1\hat{\theta}_1\right)^\beta\right]} + \frac{\bar{b}\left(\gamma_1\hat{\theta}_1{}^2\right)^\beta}{\left[\hat{\gamma}_1{}^\beta + \hat{\theta}_1{}^\beta\right]\left[\left(\hat{\theta}_1\hat{\alpha}_1\right)^\beta + \left(\hat{\gamma}_1\hat{\alpha}_1\right)^\beta + \left(\hat{\gamma}_1\hat{\theta}_1\right)^\beta\right]} \quad (14)$$

IV. Maximum Likelihood Estimator of *R*

This Section deals with MLE of reliability $R = P(Y < X < Z)$, when X, Y and Z are independent Weibull distribution with parameters (β, δ, γ), (β, α) and (β, θ) respectively and the shape parameter β is known. To compute the MLE R_{MLE} for R, firstly the MLEs $\hat{\theta}_2, \hat{\alpha}_2$, $\hat{\gamma}_2$ and $\hat{\delta}_2$ must be obtained. The MLEs $\hat{\theta}_2$, $\hat{\alpha}_2$, $\hat{\gamma}_2$ and $\hat{\delta}_2$ of the parameters θ, α, γ and δ are the values which maximize the likelihood function.

To obtain the maximum Likelihood estimator $\hat{\alpha}_2$ of α, let $Y_1, Y_2, \ldots, Y_{n_2}$ be a random sample of size n_2 drawn from Weibull distribution with parameters β and α, the likelihood function of observed sample is given by

$$L\left(\alpha, \beta, \underline{y}\right) = \beta^{n_2}\left(\frac{1}{\alpha}\right)^{n_2\beta} e^{-\sum_{i=1}^{n_2}\left(\frac{y_i}{\alpha}\right)^\beta} \prod_{i=1}^{n_2}\left[(y_i)^{\beta-1}\right]$$

The log-likelihood function $ln\left(L\left(\alpha, \beta, \underline{y}\right)\right)$ denoted by l is given by

$$l = n_2 ln\,\beta - n_2\beta\, ln\,\alpha - \alpha^{-\beta}\sum_{i=1}^{n_2}(y_i)^\beta + (\beta-1)\sum_{i=1}^{n_2} ln y_i \qquad (15)$$

The maximum likelihood estimate of α say $\hat{\alpha}_2$, is obtained by setting the first partial derivatives of (15) to zero

$$\frac{dl}{\partial\alpha} = -\frac{n_2\beta}{\hat{\alpha}_2} + \beta\hat{\alpha}_2{}^{-\beta-1}\sum_{i=1}^{n_2}(y_i)^\beta = 0 .$$

Therefore,

$$\hat{\alpha}_2 = \left(\frac{1}{n_2}\sum_{i=1}^{n_2} y_i^\beta\right)^{\frac{1}{\beta}} \quad (16)$$

By the similar way, to obtain the maximum Likelihood estimator $\hat{\theta}_2$ of θ , let $Z_1, Z_2, \ldots, Z_{n_3}$ be a random sample of size n_3 drawn from Weibull distribution with parameters β and θ, then $\hat{\theta}_2$ takes the following form

$$\hat{\theta}_2 = \left(\frac{1}{n_3}\sum_{i=1}^{n_3} z_i^\beta\right)^{\frac{1}{\beta}} . \qquad (17)$$

To obtain the maximum Likelihood estimator $\hat{\delta}_2$ and $\hat{\gamma}_2$ of δ_2 and γ_2, let $X_1, X_2, \ldots, X_{n_1}$ be a random sample of size n_1 drawn from Weibull distribution with presence of k outliers with parameters δ and γ, the likelihood function of observed sample can be written as

$$L(\delta, \gamma, \underline{x}) \equiv l = \left(\frac{b\beta}{\delta^\beta}\right)^{n_1} e^{-\sum_{i=1}^{n_1}\left(\frac{x_i}{\delta}\right)^\beta}\left[\prod_{i=1}^{n_1} x_i{}^{\beta-1}\right]\left[\prod_{i=1}^{n_1}\psi(x_i;\delta,\gamma)\right],$$

where, $\psi(x_i;\delta,\gamma) = 1+\left(\frac{\bar{b}}{b}\right)\left(\frac{\delta}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{\delta^{\beta}}\right){x_i}^{\beta}}$

$$l = n_1 \ln(b\beta) - n_1\beta\, ln\delta - \frac{1}{\delta^{\beta}}\sum_{i=1}^{n_1} {x_i}^{\beta} + (\beta-1)\sum_{i=1}^{n_1} lnx_i + \sum_{i=1}^{n_1} ln\psi(x_i;\delta,\gamma) \quad (18)$$

The first partial derivatives of the log–likelihood (18) with respect to δ and γ are given, respectively by,

$$\frac{dl}{d\delta} = \frac{-n_1\beta}{\delta} + \beta\delta^{-\beta-1}\sum_{i=1}^{n_1} {x_i}^{\beta} + \frac{\bar{b}\beta}{b\delta}\left(\frac{\delta}{\gamma}\right)^{\beta}\sum_{i=1}^{n_1}\frac{\left(1-\left(\frac{x_i}{\delta}\right)^{\beta}\right)e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{\delta^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;\delta,\gamma)} \quad (19)$$

$$\frac{dl}{d\gamma} = \frac{\bar{b}\beta}{b\gamma}\left(\frac{\delta}{\gamma}\right)^{\beta}\sum_{i=1}^{n_1}\frac{\left(\left(\frac{x_i}{\gamma}\right)^{\beta}-1\right)e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{\delta^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;\delta,\gamma)} \quad (20)$$

MLE's of δ and γ denoted by $\hat{\delta}_2$ and $\hat{\gamma}_2$ are solution to the system of equations obtained by setting the partial derivatives of the logarithm of likelihood function (19) and (20) to be zero. Obviously, it is not easy to obtain a closed form solution to this system of equations. Therefore, an iterative method must be applied to solve this equation numerically to estimate $\hat{\delta}_2, \hat{\gamma}_2$. The MLE of R, denoted by R_{MLE} is obtained by substitute $\hat{\theta}_2, \hat{\alpha}_2, \hat{\delta}_2$ and $\hat{\gamma}_2$ in (7).

V. Mixture Estimator of *R*

To avoid the difficulty of complicated in the system of likelihood equations, the mixture estimator of R denoted by R_{Mix} will be obtained. Following Read [11], the mixture estimator of α, γ and δ denoted by $\hat{\alpha}_3$, $\hat{\gamma}_3$ and $\hat{\delta}_3$ will be derived by mixing between moment estimates and MLEs which are obtained previously in Sections 3 and 4.

The mixture estimator of $\hat{\theta}_3, \hat{\alpha}_3$ can be obtained from moment estimator as follows

$$\hat{\theta}_3 = \frac{\bar{z}}{\Gamma\left(\frac{1}{\beta}+1\right)}, \hat{\alpha}_3 = \frac{\bar{y}}{\Gamma\left(\frac{1}{\beta}+1\right)} \quad (21)$$

The mixture estimator of $\hat{\delta}_3$ and $\hat{\gamma}_3$ can be obtained from likelihood estimators by substitute as follows

$$\frac{dl}{d\delta} = \frac{-n_1\beta}{\hat{\delta}_3} + \beta{\hat{\delta}_3}^{-\beta-1}\sum_{i=1}^{n_1} {x_i}^{\beta} + \frac{\bar{b}\beta}{b\hat{\delta}_3}\left(\frac{\hat{\delta}_3}{\hat{\gamma}_3}\right)^{\beta}\sum_{i=1}^{n_1}\frac{\left(1-\left(\frac{x_i}{\hat{\delta}_3}\right)^{\beta}\right)e^{-\left(\frac{1}{{\hat{\gamma}_3}^{\beta}}-\frac{1}{{\hat{\delta}_3}^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;\hat{\delta}_3,\hat{\gamma}_3)} = 0 \quad (22)$$

$$\frac{dl}{d\gamma} = \frac{\bar{b}\beta}{b\hat{\gamma}_3}\left(\frac{\hat{\delta}_3}{\hat{\gamma}_3}\right)^{\beta}\sum_{i=1}^{n_1}\frac{\left(\left(\frac{x_i}{\hat{\gamma}_3}\right)^{\beta}-1\right)e^{-\left(\frac{1}{{\hat{\gamma}_3}^{\beta}}-\frac{1}{{\hat{\delta}_3}^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;\hat{\delta}_3,\hat{\gamma}_3)} = 0 \quad (23)$$

The mixture estimator of R, denoted by R_{Mix} is obtained by substitute $\hat{\theta}_3$, $\hat{\alpha}_3$, $\hat{\gamma}_3$ and $\hat{\delta}_3$ in (7).

VI. Numerical Illustration

In this Section, an extensive numerical investigation will be carried out to compare the performance of the different estimators for different sample sizes and parameter values for Weibull distribution in the presence of k outliers. The investigated properties are biases and mean square errors (MSEs). All the computation is performed via MathCAD (14) statistical package. The algorithm for the $R = P(Y < X < Z)$ parameter estimation can be summarized in the following steps:

Step (1): Generate 1000 random samples $X_1, \ldots X_{n_1}, Y_1, \ldots Y_{n_2}$ and $Z_1, \ldots Z_{n_3}$ from Weibull distribution with the sample sizes; $(n_1, n_2, n_3) =$ (15,15,15),(20,20,20),(25,25,25), (15,15,20), (15,15,25), (15,15,20), (15,25,20), (15,25,25), (20,15,15), (20,15,20), (20,15,25), (20,25,20), (20,25,25), (25,15,15), (25,15,20), (25,15,25), (25,20,20), (25,20,25).

Step (2): The parameter values are selected as $k = (1,2),\ \beta = 2,\ \alpha = (0.5,1.5),\ \gamma = (2, 0.8)\ ,\ \delta = (0.5, 1.8)\ and\ \theta = (2,2.5)$.

Step (3): The moment estimator of θ and α are obtained by solving (11). ME of γ is obtained numerically from (12). The ME of δ is obtained by substitute $\hat{\gamma}_1$ in (13). Once the estimate of these estimator are computed, then R_{ME} will be obtainedusing (7).

Step (4): The MLE of α and θ are obtained from (16) and (17). The nonlinear equations (19) and (20) of the MLEs will be solved iteratively using NetwonRaphson method. The MLE of R will be obtained by substitute the MLEs of θ, α, δ and γ in (7).

Step (5): The mixture estimator of θ and α are computed by using (21). NetwonRaphson method is used for solving (22) and (23), to obtain the estimators of δ and γ. The mixture estimator of R will be obtained by substitute $\hat{\theta}_3, \hat{\alpha}_3\hat{\delta}_3$ and $\hat{\gamma}_3$ in (7).

Step (6): The performance of the estimators can be evaluated through some measures of accuracy which are Biases and MSEs of R.

Simulation results are summarized in Tables 1-4.From these Tables, the following conclusions can be observed on the properties of estimated parameters from R.

1. The estimated value of R increases as the value of outliers, k, increases.
2. The estimated values of R based on moment method is the smallest value, on the other hand the estimated values of R based on mixture method is the highest one.
3. The biases of R_{Mix} are the smallest relative to the biases of R_{ML} and R_{ME}.
4. Comparing the MSEs of all estimators, the mixture estimators perform the best estimator.
5. The biases and MSEs of R based on different estimators increase as the value of outliers increases in almost all cases expect for some few cases.

VII. Conclusions

This article considered the problem of estimating the reliability $R = P(Y < X < Z)$ for the Weibull distribution with presence of k outliers in strength X. Assuming that, X,Y and Z are independent with common known shape parameter but with different scale parameters. The moment, maximum likelihood and mixture estimators of $R$are derived. Performance of estimators is usually evaluated through their biases and MSEs.

Comparison study revealed that the mixture estimator works the best with respect to biases and MSEs, so the researcher strongly feels that mixture estimator is better and easy to calculate than the maximum likelihood and moment estimators.

In general, the mixture method for estimating $R = P(Y < X < Z)$of the Weibull distribution in the presence of k outliers issuggested to be used.

Table 1: Estimates of *R*, Biases and MSE's of the point estimates from Weibull Distribution, when $k = 1,\quad \beta = 2,\quad \alpha = 0.5,\quad \gamma = 2,\quad \delta = 0.5\ and\quad \theta = 2$

Sample Size	Estimates of R			Bias			MSE		
(n_1, n_2, n_3)	R_{ML}	R_{ME}	R_{Mix}	MLE	Mom	Mix	MLE	Mom	Mix
(15,15,15)	0.358	0.298	0.459	0.088	0.148	-0.013	0.073	0.112	0.019
(20,20,20)	0.382	0.324	0.455	0.064	0.122	-0.009	0.066	0.123	0.017
(25,25,25)	0.356	0.328	0.434	0.089	0.117	0.011	0.079	0.117	0.016
(15,15,20)	0.356	0.311	0.434	0.090	0.135	0.012	0.068	0.129	0.021
(15,15,25)	0.386	0.307	0.462	0.060	0.139	-0.016	0.073	0.122	0.015
(15,20,20)	0.356	0.302	0.429	0.090	0.144	0.017	0.071	0.119	0.018
(15,25,20)	0.349	0.297	0.455	0.097	0.149	-0.009	0.078	0.126	0.021
(15,25,25)	0.354	0.309	0.461	0.092	0.137	-0.015	0.077	0.117	0.017
(20,15,15)	0.359	0.293	0.428	0.087	0.153	0.018	0.074	0.122	0.016
(20,15,20)	0.356	0.299	0.459	0.090	0.147	-0.013	0.075	0.119	0.024
(20,15,25)	0.362	0.309	0.454	0.084	0.137	-0.008	0.069	0.111	0.018
(20,25,20)	0.353	0.303	0.431	0.093	0.143	0.015	0.078	0.124	0.013
(20,25,25)	0.354	0.317	0.460	0.092	0.129	-0.014	0.081	0.123	0.015
(25,15,15)	0.355	0.307	0.436	0.090	0.138	0.009	0.082	0.119	0.023
(25,15,20)	0.360	0.317	0.458	0.085	0.128	-0.013	0.076	0.114	0.019
(25,15,25)	0.390	0.319	0.428	0.055	0.126	0.017	0.068	0.116	0.017
(25,20,20)	0.348	0.312	0.456	0.097	0.133	-0.011	0.079	0.118	0.014
(25,20,25)	0.387	0.318	0.458	0.058	0.127	-0.013	0.069	0.111	0.011

Table 2: Estimates of *R*, Biases and MSE's of the point estimates from Weibull Distribution, when $k = 2, \quad \beta = 2, \quad \alpha = 0.5, \quad \gamma = 2\,, \quad \delta = 0.5 \;\; and \quad \theta = 2$

Sample Size	Estimates of *R*			*Bias*			*MSE*		
(n_1, n_2, n_3)	R_{ML}	R_{ME}	R_{Mix}	*MLE*	*Mom*	*Mix*	*MLE*	*Mom*	*Mix*
(15,15,15)	0.378	0.333	0.467	0.067	0.112	-0.022	0.054	0.095	0.012
(20,20,20)	0.368	0.322	0.464	0.077	0.123	-0.019	0.061	0.098	0.014
(25,25,25)	0.377	0.336	0.462	0.068	0.109	-0.017	0.049	0.078	0.009
(15,15,20)	0.376	0.334	0.422	0.069	0.111	0.023	0.052	0.082	0.014
(15,15,25)	0.372	0.338	0.471	0.073	0.107	-0.026	0.059	0.095	0.013
(15,20,20)	0.371	0.346	0.473	0.074	0.099	-0.028	0.058	0.097	0.016
(15,25,20)	0.368	0.340	0.463	0.077	0.105	-0.018	0.062	0.095	0.012
(15,25,25)	0.371	0.333	0.472	0.074	0.112	-0.027	0.061	0.098	0.009
(20,15,15)	0.364	0.344	0.416	0.081	0.101	0.029	0.055	0.089	0.011
(20,15,20)	0.376	0.331	0.424	0.069	0.114	0.021	0.064	0.096	0.014
(20,15,25)	0.369	0.328	0.470	0.076	0.117	-0.025	0.059	0.097	0.011
(20,25,20)	0.361	0.325	0.426	0.084	0.120	0.019	0.048	0.087	0.013
(20,25,25)	0.379	0.327	0.467	0.066	0.118	-0.022	0.052	0.091	0.016
(25,15,15)	0.370	0.341	0.420	0.075	0.104	0.025	0.061	0.092	0.011
(25,15,20)	0.365	0.324	0.464	0.08	0.121	-0.019	0.056	0.090	0.010
(25,15,25)	0.373	0.329	0.467	0.072	0.116	-0.022	0.051	0.087	0.013
(25,20,20)	0.376	0.332	0.463	0.069	0.113	-0.018	0.057	0.088	0.009
(25,20,25)	0.374	0.343	0.462	0.071	0.102	-0.017	0.053	0.087	0.011

Table 3: Estimates of *R*, Biases and MSE's of the point estimates from Weibull Distribution, when $k = 1, \quad \beta = 2, \quad \alpha = 1.5, \quad \gamma = 0.8\,, \quad \delta = 1.8 \;\; and \quad \theta = 2.5$

Sample Size	Estimates of *R*			*Bias*			*MSE*		
(n_1, n_2, n_3)	R_{ML}	R_{ME}	R_{Mix}	*MLE*	*Mom*	*Mix*	*MLE*	*Mom*	*Mix*
(15,15,15)	0.445	0.398	0.546	0.085	0.132	-0.016	0.071	0.088	0.023
(20,20,20)	0.468	0.413	0.543	0.062	0.117	-0.013	0.073	0.084	0.031
(25,25,25)	0.451	0.411	0.541	0.079	0.119	-0.011	0.072	0.073	0.028
(15,15,20)	0.445	0.404	0.516	0.085	0.126	0.014	0.081	0.079	0.034
(15,15,25)	0.456	0.399	0.540	0.074	0.131	-0.01	0.078	0.078	0.027
(15,20,20)	0.447	0.395	0.518	0.083	0.135	0.012	0.076	0.091	0.039
(15,25,20)	0.439	0.396	0.545	0.091	0.134	-0.015	0.073	0.087	0.028
(15,25,25)	0.448	0.401	0.541	0.082	0.129	-0.011	0.071	0.083	0.034
(20,15,15)	0.453	0.391	0.517	0.077	0.139	0.013	0.079	0.079	0.027
(20,15,20)	0.447	0.393	0.521	0.083	0.137	0.009	0.084	0.088	0.036
(20,15,25)	0.449	0.401	0.538	0.081	0.129	-0.008	0.079	0.090	0.031
(20,25,20)	0.446	0.395	0.519	0.084	0.135	0.011	0.068	0.082	0.028
(20,25,25)	0.444	0.398	0.547	0.086	0.132	-0.017	0.081	0.076	0.033
(25,15,15)	0.448	0.402	0.516	0.082	0.128	0.014	0.087	0.084	0.027
(25,15,20)	0.451	0.403	0.543	0.079	0.127	-0.013	0.086	0.079	0.030
(25,15,25)	0.466	0.398	0.541	0.064	0.132	-0.011	0.075	0.081	0.029
(25,20,20)	0.462	0.394	0.545	0.068	0.136	-0.015	0.079	0.086	0.018
(25,20,25)	0.456	0.406	0.520	0.074	0.124	0.01	0.073	0.076	0.022

Table 4: Estimates of *R*, Biases and MSE's of the point estimates from Weibull Distribution, when $k = 2,\quad \beta = 2,\quad \alpha = 1.5,\quad \gamma = 0.8,\quad \delta = 1.8\quad and\quad \theta = 2.5$

Sample Size	Estimates of *R*			*Bias*			*MSE*		
(n_1, n_2, n_3)	R_{ML}	R_{ME}	R_{Mix}	*MLE*	*Mom*	*Mix*	*MLE*	*Mom*	*Mix*
(15,15,15)	0.453	0.406	0.556	0.077	0.124	-0.026	0.074	0.076	0.019
(20,20,20)	0.466	0.419	0.553	0.064	0.111	-0.023	0.071	0.064	0.018
(25,25,25)	0.458	0.417	0.511	0.072	0.113	0.019	0.068	0.066	0.018
(15,15,20)	0.449	0.408	0.516	0.081	0.122	0.014	0.076	0.081	0.015
(15,15,25)	0.459	0.403	0.514	0.071	0.127	0.016	0.072	0.073	0.014
(15,20,20)	0.456	0.405	0.548	0.074	0.125	-0.018	0.070	0.084	0.019
(15,25,20)	0.447	0.406	0.542	0.083	0.124	-0.012	0.068	0.081	0.017
(15,25,25)	0.453	0.402	0.513	0.077	0.128	0.017	0.064	0.089	0.015
(20,15,15)	0.457	0.398	0.511	0.073	0.132	0.019	0.071	0.071	0.018
(20,15,20)	0.451	0.399	0.509	0.079	0.131	0.021	0.079	0.082	0.014
(20,15,25)	0.458	0.408	0.548	0.072	0.122	-0.018	0.074	0.077	0.019
(20,25,20)	0.448	0.404	0.517	0.082	0.126	0.013	0.069	0.076	0.014
(20,25,25)	0.449	0.401	0.543	0.081	0.129	-0.013	0.077	0.071	0.010
(25,15,15)	0.459	0.412	0.553	0.071	0.118	-0.023	0.082	0.073	0.013
(25,15,20)	0.458	0.409	0.517	0.072	0.121	0.013	0.081	0.075	0.017
(25,15,25)	0.461	0.404	0.545	0.069	0.126	-0.015	0.074	0.081	0.011
(25,20,20)	0.468	0.397	0.519	0.062	0.133	0.011	0.076	0.080	0.017
(25,20,25)	0.461	0.408	0.514	0.069	0.122	0.016	0.071	0.077	0.015

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