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Phase Trajectories of the Fractal Parametric Oscillator

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ABSTRACT
In this paper we consider a model of fractal parametric oscillator. Conducted a phase analysis of the decision model, and built its phase trajectories.
Keywords: parametric oscillator, the phase trajectories, the Mittag-Leffler function, the operator Gerasimov – Caputo

I. INTRODUCTION

In investigation nonlinear dynamical systems often use the concept of a fractal, which is associated with the study of fractal geometry and processes such as, deterministic chaos, which is consistent with the theory of physics of open systems [1].

The development of nonlinear dynamical systems has led to develop new methods of analysis. One of these is the method of fractional derivatives [2]. This method to determine the natural phenomenon, in nonlinear dynamic systems, and other: economic, social, humanitarian. If in this method change the order of the fractional derivative confirmed the known results, but also study new results.

Study presents a model of fractal parametric oscillation system and its phase trajectories. Development the theory of fractal oscillatory systems can be a method of Radio physics dynamic processes, such as ionosferno-magnetospheric plasma.

II. STATEMENT OF THE PROBLEM AND SOLUTION METHOD

In [3], the Cauchy problem for the fractional Mathieu equation: to find a solution, $u(t)$, $t \in [0, T_\alpha]$:

$$\partial_{0t}^{\beta} u(\tau) + \left[\delta + \xi \cos_\alpha(\omega t)\right] u(t) = 0, \tag{1}$$

$$u(0) = u_1, u'(0) = u_2. \tag{2}$$

Here $\cos_\alpha(\omega t) = E_{\alpha,1}\left[-(\omega t)^\alpha\right] = \sum_{k=0}^{\infty} \frac{(-1)^k (t\omega)^{\alpha k}}{\Gamma(\alpha k + 1)}$ - the generalized cosine (function of Mittag-Leffler) with a parameter $1 < \alpha < 2$. Put $\alpha = 2$, get the usual cosine, i.e. $\cos_2(\omega t) = \cos(\omega t)$. The left side of equation (1) is the fractional derivative of order $1 < \beta < 2$, which is defined as

$$\partial_{0t}^{\beta} u(\tau) = \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{u'(\tau)\, d\tau}{(t-\tau)^{\beta-1}}. \tag{3}$$

The parameters δ, ξ, u_1, u_2, defined constants.

If in (1) put $\xi = 0$ and $\delta = \varphi^\alpha$ then it is the well known equation of fractional oscillator, which is studied in [4]-[6]. In [4] investigated of the fractal oscillator was by using the fractional derivatives of Riemann-Liouville. In [5] and [6] - by using the Caputo operator (3), but the more correct to call it the Gerasimov-Caputo operator. Mention of this statement can be found in A. Gerasimov (1948) [7].

Equation (1) is a generalization of the Mathieu equation, which describes the parametric excitation of oscillation in mechanical systems, as well as the related phenomenon of parametric resonance. The system described by equation (1) call fractal parametric oscillator.

The solution of the problem (1-2) is a Volterra integral equation of the second kind [3]

$$u(t) - \int_0^t K(t-\tau) u(\tau) d\tau = g(t).$$

Here the kernel $K(t,\tau) = \xi (t-\tau)^{\beta-1} E_{\beta,\beta}\left[-\delta (t-\tau)^\beta\right] \cos_\alpha(\omega\tau)$ - and the right side of (2) $g(t) = u_1 E_{\beta,1}\left(-\delta t^\beta\right) + u_2 t E_{\beta,2}\left(-\delta t^\beta\right)$.

Note that if in (2) to put $\xi = 0$, obtain a well-known solution to the equation of fractional oscillator. [6]:

$$u(t) = u_1 E_{\beta,1}\left(-\delta t^\beta\right) + u_2 t E_{\beta,2}\left(-\delta t^\beta\right)$$

Solution of the integral equation (2) can be obtained using the composite trapezoidal quadrature formula:

$$u_1 = g_1,\ u_i = \frac{g_i + h\sum_{j=1}^{i-1}\gamma_j K_{i,j}u_j}{1-\frac{hK_{i,i}}{2}},$$

$$i = 2,3,...,n;\ h = \frac{T_\alpha}{n-1};\ \gamma_j = \begin{cases}1/2, j=1\\ 1, j>1\end{cases}. \quad (4)$$

In (4) needed that the denominator $1-\frac{hK_{i,i}}{2}\neq 0$. This can always be achieved by reducing the value of the step h.

III. THE PHASE TRAJECTORIES OF FRACTAL PARAMETRIC OSCILLATOR

Write the solution (2) as follows:

$$u(t) = u_1 E_{\beta,1}\left(-\delta t^\beta\right) + u_2 t E_{\beta,2}\left(-\delta t^\beta\right) + $$
$$+\xi\int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}\left[-\delta(t-\tau)^\beta\right]\cos_\alpha(\omega\tau)u(\tau)d\tau.$$

This equation can be written in following form:

$$u(t) = u_1 E_{\beta,1}\left(-\delta t^\beta\right) + u_2 t E_{\beta,2}\left(-\delta t^\beta\right) + \xi E_{0t,-\delta}^{-\beta,\beta} f(\tau)$$

(4)

Applying [8], there was used the following operator:

$$E_{0t,-\delta}^{-\beta,\beta} f(\tau) = \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}\left[-\delta(t-\tau)^\beta\right] f(\tau)d\tau,$$

$$f(\tau) = \cos_\alpha(\omega\tau)u(\tau).$$

(5)

Some properties of (5) are considered in [8]. Apply the Gerasimov-Caputo operator fractional differential form $\partial_{0t}^{\beta-1}$ to the solution (4), obtain:

$$w(t) = \partial_{0t}^{\beta-1}u(\tau) = \partial_{0t}^{\beta-1}\left[u_1 E_{\beta,1}\left(-\delta t^\beta\right)\right] +$$
$$+\partial_{0t}^{\beta-1}\left[u_2 t E_{\beta,2}\left(-\delta t^\beta\right)\right] + \xi\partial_{0t}^{\beta-1}\left[E_{0t,-\delta}^{-\beta,\beta} f(\tau)\right].$$

(6)

Consider each of the terms in equation (6).

$$\partial_{0t}^{\beta-1}\left[u_1 E_{\beta,1}\left(-\delta t^\beta\right)\right] = u_1\partial_{0t}^{\beta-1}\sum_{k=0}^\infty\frac{(-\delta)^k t^{\beta k}}{\Gamma(\beta k+1)} =$$
$$= u_1\sum_{k=0}^\infty\frac{(-\delta)^k\partial_{0t}^{\beta-1}\left[t^{\beta k}\right]}{\Gamma(\beta k+1)} = -u_1\delta t E_{\beta,2}\left(-\delta t^\beta\right).$$

(7)

In (7) have used the property [9]:

$$\partial_{0t}^{\alpha}\left[t^{\beta-1}\right] = \frac{\Gamma(\beta)t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

Similarly, obtain for the second term in (6)

$$\partial_{0t}^{\alpha}\left[t^{\beta-1}\right] = \frac{\Gamma(\beta)t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

(8)

Consider the last term in (6) $\xi\partial_{0t}^{\beta-1}\left[E_{0t,-\delta}^{-\beta,\beta} f(\tau)\right]$. In [8] proved the semigroup property of the operator:

$$D_{0t}^{-\alpha}E_{0t,-\delta}^{-\beta,\beta} f(\tau) = E_{0t,-\delta}^{-(\beta+\alpha),\beta} f(\tau), \alpha>0, \beta>0,$$

(9)

where $D_{0t}^{-\alpha} f(\tau) = \frac{1}{\Gamma(-\alpha)}\int_0^t\frac{f(\tau)d\tau}{(t-\tau)^{1-\alpha}}$ - the fractional integration operator. The Gerasimov-Caputo operator can be expressed through the fractional integral: $\partial_{0t}^{\beta-1} f(\tau) = D_{0t}^{-(2-\beta)}D_{0t}^1 f(\tau)$.

In view of this property and the property (9), obtain:

$$\xi\partial_{0t}^{\beta-1}\left[E_{0t,-\delta}^{-\beta,\beta} f(\tau)\right]\xi D_{0t}^{-(2-\beta)}D_{0t}^1\left[E_{0t,-\delta}^{-\beta,\beta} f(\tau)\right] =$$
$$= \xi D_{0t}^{-(2-\beta)}D_{0t}^1 D_{0t}^{-1}\left[E_{0t,-\delta}^{-(\beta-1),\beta} f(\tau)\right] =$$
$$= \xi\int_0^t E_{\beta,1}\left(-\delta(t-\tau)^\delta\right)\cos_\alpha(\omega\tau)u(\tau)d\tau. \quad (10)$$

Finally, obtain the equation for the derivative of the order $\beta-1$:

$$w(t) = -u_1\delta t E_{\beta,2}\left(-\delta t^\beta\right) - u_2\delta t^2 E_{\beta,3}\left(-\delta t^\beta\right) +$$
$$+\xi\int_0^t E_{\beta,1}\left(-\delta(t-\tau)^\delta\right)\cos_\alpha(\omega\tau)u(\tau)d\tau.$$

(11)

Equation (11) is an integral Volterra equation of the second kind, and it can be solved numerically using the method (3). Phase trajectories are based on the respective pairs of points (u_i, w_i).

IV. THE RESULTS OF THE CALCULATIONS

In the numerical simulation, for simplicity, put $u_2 = 0$ and $u_1 = \omega = 1$.

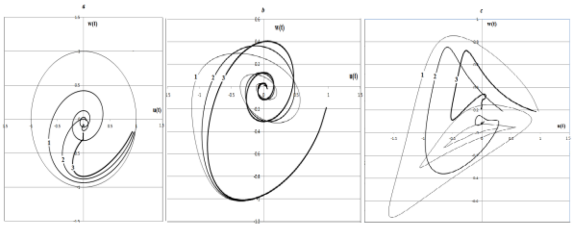


Fig. 1. Phase trajectories of: a) a harmonic oscillator and a fractal oscillator $\xi = 0$: curve 1 - $\beta = 1.8$, curve 2 - $\beta = 1.6$, curve 3 - $\beta = 1.4$; b): fractal parametric oscillator $\alpha = 2, \xi = 1$: curve 1 - $\beta = 1.8$, curve 2 - $\beta = 1.6$, curve 3 $\beta = 1.4$; c) $\beta = 1.8, \xi = 1$: curve 1 - $\alpha = 1.8$, curve 2 - $\alpha = 1.6$, curve 3 - $\alpha = 1.4$.

In Fig. 1 presents phase trajectories with the parameters $n = 100$, $t\in(0,2\pi)$ for the following cases: (Fig. 1a) harmonic oscillator - a circle, plot with the values of the parameters $\beta = 2, \xi = 0$, the fractal oscillator - (Fig. 1b) fractal parametric oscillator - and (Fig. 1c).

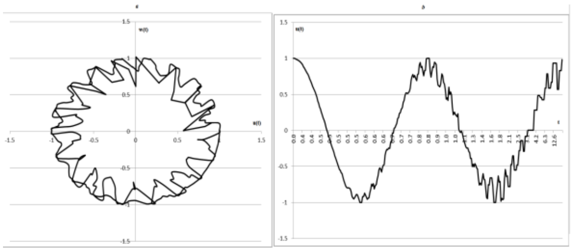
Fig. 1a (state trajectory is a circle) corresponds to the classical harmonic oscillations of the system. The phase trajectories of the fractal oscillator are coordinated with phase trajectories plotted in [4].

New results are shown in Fig.1b and Fig.1c. In the first case, for $\beta = 1{,}8$ and for different values α, the phase trajectories of parametric oscillator are similar to the phase trajectories of fractal oscillator (Fig. 1a), but have a decaying type view of sustainable focus. Seen (Fig. 1b) sometimes are regrouping trajectories. This is due to the properties of the generalized cosine function, which is included in the original equation (1). Phase trajectory in Fig. 1c was plotted with $\alpha = 2$ and different values β. In this case phase trajectories have a different view.

In Fig. 2 shows the phase trajectory in Fig. 2a and the solution of the original Cauchy problem (1) - (2) According to [4] for the parameter value, the fractional parameter depends on time according to the law. Parameters $\varepsilon \ge 0$ and $g \ge 0$ define the limits of values $\beta(t)$: $1+\varepsilon < \beta(t) < 2-g$ and satisfy inequality $g+\varepsilon < 1$, m – any number.

For the calculations were selected following parameters:

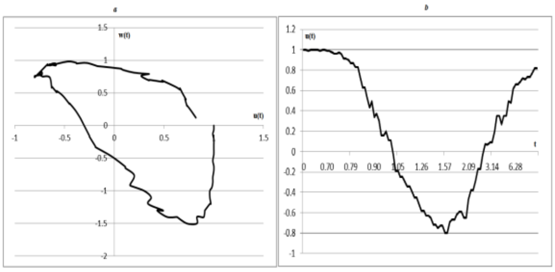
$$n = 300,\ t \in (0, 4\pi),\ \varepsilon = 0{,}95,\ g = 0, m = 18.$$



In Fig. 2a shows that the phase trajectory has multiple return points, which corresponds to the results of [4]. In Fig. 2b plotted the calculated curve - the shift function. The curve has an oscillating form of constant amplitude. This is due to the introduction in equation (1) the differentiation operator of fractional order β.

In Fig. 3 shows the phase trajectory and the offset $u(t)$ for the fractal parametric oscillator. For the calculation put:

$$n = 100,\ t \in (0, 2\pi), g = 0.01,\ \xi = 1, \varepsilon = 0.95, k = 18, \alpha = 1.8 \cdot$$



In Fig. 3a shows the difference between the phase trajectory of the fractal parametric oscillator and the phase trajectory of the fractal oscillator (Fig. 3a). In this case the phase trajectory is not closed. An offset $u(t)$ (Fig. 3b) has a more complex structure: more frequent oscillations with variable amplitude.

If changing β from 1 to 2, get different fractional equation (1), each has a family of solutions and properties. This allows us to make the following observation.

Notice. In study [4], the authors concluded that the nonlinear signal any oscillating system can be parameterized by a solution of a fractal oscillator. Similarly, can assume that the complex signal represented by a solution of the fractal parametric oscillator. Build a model of a signal $f(t)$, then by (2) get:

$$f(t) = g(t) - \int_0^t K(t-\tau) f(\tau) d\tau =$$

$$= u_1 E_{\beta,1}\left(-\delta t^{\beta}\right) + u_2 t E_{\beta,2}\left(-\delta t^{\beta}\right) +$$

$$+\xi \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}\left[-\delta (t-\tau)^{\beta}\right] E_{\alpha,1}\left(-(\omega\tau)^{\alpha}\right) f(\tau) d\tau$$

Function values $f(t)$ at discrete points in time are known from the experiment. Then the problem of identification of the model is: to define $\beta = \beta(u_1, u_2, \alpha, \omega, t)$. In contrast to study [4] the value β also depends on the fractional parameter α, it led more flexibility to present the original signal. This problem is quite complex and deserves attention in the future study of the properties of fractal parametric oscillator.

V. CONCLUSION

Solution the Cauchy problem (1-2) generalizes the well-known solutions for the harmonic oscillator and fractal oscillator. Were plotted phase trajectories. Calculations confirmed the earlier results [4] and led to new results. The phase trajectories of the parametric oscillator differ from the previous ones and have a more complex structure. So there may be non-linear effects in such systems. Suggested that the nonlinear signal can be represented by a fractal parametric oscillator solutions that deserve further study

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