RESEARCH ARTICLE OPEN ACCESS

Solutions To The Pell Equation $x^2-D.y^2=2^k$ Where $D=r^2s^2+2.s$ And Recurrences

Özen Özer*, Yılmaz Çeven**
*Köyceğiz Vocational High School, Muğla University, Muğla, 48800, Turkey
Department of Mathematics, Faculty of Science and Arts, Süleyman Demirel University, Isparta,32000, Turkey

Abstract
Let $r,s\geq 1$ and $k\geq 0$ be arbitrary integers and also $D=r^2s^2+2.s$ be a positive non- square integer. In this paper, we consider the Pell equation $x^2-D.y^2=2^k$ and we get all positive integer solutions of this equation for all $k\geq 0$ integers. Moreover, we derive recurrence relations on the solutions of the Pell equation $x^2-D.y^2=2^k$.
Mathematics Subject Classification: 11A55, 11D09, 11D61, 11D99,
Keywords: Continued Fraction, Quadratic Equation, Exponential Equation, Solutions of Pell Equation

I. Introduction

The equation

$$x^2-Dy^2=\mp N \qquad (1)$$

with given integers D , N and unknowns x,y is called Pell's Equation. if D is negative, it can have only a finite number of solutions. if D is a perfect square, say $D=t^2$, the equation reduces to $(x-ty).(x+ty)=\mp N$ andthere is only a finite number of solutions. The most interesting case of equation arises when $D\neq 1$ be a positive non- square integer.

For $N=1$, the Pell equation

$$x^2-Dy^2=\mp 1 \qquad (2)$$

İs known as classical Pell equation and it has infinitely many solutions (x_n,y_n) for $n\in N$. There are different methods for finding the first non- trivial (x_1,y_1) solution called the fundamental solution from which all other solutions are easily computed $(see\ [2]-[3])$.

Also, there are many papers in which details on Pell equations and different types of Pell's equation are considered $(see\ [1]-[4]-[5]-[6])$.

In this paper, in the case of $D=r^2s^2+2.s$ where $r,s\geq 1$, we consider the Pell equation $x^2-D.y^2=2^k$ when $k\geq 0$ integer and by constructing some criteria we get all positive solutions of this equation. We consider the problem in three cases:

$(i)\ r=s=1$
$(ii)\ r\geq 2,\ s=1$
$(iii)\ r,s\geq 2$

for $k=0$ and $k\geq 1$ respectively. Moreover, we give nümerical examplesto all new constructed theorems and also by using method of $([5])$, we derive recurrences relations on the solutions of this equation.

II. Preliminary Notes

We need the following theorems for the proof of our theorems.

Theorem 2.1. If N is a quadratic non- residue modulo D, then the Pell equation $x^2-Dy^2=N$ has no integer solution . $([5])$

Theorem 2.2. Let (x_1,y_1) be a fundamental solution to the equation $x^2-Dy^2=+1$. Then all positive integer solutions of the equation $x^2-Dy^2=+1$ are given by

$$x_n+\sqrt{D}\,y_n=\left(x_1+\sqrt{D}y_1\right)^n \qquad (3)$$

with $n\geq 2$. $([3])$

Theorem 2.3. Let D be a positive integer, that is not a perfect square. Then the continued fraction expansion of $\sqrt{D}$ such that $\sqrt{D}=\left[a_0;\overline{a_1,........,a_{l-1},2a_0}\right]$ where is $l(\sqrt{D})=l$ is the period length and the a_j 's are given by the recursion formulas;

$$a_0 = \lfloor \sqrt{D} \rfloor,$$

$$a_t = \lfloor \alpha_t \rfloor \text{ and } \alpha_{t+1} = \frac{1}{\alpha_t - a_t}, \quad t = 0,1,2,...$$

Recall that $a_l = 2a_0$ and $a_t = a_{l+t}$ for $t \geq 1$. The n^{th} convergent of $\sqrt{D}$ for $n \geq 0$ is given by

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

By means of the n^{th} convergent of $\sqrt{D}$, we can give the fundamental solution of the $x^2 - Dy^2 = \mp 1$. Let $p_{-1} = 1$, $p_0 = a_0$ and $q_{-1} = 0$, $q_0 = 1$.In general

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2} \qquad (4)$$

for $n \geq 1$. Then the fundamental solution of $x^2 - Dy^2 = +1$ is

$$(x_1, y_1) = \begin{cases} (p_{l-1}, q_{l-1}) & ;if\ l \text{ is even} \\ (p_{2l-1}, q_{2l-1}) & ;if\ l \text{ is odd} \end{cases} \qquad (5)$$

$([1],[2, p.154])$.

Theorem 2.4. If (u_1, v_1) and (x_{n-1}, y_{n-1}) are integer solutions of $x^2 - Dy^2 = \mp N$ and $x^2 - Dy^2 = +1$, respectively, then (u_n, v_n) is also a positive solution of $x^2 - Dy^2 = \mp N$, where

$$u_n + \sqrt{D} v_n = (x_{n-1} + \sqrt{D} y_{n-1}).(u_1 + \sqrt{D} v_1) \qquad (6)$$

for $n \geq 2$. $([4])$

III. The Main Results on The Pell Equation $x^2 - D.y^2 = 2^k$

By using recurrence on infinite sequence of positive solutions of the Pell equation $x^2 - D.y^2 = 2^k$ where $D = r^2 s^2 + 2.s$ with $r, s \geq 1$ integers and $k \geq 0$ is also an integer.First we consider the case $k = 0$, that is the classical Pell equation $x^2 - (r^2 s^2 + 2s).y^2 = 1$. Then, we can give following theorem.

Theorem 3.1. Let $D = r^2 s^2 + 2.s$ with $r, s \geq 1$ integers . Then the following conditions satisfy:

(a) The continued fraction expansion of $\sqrt{D}$ is;

$$\sqrt{D} = \begin{cases} [1; \overline{1,2}] & ;if\ r = s = 1 \\ [r; \overline{r, 2r}] & ;if\ r \geq 2,\ s = 1 \\ [rs; \overline{r, 2rs}] & ;if\ r \geq 2,\ s \geq 2 \end{cases}$$

(b) The fundamental solution of $x^2 - Dy^2 = 1$ is;

$$(x_1, y_1) = \begin{cases} (2,1) & ;if\ r = s = 1 \\ (r^2 + 1, r) & ;if\ r \geq 2,\ s = 1 \\ (r^2 s + 1, r) & ;if\ r \geq 2,\ s \geq 2 \end{cases}$$

(c) For $n \geq 4$,

$$(x_n) = \begin{cases} (3(x_{n-1} + x_{n-2}) - x_{n-3}) & ;if\ r = s = 1 \\ ((2r^2 + 1).(x_{n-1} + x_{n-2}) - x_{n-3}) & ;if\ r \geq 2,\ s = 1 \\ ((2r^2 s + 1).(x_{n-1} + x_{n-2}) - x_{n-3}) & ;if\ r \geq 2,\ s \geq 2 \end{cases}$$

and

$$(y_n) = \begin{cases} (3(y_{n-1} + y_{n-2}) - y_{n-3}) & ;if\ r = s = 1 \\ ((2r^2 + 1).(y_{n-1} + y_{n-2}) - y_{n-3}) & ;if\ r \geq 2,\ s = 1 \\ ((2r^2 s + 1).(y_{n-1} + y_{n-2}) - y_{n-3}) & ;if\ r \geq 2,\ s \geq 2 \end{cases}$$

Proof: (a) Assume that $r = s = 1$,by Theorem 2.3., it is easily seen that the continued fraction $\sqrt{3}$ is $\sqrt{3} = [1; \overline{1,2}]$.

Now, let $r \geq 2$, $s = 1$. Then

$$\sqrt{r^2+2} = r+\left(\sqrt{r^2+2}-r\right)$$
$$= r+\cfrac{1}{\cfrac{1}{\sqrt{r^2+2}-r}} = r+\cfrac{1}{\cfrac{\sqrt{r^2+2}+r}{2}}$$
$$= r+\cfrac{1}{r+\cfrac{\sqrt{r^2+2}-r}{2}}$$
$$= r+\cfrac{1}{r+\cfrac{1}{\cfrac{2}{\sqrt{r^2+2}-r}}} = r+\cfrac{1}{r+\cfrac{1}{\sqrt{r^2+2}+r}}$$
$$= r+\cfrac{1}{r+\cfrac{1}{\sqrt{r^2+2}+r}}$$
$$= r+\cfrac{1}{r+\cfrac{1}{2r+\left(\sqrt{r^2+2}-r\right)}}$$
$$\therefore \sqrt{D}=\sqrt{r^2+2}=\left[r;\overline{r,2r}\right]$$

Similarly, it can be shown that $\sqrt{D}=\sqrt{r^2s+2s}=\left[rs;\overline{r,2rs}\right]$ for $r,s\geq 2$.

(b) Since $(x_1,y_1)=(2,1)$ is a fundamental solution of $x^2-3y^2=1$, the case of $r=s=1$ is clear. Also, for $r\geq 2,\ s=1$ by using the method defined in Theorem 2.3., we get $l=2$, $a_0=r\ a_1=r$. Hence, $(x_1,y_1)=(p_1,q_1)=\left(r^2+1,r\right)$ is the fundamental solution since $p_{-1}=1\ p_0=a_0=r$, $q_{-1}=0$, $q_0=1$ and $\begin{matrix} p_1=a_0p_0+p_{-1}=r^2+1 \\ q_1=a_0q_0+q_{-1}=r \end{matrix}$ by (4) and (5) .

Finally, we assume that $r,s\geq 2$, by using the method defined in Theorem 2.3., we get $l=2$, $a_0=rs\ a_1=r$. Hence, $(x_1,y_1)=(p_1,q_1)=\left(r^2s+1,r\right)$ is the fundamental solution since $p_{-1}=1\ p_0=a_0=rs$, $q_{-1}=0$, $q_0=1$ and $\begin{matrix} p_1=a_0p_0+p_{-1}=r^2s+1 \\ q_1=a_0q_0+q_{-1}=r \end{matrix}$ by (4) and (5) .

(c) By Theorem 2.2., we can see easily that all solutions (x_n,y_n) of $x^2-Dy^2=+1$ can be derived from the fundamental solution (x_1,y_1) of this equation. Assume that $r=s=1$. In a similar way in ([5]) it can be shown by induction on n that
$x_n=3(x_{n-1}+x_{n-2})-x_{n-3}$,
$y_n=3(y_{n-1}+y_{n-2})-y_{n-3}$
for $n\geq 4$ .Moreover, in a similar way, we get
$x_n=\left(2r^2+1\right)(x_{n-1}+x_{n-2})-x_{n-3}$,
$y_n=\left(2r^2+1\right)(y_{n-1}+y_{n-2})-y_{n-3}$
where $D=r^2+2$ for $n\geq 4$ and
$x_n=\left(2r^2s+1\right)(x_{n-1}+x_{n-2})-x_{n-3}$,
$y_n=\left(2r^2s+1\right)(y_{n-1}+y_{n-2})-y_{n-3}$
where $D=r^2s^2+2.s$ for $n\geq 4$.

Now, we consider the general case for $k\geq 1$ .Note that we denote the integer solutions of $x^2-\left(r^2s^2+2s\right).y^2=2^k$ by (u_n,v_n) and denote the integer solutionsof $x^2-\left(r^2s^2+2s\right).y^2=1$ by (x_n,y_n).Then we have following theorem.

Theorem 3.2. Let $r=s=1$,that is $D=3$,and $k\geq 1$ be a arbitrary integer. Define a sequence $\{(u_n,v_n)\}$ of positive integers by

$$(u_1,v_1)=\begin{cases} no\ solution & ;if\ \ k\ is\ odd \\ \left(2^{\frac{k}{2}+1},2^{\frac{k}{2}}\right) & ;if\ \ k\ is\ even \end{cases} \qquad (7)$$

and ,since k is even, we get

$$\begin{aligned} u_n &= 2^{\frac{k}{2}+1}x_{n-1}+3.2^{\frac{k}{2}}y_{n-1} \\ v_n &= 2^{\frac{k}{2}}x_{n-1}+2^{\frac{k}{2}+1}y_{n-1} \end{aligned} \qquad (8)$$

where $\{(x_n,y_n)\}$is the sequence of positive solutions of $x^2-3.y^2=1$ .Then the following conditions satisfy with k is even;

(a) (u_n, v_n) is a solution of $x^2 - 3.y^2 = 2^k$ for any integer $n \geq 1$.

(b) For $n \geq 2$,

$$u_{n+1} = 2u_n + 3v_n \quad , \quad v_{n+1} = u_n + 2v_n$$

(c) For $n \geq 4$,

$$u_n = 3(u_{n-1} + u_{n-2}) - u_{n-3} \ ,$$
$$v_n = 3(v_{n-1} + v_{n-2}) - v_{n-3}$$

Proof : (a) Assume that k is odd. Since 2 is a quadratic non- residue $\bmod 3$, then $\left(\frac{2}{3}\right)^k = (-1)^k = -1$.By Theorem 2.1., the Pell Equation $x^2 - 3.y^2 = 2^k$ has no integer solution. Now, let k is even.Then it easily seen that $(u_1, v_1) = \left(2^{\frac{k}{2}+1}, 2^{\frac{k}{2}}\right)$ is a solution of $x^2 - 3.y^2 = 2^k$,that is

$$u_1^2 - Dv_1^2 = \left(2^{\frac{k}{2}+1}\right)^2 - 3.\left(2^{\frac{k}{2}}\right)^2 = 2^2 2^k - 3.2^k$$
$$= 2^k$$

Also, (u_n, v_n) is a solution for $n \geq 2$. We can prove this as follows. Recall that (x_{n-1}, y_{n-1}) is a solution of $x^2 - 3.y^2 = 1$,that is,

$$x_{n-1}^2 - 3.y_{n-1}^2 = 1 \qquad (9)$$

Further, we see (u_1, v_1) is a solution of $x^2 - 3.y^2 = 2^k$,that is,

$$u_1^2 - 3.v_1^2 = 2^k \qquad (10)$$

using $(9)$and (10), we find that

$$u_n^2 - 3v_n^2 = \left(2^{\frac{k}{2}+1} x_{n-1} + 3.2^{\frac{k}{2}} y_{n-1}\right)^2 - 3.\left(2^{\frac{k}{2}} x_{n-1} + 2^{\frac{k}{2}+1} y_{n-1}\right)^2$$
$$= x_{n-1}^2.\left(2^{k+2} - 3.2^k\right) + x_{n-1}y_{n-1}.\left(3.2^{k+2} - 3.2^{k+2}\right) + y_{n-1}^2.\left(3^2.2^k - 3.2^{k+2}\right)$$
$$= 2^k x_{n-1}^2 - 2^k 3y_{n-1}^2 = 2^k\left(x_{n-1}^2 - 3y_{n-1}^2\right) = 2^k$$

Therefore, (u_n, v_n) is a solution of $x^2 - 3.y^2 = 2^k$ for even k integers

(b) By Theorem 2.2. and Tehorem 2.4., we get

$$u_{n+1} + \sqrt{D}\, v_{n+1} = \left(x_n + \sqrt{D} y_n\right).\left(u_1 + \sqrt{D} v_1\right)$$
$$= \left(x_1 + \sqrt{D} y_1\right)^n.\left(u_1 + v_1\sqrt{D}\right)$$
$$= \left(x_1 + \sqrt{D} y_1\right).\left(u_n + \sqrt{D} v_n\right)$$

Since $(x_1, y_1) = (2,1)$is a fundamental solution of the Pell equation $x^2 - 3.y^2 = 1$, we get that

$$u_{n+1} = 2u_n + 3v_n \quad , \quad v_{n+1} = u_n + 2v_n$$

for $n \geq 2$.

(c) We see as above that

$$u_n = 2^{\frac{k}{2}+1} x_{n-1} + 3.2^{\frac{k}{2}} y_{n-1} \ , \ v_n = 2^{\frac{k}{2}} x_{n-1} + 2^{\frac{k}{2}+1} y_{n-1}$$

also $u_{n+1} = 2u_n + 3v_n$. In a similar way in ([5]), by induction on n and combining these two results, it can be shown that

$$u_n = 3(u_{n-1} + u_{n-2}) - u_{n-3}$$

for $n \geq 4$.

Similarly, combining (8) and $v_{n+1} = u_n + 2v_n$ results, we get $v_n = 3(v_{n-1} + v_{n-2}) - v_{n-3}$ for $n \geq 4$.

Example: Let $r = s = 1$ and $k = 4$ .Then, by Theorem 3.2., $(u_1, v_1) = (8, 4)$ is a solution of $x^2 - 3.y^2 = 2^4 = 16$, and some other solutions are;

$(u_2, v_2) = (28,16)$,
$(u_3, v_3) = (104, 60)$,
$(u_4, v_4) = (388, 224)$
$(u_5, v_5) = (1448, 836)$,
$(u_6, v_6) = (5404, 3120)$

Remark: Note that in Theorem 3.3. and Theorem 3.4., we will consider the case k is even . When we consider the case k is odd, then we find that there is a solution (u_1, v_1) of $x^2 - (r^2 + 2).y^2 = 2^k$ and $x^2 - (r^2 s^2 + 2s).y^2 = 2^k$ respectively, for some values of k , or there is no solution.

Forexample, $r = 4, s = 2$ and $k = 3$, we can not find solution of the Pell equation $x^2 - 68y^2 = 2^3 = 8$. But for $k = 5$,we find that $(u_1, v_1) = (10,1)$ is a solution of $x^2 - 68y^2 = 2^5 = 32$.

Moreover, for $r = 5, s = 3$ and for every odd k ,there is no solution of $x^2 - 301.y^2 = 2^k$.

Also, we can see that Keith Mathews' " Some Bc Math/ PHP Nunber Theory Programs" , 2013.

Theorem 3.3. Let $s=1$, $r\geq 2$ and k be arbitrary integers with $k\geq 1$ is even. Define a sequence $\{(u_n,v_n)\}$ of positive integers by

$$(u_1,v_1)=\left(2^{\frac{k}{2}}.(r^2+1),2^{\frac{k}{2}}r\right) \qquad (11)$$

and ,since k is even, we get

$$u_n=2^{\frac{k}{2}}(r^2+1)x_{n-1}+2^{\frac{k}{2}}r(r^2+2)y_{n-1}$$
$$v_n=2^{\frac{k}{2}}rx_{n-1}+2^{\frac{k}{2}}(r^2+1)y_{n-1} \qquad (12)$$

where $\{(x_n,y_n)\}$ is the sequence of positive solutions of $x^2-(r^2+2)y^2=1$ .Then the following conditions satisfy with k is even;

(a) (u_n,v_n) is a solution of $x^2-(r^2+2)y^2=2^k$ for any integer $n\geq 1$.

(b) For $n\geq 2$,

$$u_{n+1}=(r^2+1)u_n+r(r^2+2)v_n \quad , \quad v_{n+1}=r.u_n+(r^2+1)v_n$$

(c) For $n\geq 4$,

$$u_n=(2r^2+1)(u_{n-1}+u_{n-2})-u_{n-3} \ ,$$
$$v_n=(2r^2+1)(v_{n-1}+v_{n-2})-v_{n-3}$$

Proof : (a) Assume that k is even. Then , it easily seen that $(u_1,v_1)=\left(2^{\frac{k}{2}}.(r^2+1),2^{\frac{k}{2}}r\right)$ is a solution of $x^2-(r^2+2)y^2=2^k$ since

$$u_1^2-Dv_1^2=\left(2^{\frac{k}{2}}(r^2+1)\right)^2-(r^2+2)\left(2^{\frac{k}{2}}r\right)^2$$
$$=(r^2+1)^2 2^k-(r^2+2)r^2 2^k=2^k$$

Also, (u_n,v_n) is a solution for $n\geq 2$. We can prove this as follows. Note that by definition, (x_{n-1},y_{n-1}) is a solution of $x^2-(r^2+2)y^2=1$,that is,

$$x_{n-1}^2-(r^2+2)y_{n-1}^2=1 \qquad (13)$$

Further, we see above that (u_1,v_1) is a solution of $x^2-(r^2+2)y^2=2^k$,that is,

$$u_1^2-(r^2+2)v_1^2=2^k \qquad (14)$$

applying (13) and (14), we get

$$u_n^2-(r^2+2)v_n^2=\left(2^{\frac{k}{2}}(r^2+1)x_{n-1}+2^{\frac{k}{2}}r(r^2+2)y_{n-1}\right)^2-(r^2+2)\left(2^{\frac{k}{2}}rx_{n-1}+2^{\frac{k}{2}}(r^2+1)y_{n-1}\right)^2$$

$$=x_{n-1}^2.\left(2^k(r^2+1)^2-r^2(r^2+2)2^k\right)+x_{n-1}y_{n-1}.\left(2^{k+1}r(r^2+1)(r^2+2)\right)(1-1)$$
$$-y_{n-1}^2(r^2+2)2^k.\left(-r^2(r^2+2)+(r^2+1)^2\right)$$

$$=2^k x_{n-1}^2-2^k(r^2+2)y_{n-1}^2$$
$$=2^k\left(x_{n-1}^2-(r^2+2)y_{n-1}^2\right)=2^k$$

Therefore, (u_n,v_n) is a solution of $x^2-(r^2+2)y^2=2^k$ for even k integers.

(b) By Theorem 2.2. and Tehorem 2.4., we get

$$u_{n+1}+\sqrt{D}v_{n+1}=(x_n+\sqrt{D}y_n).(u_1+\sqrt{D}v_1)$$
$$=(x_1+\sqrt{D}y_1)^n.(u_1+v_1\sqrt{D})$$
$$=(x_1+\sqrt{D}y_1).(u_n+\sqrt{D}v_n)$$

Since $(x_1,y_1)=(r^2+1,r)$ is a fundamental solution of the Pell equation $x^2-(r^2+2)y^2=1$, we find that

$$u_{n+1}=(r^2+1)u_n+r(r^2+2)v_n \quad , \quad v_{n+1}=r.u_n+(r^2+1)v_n$$

for $n\geq 2$.

(c) Recall that

$$u_n=2^{\frac{k}{2}}(r^2+1)x_{n-1}+2^{\frac{k}{2}}r(r^2+2)y_{n-1} \ , \ v_n=2^{\frac{k}{2}}rx_{n-1}+2^{\frac{k}{2}}(r^2+1)y_{n-1}$$

by (12), also $u_{n+1}=(r^2+1)u_n+r(r^2+2)v_n$.
In a similar way in $([5])$, by induction on n and combining these two results, it can be shown that

$$u_n=(2r^2+1)(u_{n-1}+u_{n-2})-u_{n-3}$$

for $n\geq 4$.

Similarly, combining (12) and $v_{n+1}=r.u_n+(r^2+1)v_n$ results, we get

$$v_n=(2r^2+1)(v_{n-1}+v_{n-2})-v_{n-3}$$

for $n\geq 4$.

Example: Let $r=3,\ s=1$ and let $k=2$ .Then, we get $D=11$ and $x^2-11y^2=4$. By Theorem 3.3., $(u_1,v_1)=(20,6)$ is a solution of $x^2-11y^2=4$, and some other solutions are;

$(u_2,v_2)=(398,120)$,
$(u_3,v_3)=(7940,2394)$,
$(u_4,v_4)=(158402,47760)$,
$(u_5,v_5)=(3160100,952806)$

Theorem 3.4. Let $r,s\geq 2$ and k be arbitrary integers with $k\geq 1$ is even. Define a sequence $\{(u_n,v_n)\}$ of positive integers by

$$(u_1,v_1)=\left(2^{\frac{k}{2}}.(r^2s+1),2^{\frac{k}{2}}r\right) \qquad (15)$$

and ,since k is even, we get

$$u_n=2^{\frac{k}{2}}(r^2s+1)x_{n-1}+2^{\frac{k}{2}}r(r^2s^2+2s)y_{n-1}$$
$$v_n=2^{\frac{k}{2}}rx_{n-1}+2^{\frac{k}{2}}(r^2s+1)y_{n-1} \qquad (16)$$

where $\{(x_n,y_n)\}$ is the sequence of positive solutions of $x^2-(r^2s^2+2s).y^2=1$ .Then the following conditions hold;

(a) (u_n,v_n) is a solution of $x^2-(r^2s^2+2s).y^2=2^k$ for any integer $n\geq 1$.

(b) For $n\geq 2$,

$$u_{n+1}=(r^2s+1)u_n+r(r^2s^2+2s)v_n \quad , \quad v_{n+1}=r.u_n+(r^2s+1)v_n$$

(c) For $n\geq 4$,

$$u_n=(2r^2s+1)(u_{n-1}+u_{n-2})-u_{n-3} ,$$
$$v_n=(2r^2s+1)(v_{n-1}+v_{n-2})-v_{n-3}$$

Proof : (a) Assume that k is even. Then , it easily seen that $(u_1,v_1)=\left(2^{\frac{k}{2}}.(r^2s+1),2^{\frac{k}{2}}r\right)$ is a solution of $x^2-(r^2s^2+2s).y^2=2^k$ since

$$u_1^2-Dv_1^2=\left(2^{\frac{k}{2}}(r^2s+1)\right)^2-(r^2s^2+2s).\left(2^{\frac{k}{2}}r\right)^2$$
$$=(r^2s+1)^2 2^k-(r^2s^2+2s)r^2 2^k=2^k$$

Also, (u_n,v_n) is a solution for $n\geq 2$. We can prove this as follows. Recall that (x_{n-1},y_{n-1}) is a solution of $x^2-(r^2s^2+2s).y^2=1$,that is,

$$x_{n-1}^2-(r^2s^2+2s)y_{n-1}^2=1 \qquad (17)$$

Further, we see above that (u_1,v_1) is a solution of $x^2-(r^2s^2+2s).y^2=2^k$,that is,

$$u_1^2-(r^2s^2+2s).v_1^2=2^k \qquad (18)$$

applying (17) and (18), we get

$$u_n^2-(r^2s^2+2s)v_n^2=\left(2^{\frac{k}{2}}(r^2s+1)x_{n-1}+2^{\frac{k}{2}}r(r^2s^2+2s)y_{n-1}\right)^2-(r^2s^2+2s).\left(2^{\frac{k}{2}}rx_{n-1}+2^{\frac{k}{2}}(r^2s+1)y_{n-1}\right)^2$$
$$=x_{n-1}^2.\left(2^k(r^2s+1)^2-r^2(r^2s^2+2s)2^k\right)+x_{n-1}y_{n-1}.\left(2^{k+1}r(r^2s+1)(r^2s^2+2s)\right)(1-1)$$
$$-y_{n-1}^2(r^2s^2+2s)2^k.\left(-r^2(r^2s^2+2s)+(r^2s+1)^2\right)$$
$$=2^kx_{n-1}^2-2^k(r^2s^2+2s)y_{n-1}^2$$
$$=2^k\left(x_{n-1}^2-(r^2s^2+2s)y_{n-1}^2\right)=2^k$$

Hence, (u_n,v_n) is a solution of $x^2-(r^2s^2+2s).y^2=2^k$ for even k integers.

(b) By Theorem 2.2. and Tehorem 2.4., we get

$$u_{n+1}+\sqrt{D}v_{n+1}=\left(x_n+\sqrt{D}y_n\right)\left(u_1+\sqrt{D}v_1\right)$$
$$=\left(x_1+\sqrt{D}y_1\right)^n.\left(u_1+v_1\sqrt{D}\right)$$
$$=\left(x_1+\sqrt{D}y_1\right)\left(u_n+\sqrt{D}v_n\right)$$

Since $(x_1, y_1)=(r^2s+1, r)$ is a fundamental solution of the Pell equation $x^2-(r^2s^2+2s).y^2=1$, we find that

$$u_{n+1}=(r^2s+1)u_n+r(r^2s^2+2s)v_n \quad , \quad v_{n+1}=r.u_n+(r^2s+1)v_n$$

for $n\geq 2$.

(c) Recall that

$$u_n=2^{\frac{k}{2}}(r^2s+1)x_{n-1}+2^{\frac{k}{2}}r(r^2s^2+2s)y_{n-1} \quad , \quad v_n=2^{\frac{k}{2}}rx_{n-1}+2^{\frac{k}{2}}(r^2s+1)y_{n-1}$$

by (16), also

$$u_{n+1}=(r^2s+1)u_n+r(r^2s^2+2s)v_n \quad , \quad v_{n+1}=r.u_n+(r^2s+1)v_n$$

.

Combining these results as ([5]), we find by induction on n that

$$u_n=(2r^2s+1)(u_{n-1}+u_{n-2})-u_{n-3} ,$$
$$v_n=(2r^2s+1)(v_{n-1}+v_{n-2})-v_{n-3}$$

for $n\geq 4$.

Example: Let $r=3$, $s=2$ and let $k=6$ .Then, we get $D=40$ and $x^2-40y^2=64$. By Theorem 3.4., $(u_1,v_1)=(152,24)$ is a solution of $x^2-40y^2=64$, and some other solutions are;

$(u_2,v_2)=(5768,912)$,

$(u_3,v_3)=(219032,34632)$,

$(u_4,v_4)=(8317448,1315104)$,

$(u_5,v_5)=(315843992,49939320)$

IV. Acknowledgement

The research was supported by a grant from the coordination of scientific research projects with project number: 3466-D1-13 of Süleyman Demirel University in 2013.

References

[1] Baltus.C., Continued Fractions and the Pell Equations: The work of Euler and Lagrange, Comm.Anal. Theory Continued Fractions, 3, 1994, 4 – 31.

[2] Koblitz, N., A Course in Number Theory and Crytography Graduate Texts in Mathematics, Second Edition, Springer, 1994

[3] Leveque, J.W., Topics in Number Theory, Vol.1 and 2, Dover Publications, 2002.

[4] Nagell, T., Introduction to Number Theory , Chelsea Publishing Company, New York, 2001.

[5] Özer, Ö., Disproof the Conjecture Related to Pell Equation $x^2-Dy^2=2$, Advances in Theoretical and Applied Mathematics, Vol. 7, No.2, 2012, 197- 207

[6] Wright, M., Solving Pell Equations, Undergrad Honors Project, 2006, 1-38.