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Multi - Anti Fuzzy Group And Its Lower Level Subgroups

R. Muthuraj, S. Balamurugan
PG and Research Department of Mathematics, H.H. The Rajah's College, Pudukkottai-622 001, Tamilnadu, India.
Department of Mathematics, Velammal College of Engineering & Technology, Madurai-625 009, Tamilnadu, India.

Abstract
In this paper, we define the algebraic structures of multi-anti fuzzy subgroup and some related properties are investigated. The purpose of this study is to implement the fuzzy set theory and group theory in multi-anti fuzzy subgroups. Characterizations of multi-lower level subsets of a multi-anti fuzzy subgroup of a group are given.
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I. Introduction

S.Sabu and T.V.Ramakrishnan [5] proposed the theory of multi-fuzzy sets in terms of multi-dimensional membership functions and investigated some properties of multi-level fuzziness. L.A.Zadeh [8] introduced the theory of multi-fuzzy set is an extension of theories of fuzzy sets. R.Muthuraj and S.Balamurugan [2] proposed multi-fuzzy group and its level subgroups. In this paper we define a new algebraic structure of multi-anti fuzzy subgroups and study some of their related properties.

II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel.

2.1 Definition

Let X be any non-empty set. A fuzzy subset μ of X is $\mu : X \to [0,1]$.

2.2 Definition

Let X be a non-empty set. A multi-fuzzy set A in X is defined as a set of ordered sequences:

$A = \{ (x, \mu_1(x), \mu_2(x), ..., \mu_i(x), ...) : x \in X\}$, where $\mu_i : X \to [0, 1]$ for all i.

Remark

i. If the sequences of the membership functions have only k-terms (finite number of terms), k is called the dimension of A.
ii. The set of all multi-fuzzy sets in X of dimension k is denoted by $M^k FS(X)$.
iii. The multi-fuzzy membership function μ_A is a function from X to $[0,1]^k$ such that for all x in X, $\mu_A(x) = (\mu_1(x), \mu_2(x), ..., \mu_k(x))$.
iv. For the sake of simplicity, we denote the multi-fuzzy set

$A = \{ (x, \mu_1(x), \mu_2(x), ..., \mu_k(x)) : x \in X\}$ as $A= (\mu_1, \mu_2, ..., \mu_k)$.

2.3 Definition

Let k be a positive integer and let A and B in $M^k FS(X)$, where $A= (\mu_1, \mu_2, ..., \mu_k)$ and $B = (\nu_1, \nu_2..., \nu_k)$, then we have the following relations and operations:

i. $A \subseteq B$ if and only if $\mu_i \le \nu_i$, for all i = 1, 2, ..., k;
ii. $A = B$ if and only if $\mu_i = \nu_i$, for all i = 1, 2, ..., k;
iii. $A \cup B = (\mu_1 \cup \nu_1, ..., \mu_k \cup \nu_k) = \{(x, \max(\mu_1(x), \nu_1(x)), ..., \max(\mu_k(x), \nu_k(x))) : x \in X\}$;
iv. $A \cap B = (\mu_1 \cap \nu_1, ..., \mu_k \cap \nu_k) = \{(x, \min(\mu_1(x), \nu_1(x)), ..., \min(\mu_k(x), \nu_k(x))) : x \in X\}$;
v. $A+B = (\mu_1+\nu_1, ..., \mu_k+\nu_k) = \{ (x, \mu_1(x)+\nu_1(x) - \mu_1(x)\nu_1(x), ..., \mu_k(x)+\nu_k(x) - \mu_k(x)\nu_k(x)) : x \in X\}$.

2.4 Definition

Let $A= (\mu_1, \mu_2, ..., \mu_k)$ be a multi-fuzzy set of dimension k and let μ_i' be the fuzzy complement of the ordinary fuzzy set μ_i for i = 1, 2, ..., k. The Multi-fuzzy Complement of the multi-fuzzy set A is a multi-fuzzy set $(\mu_1', ... , \mu_k')$ and it is denoted by C(A) or A' or A^C.

That is, $C(A) = \{(x, c(\mu_1(x)), ..., c(\mu_k(x))) : x \in X\} = \{(x, 1-\mu_1(x), ... ,1-\mu_k(x)) : x \in X\}$, where c is the fuzzy complement operation.

2.5 Definition

Let A be a fuzzy set on a group G. Then A is said to be a fuzzy subgroup of G if for all $x, y \in G$,

i. $A(xy) \ge \min \{ A(x) , A(y)\}$
ii. $A(x^{-1}) = A(x)$.

2.6 Definition

A multi-fuzzy set A of a group G is called a multi-fuzzy subgroup of G if for all x, y$\in$G,

i. $A(xy) \geq \min\{A(x), A(y)\}$
ii. $A(x^{-1}) = A(x)$

2.7 Definition

Let A be a fuzzy set on a group G. Then A is called an anti fuzzy subgroup of G if for all x,y$\in$G,

i. $A(xy) \leq \max\{A(x), A(y)\}$
ii. $A(x^{-1}) = A(x)$

2.8 Definition

A multi-fuzzy set A of a group G is called a multi-anti fuzzy subgroup of G if for all x,y$\in$G,

i. $A(xy) \leq \max\{A(x), A(y)\}$
ii. $A(x^{-1}) = A(x)$

2.9 Definition

Let A and B be any two multi-fuzzy sets of a non-empty set X. Then for all x$\in$X,

i. $A \subseteq B$ iff $A(x) \leq B(x)$,
ii. $A = B$ iff $A(x) = B(x)$,
iii. $A \cup B(x) = \max\{A(x), B(x)\}$,
iv. $A \cap B(x) = \min\{A(x), B(x)\}$.

2.10 Definition

Let A and B be any two multi-fuzzy sets of a non-empty set X. Then

i. $A \cup A = A$, $A \cap A = A$,
ii. $A \subseteq A \cup B$, $B \subseteq A \cup B$, $A \cap B \subseteq A$ and $A \cap B \subseteq B$,
iii. $A \subseteq B$ iff $A \cup B = B$,
iv. $A \subseteq B$ iff $A \cap B = A$.

III. Properties of multi-anti fuzzy subgroups

In this section, we discuss some of the properties of multi-anti fuzzy subgroups.

3.1 Theorem

Let 'A' be a multi-anti fuzzy subgroup of a group G and 'e' is the identity element of G. Then

i. $A(x) \geq A(e)$ for all $x \in G$.
ii. The subset $H = \{x \in G \,/\, A(x) = A(e)\}$ is a subgroup of G.

Proof

i. Let x$\in$G.

$$\begin{aligned} A(x) &= \max\{A(x), A(x)\} \\ &= \max\{A(x), A(x^{-1})\} \\ &\geq A(xx^{-1}) \\ &= A(e). \end{aligned}$$

Therefore, $A(x) \geq A(e)$, for all x$\in$G.

ii. Let $H = \{x \in G \,/\, A(x) = A(e)\}$

Clearly H is non-empty as e$\in$H.

Let x,y$\in$H. Then, $A(x) = A(y) = A(e)$

$$\begin{aligned} A(xy^{-1}) &\leq \max\{A(x), A(y^{-1})\} \\ &= \max\{A(x), A(y)\} \\ &= \max\{A(e), A(e)\} \\ &= A(e) \end{aligned}$$

That is, $A(xy^{-1}) \leq A(e)$ and obviously $A(xy^{-1}) \geq A(e)$ by i.

Hence, $A(xy^{-1}) = A(e)$ and $xy^{-1} \in H$.

Clearly, H is a subgroup of G.

3.2 Theorem

A is a multi-fuzzy subgroup of G iff A^C is a multi-anti fuzzy subgroup of G.

Proof

Suppose A is a multi-fuzzy subgroup of G. Then for all x,y$\in$G,

$$A(xy) \geq \min\{A(x), A(y)\}$$
$$\Leftrightarrow 1 - A^c(xy) \geq \min\{(1 - A^c(x)), (1 - A^c(y))\}$$
$$\Leftrightarrow A^c(xy) \leq 1 - \min\{(1 - A^c(x)), (1 - A^c(y))\}$$
$$\Leftrightarrow A^c(xy) \leq \max\{A^c(x), A^c(y)\}.$$

We have, $A(x) = A(x^{-1})$ for all x in G

$$\Leftrightarrow 1 - A^c(x) = 1 - A^c(x^{-1})$$

Therefore $A^c(x) = A^c(x^{-1})$.

Hence A^c is a multi-anti fuzzy subgroup of G.

3.3 Theorem

Let 'A' be any multi-anti fuzzy subgroup of a group G with identity 'e'. Then $A(xy^{-1}) = A(e) \Rightarrow A(x) = A(y)$ for all x,y in G.

Proof

Given A is a multi-anti fuzzy subgroup of G and $A(xy^{-1}) = A(e)$.

Then for all x,y in G,

$$\begin{aligned} A(x) &= A(x(y^{-1}y)) \\ &= A((xy^{-1})y) \\ &\leq \max\{A(xy^{-1}), A(y)\} \\ &= \max\{A(e), A(y)\} \\ &= A(y). \end{aligned}$$

That is, $A(x) \leq A(y)$.

Now, $A(y) = A(y^{-1})$, since A is a multi-anti fuzzy subgroup of G.

$$\begin{aligned} &= A(ey^{-1}) \\ &= A((x^{-1}x)y^{-1}) \\ &= A(x^{-1}(xy^{-1})) \\ &\leq \max\{A(x^{-1}), A(xy^{-1})\} \\ &= \max\{A(x), A(e)\} \\ &= A(x). \end{aligned}$$

That is, $A(y) \leq A(x)$.

Hence, $A(x) = A(y)$.

3.4 Theorem

A is a multi-anti fuzzy subgroup of a group G $\Leftrightarrow A(xy^{-1}) \leq \max\{A(x), A(y)\}$, for all x,y in G .

Proof

Let A be a multi-anti fuzzy subgroup of a group G. Then for all x,y in G,

$$A(xy) \leq \max\{A(x), A(y)\}$$
and $A(x) = A(x^{-1})$.

Now, $A(xy^{-1}) \leq \max\{A(x), A(y^{-1})\}$
$= \max\{A(x), A(y)\}$
$\Leftrightarrow A(xy^{-1}) \leq \max\{A(x), A(y)\}$.

IV. Properties of Multi-lower level subsets of a multi-anti fuzzy subgroup

In this section, we introduce the concept of multi-lower level subset of a multi-anti fuzzy subgroup and discuss some of its properties.

4.1 Definition

Let A be a multi-anti fuzzy subgroup of a group G. For any $t = (t_1,t_2, \dots , t_k , \dots)$ where $t_i \in [0,1]$, for all i, we define the multi-lower level subset of A is the set, $L(A ; t) = \{ x \in G / A(x) \leq t \}$.

4.1 Theorem

Let A be a multi-anti fuzzy subgroup of a group G. Then for any $t = (t_1,t_2, \dots , t_k , \dots)$, where $t_i \in [0,1]$ for all i such that $t \geq A(e)$, where 'e' is the identity element of G, L (A ; t) is a subgroup of G.

Proof

For all $x, y \in L(A ; t)$, we have,

$A(x) \leq t$; $A(y) \leq t$.

Now, $A(xy^{-1}) \leq \max \{A(x), A(y)\}$.

$\leq \max \{ t , t \} = t$

That is, $A(xy^{-1}) \leq t$

Therefore, $xy^{-1} \in L(A ; t)$.

Hence L(A ; t) is a subgroup of G.

4.2 Theorem

Let G be a group and A be a multi-fuzzy subset of G such that L(A ; t) is a subgroup of G. Then for $t = (t_1,t_2, \dots , t_k , \dots)$, where $t_i \in [0,1]$ for all i such that $t \geq A(e)$ where 'e' is the identity element of G, A is a multi-anti fuzzy subgroup of G.

Proof

Let $x,y \in G$ and $A(x)= r$ and $A(y)= s$, where $r = (r_1,r_2, \dots , r_k , \dots)$, $s = (s_1,s_2, \dots , s_k , \dots)$, $r_i ,s_i \in [0,1]$ for all i.

Suppose $r < s$.

Now $A(y) = s$ which implies $y \in L(A ; s)$.

And now $A(x) = r < s$ which implies $x \in L(A ; s)$.

Therefore $x,y \in L(A ; s)$.

As L(A ; s) is a subgroup of G, $xy^{-1} \in L(A ; s)$.

Hence, $A(xy^{-1}) \leq s = \max\{r, s\}$

$\leq \max\{A(x), A(y)\}$

That is, $A(xy^{-1}) \leq \max\{A(x), A(y)\}$.

Hence A is a multi-anti fuzzy subgroup of G.

4.2 Definition

Let A be a multi-anti fuzzy subgroup of a group G. The subgroups L(A ; t) for $t = (t_1,t_2, \dots , t_k , \dots)$ where $t_i \in [0,1]$ for all i and $t \geq A(e)$ where 'e' is the identity element of G, are called multi-lower level subgroups of A.

4.3 Theorem

Let A be a multi-anti fuzzy subgroup of a group G and 'e' is the identity element of G. If two multi-lower level subgroups L(A ; r), L(A ; s), for $r = (r_1,r_2, \dots , r_k , \dots)$, $s = (s_1,s_2, \dots , s_k , \dots)$, $r_i, s_i \in [0,1]$ for all i and $r, s \geq A(e)$ with $r < s$ of A are equal, then there is no x in G such that $r < A(x) \leq s$.

Proof

Let L(A ; r) = L(A ; s).

Suppose there exists a $x \in G$ such that $r < A(x) \leq s$.

Then $L(A ; r) \subseteq L(A ; s)$.

That is, $x \in L(A ; s)$, but $x \notin L(A ; r)$, which contradicts the assumption that, L(A ; r) = L(A ; s).

Hence there is no x in G such that $r < A(x) \leq s$.

Conversely, Suppose that there is no x in G such that $r < A(x) \leq s$.

Then, by definition, $L(A ; r) \subseteq L(A ; s)$.

Let $x \in L(A ; s)$ and there is no x in G such that $r < A(x) \leq s$.

Hence $x \in L(A ; r)$ and $L(A ; s) \subseteq L(A ; r)$.

Hence L(A ; r) = L(A ; s).

4.4 Theorem

A multi-fuzzy subset A of G is a multi-anti fuzzy subgroup of a group G if and only if the multi-lower level subsets L(A ; t), for $t = (t_1,t_2, \dots , t_k , \dots)$ where $t_i \in [0,1]$ for all i and $t \geq A(e)$, are subgroups of G.

Proof It is clear.

4.5 Theorem

Any subgroup H of a group G can be realized as a multi-lower level subgroup of some multi-anti fuzzy subgroup of G.

Proof

Let A be a multi-fuzzy subset and $x \in G$.

Define,

$$A(x) = \begin{cases} 0 & \text{if } x \in H \\ t & \text{if } x \notin H , \text{ for } t = (t_1,t_2, \dots , t_k , \dots) \end{cases}$$

where $t_i \in [0,1]$ for all i and $t \geq A(e)$.

we shall prove that A is a multi-anti fuzzy subgroup of G. Let $x,y \in G$.

i. Suppose $x,y \in H$. Then $xy \in H$ and $xy^{-1} \in H$.

$A(x) = 0$, $A(y) = 0$, $A(xy) = 0$ and $A(xy^{-1}) = 0$.

Hence $A(xy^{-1}) \leq \max\{A(x), A(y)\}$.

ii. Suppose $x \in H$ and $y \notin H$. Then $xy \notin H$ and $xy^{-1} \notin H$.

$A(x) = 0$, $A(y) = t$ and $A(xy^{-1}) = t$.

Hence $A(xy^{-1}) \leq \max\{A(x), A(y)$.

iii. Suppose $x,y \notin H$. Then $xy^{-1} \in H$ or $xy^{-1} \notin H$.

$A(x) = t$, $A(y) = t$ and $A(xy^{-1}) = 0$ or t.

Hence $A(xy^{-1}) \leq \max\{A(x), A(y)\}$.

Thus in all cases, A is a multi-anti fuzzy subgroup of G. For this multi-anti fuzzy subgroup A, L(A ; t) = H.

Remark

As a consequence of the Theorem 4.3, the multi-lower level subgroups of a multi-anti fuzzy subgroup A of a group G form a chain. Since $A(e) \leq A(x)$ for all x in G where 'e' is the identity element of G, therefore $L(A; t_0)$, where $A(e) = t_0$ is the smallest and we have the chain :

$\{e\} \subseteq L(A ; t_0) \subset L(A ; t_1) \subset L(A ; t_2) \subset \dots \subset L(A ; t_n) = G$, where $t_0 < t_1 < t_2 < \dots < t_n$.

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