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The Spinning Motion of a Gyrostat under the Influence of Newtonian Force Field and a Gyrostatic Moment Vector

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In this paper, the rotational motion of a gyrostat about a fixed point in a central Newtonian force field is considered. This body is acted upon by a gyrostatic moment vector $\underline{\ell}$. We consider the motion of the body in a case analogous to Lagrange's case. The analytical periodic solutions of the equations of motion are obtained using the Poincaré's small parameter method. A geometric interpretation of motion is given by using Euler's angles to describe the orientation of the body at any instant of time. The graphical representations of these solutions are presented when the different parameters of the body are acted. The fourth order Runge-Kutta method is applied to investigate the numerical solutions of the autonomous system. A comparison between the analytical and the numerical solutions shows a good agreement between them and the deviations are very small.
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I. Introduction

The rotational motion of a gyrostat about a fixed point in a uniform force field or in a Newtonian one is one of the important problems in the theoretical classical mechanics. This problem had shed the interest of many outstanding researchers e. g. [1-14]. In fact this problems require complicated mathematical techniques. It is known that this motion is governed by six non-linear differential equations with three first integrals [15].

Many attempts were made by outstanding scientists to find the solution of these equations but they have not found it in its full generality, except for three special cases (Euler-Poinsot, Lagrange-Poisson and Kovalevskaya). These cases have certain restrictions on the location of the body's centre of mass and on the values of the principal moments of inertia [1-3]. Arkhangel'skii, Iu. A. [1] showed that this fourth algebraic integral exists only in two special cases analogous to those of Euler and Lagrange and that, other cases with single-valued integrals are not additional cases but it can be reduced to previous cases. The necessary and sufficient condition for some functions to be a first integral for the Euler- Poisson equations when the motion of a rigid body is acted upon by a central Newtonian force field is investigated in [4]. The Hess's case for the motion of a rigid body was studied in [5] having the assumption of giving initial high value for the angular velocity about some axis, is imparted to the body.

The motion of Kovalevskaya gyroscope was studied in [6-11]. In [8], the existance of periodic solutions for the equation of motion of a rigid body in a Kovalevskaya top are obtained and it has been extended in [9]. The periodic solutions nearby equilibrium points for the same problem are investigated in [10] using the Liapunov theorem of holomorphic integral when the body moves under the influence of a central Newtonian field. The author generalized this problem in [11] when the body acted by potential and groscopic forces. An exceptional case of motion of this gyroscope was treated in [12].

In [16], the authors have obtained the ten classical integrals for the generalized problem of the roto-translatory motion of n gyrostats $n \geq 2$. This problem was studied in [17], when a system was made of two gyrostats attracting one another according to Newton's law. The problem of the earth's rotation, using a symmetrical gyrostat as a model was considered in [18]. The authors considered the first two components of the gyrostatic moment are null and the third component is chosen as a constant. This study was extended and was generalized in [19].

The small parameter method of Poincaré [20] was used to find the first terms of the series expansion of the periodic solutions of the equations of motion of a rotating heavy rigid body about a fixed point when the body spins rapidly about the dynamically symmetric axis [21,22] and acted by the gravitaional and Newtonian force field respectively. This problem was generalized in [23] when the body moves under the unfluence of Newtonian force field and the third component of gyrostatic moment vector.

The problem of a perturbed rotational motion of a heavy solid close to regular precession with constant restoring moment was treated in [13] and [14]. It is assumed that the angular velocity of the body is sufficiently high, its direction is close to the axis of dynamic symmetry of the body, and the perturbing moments are small in comparison with the gravity moments. Averaged systems of the equations of motion are obtained in the first and second approximations in terms of the small parameter. The perturbed problem of the rotatory motion of a symmetric gyrostat about a fixed point with the third non-zero component of a gyrostatic moment vector (about the axis of symmetry) and under the action of some moments was considered in [24]. This problem is generalized in [25]. The problem of existence of periodic motions of a solid was studied in [26]. The author used the Poincaré's method of small parameter to obtain the periodic solutions of the equations of motion. It was assumed that the center of mass of the solid differs little from a dynamically symmetric axis. This problem was generalized in [27], when the body rotates under the action of a central Newtonian force field and the third component of the gyrostatic moment vector.

In this work, the rotational motion of a gyrostat about a fixed point in a central Newtonian force field analogous to Lagrange's case is studied when the body is acted upon by a gyrostatic moment vector about the moving axes. The equations of motion and their first integrals are obtained and have been reduced to a quasilinear autonomous system of two degrees of freedom with one first integral. Poincaré's small parameter method [20] is applied to investigate the analytical periodic solutions of the equations of motion of the body with one point fixed, rapidly spinning about one of the principal axes of the ellipsoid of inertia. A geometric interpretation of motion is given by using Euler's angles [28] to describe the orientation of the body at any instant of time. The numerical solutions of the autonomous system are obtained using the fourth order Runge-Kutta method [29]. The phase plane diagrams describe the stability are presented. A comparison between the analytical and the numerical solutions shows a good agreement between them and the deviations are very small.

The model of a gyrostat has a wide range of applications in various fields such as satellite, robot manipulators, and spacecraft. Moreover, the study of the rotational motion of a gyrostat has been motivated by industrial applications in many fields. This is because the gyrostat provides a convenient model for the satellite-gyrostat, spacecraft and like; see [30,31]

II. Equations of Motion and Change of Variables

Consider a rigid body (gyrostat) of mass M, with one fixed point O; its ellipsoid of inertia is arbitrary and acted upon by a central Newtonian force field arising from an attracting centre O_1 being located on a downward fixed axis OZ passing through the fixed point with gyrostatic moment vector $\underline{\ell} \equiv (\ell_1, \ell_2, \ell_3)$ about x, y and z axes respectively.

It is taken into consideration that at the initial time, the body rotates about $z-$axis with a high angular velocity r_0, and that this axis makes an angle $\theta_0 \neq n\pi/2$ $(n = 0, 1, 2, ...)$ with the $Z-$axis. Without loss of generality, we select the positive branches of the $z-$ axis and of the $x-$ axis in a way to avoid an obtuse angle with the direction of the $Z-$ axis. The equations of motion and their three first integrals similar to Lagrange case take the forms [32,33]

$$\begin{aligned}
&\dot{p}_1 + A_1 q_1 r_1 + A^{-1}[r_0^{-1} q_1 \ell_3 - (c\sqrt{\gamma_0''})^{-1} r_1 \ell_2] = -\varepsilon a^{-1}(z_0' \gamma_1' - k a A_1 \gamma_1' \gamma_1''),\\
&\dot{q}_1 + B_1 p_1 r_1 - B^{-1}[r_0^{-1} p_1 \ell_3 - (c\sqrt{\gamma_0''})^{-1} r_1 \ell_1] = \varepsilon b^{-1}(z_0' \gamma_1 + a B_1 \gamma_1 \gamma_1''),\\
&\dot{r}_1 = \varepsilon^2 [(cC\sqrt{\gamma_0''})^{-1} (q_1 \ell_1 - p_1 \ell_2) - (C_1 p_1 q_1 - k C_1 \gamma_1 \gamma_1')],\\
&\dot{\gamma}_1 = r_1 \gamma_1' - \varepsilon q_1 \gamma_1'', \qquad \dot{\gamma}_1' = \varepsilon p_1 \gamma_1'' - r_1 \gamma_1, \qquad \dot{\gamma}_1'' = \varepsilon (q_1 \gamma_1 - p_1 \gamma_1');
\end{aligned} \tag{1}$$

$$r_1^2 = 1 + \varepsilon^2 S_1, \quad r_1 \gamma_1'' = 1 + \varepsilon S_2, \quad \gamma_1^2 + \gamma_1'^2 + \gamma_1''^2 = (\gamma_0'')^{-2}; \tag{2}$$

where

$$\begin{aligned}
S_1 &= a\{(p_{10}^2 - p_1^2) + (q_{10}^2 - q_1^2) + k[(\gamma_{10}^2 - \gamma_1^2) + (\gamma_{10}'^2 - \gamma_1'^2) + (1 - \gamma_1''^2)]\} - 2 z_0' (1 - \gamma_1'')\\
S_2 &= a[(p_{10}\gamma_{10} - p_1 \gamma_1) + (q_{10}\gamma_{10}' - q_1 \gamma_1')] + (cC\sqrt{\gamma_0''})^{-1}[\ell_1(\gamma_{10} - \gamma_1) + \ell_2(\gamma_{10}' - \gamma_1')\\
&\quad + \ell_3 (1 - \gamma_1'')];
\end{aligned} \tag{3}$$

$$\begin{aligned}
&p = c\sqrt{\gamma_0''}\, p_1, \quad q = c\sqrt{\gamma_0''}\, q_1, \quad r = r_0 r_1, \quad k = N/c^2 \qquad (\,.\equiv d/d\tau\,),\\
&\gamma = \gamma_0'' \gamma_1, \quad \gamma' = \gamma_0'' \gamma_1', \quad \gamma'' = \gamma_0'' \gamma_1'', \quad t = \tau/r_0, \quad \gamma_0 > 0, \quad 0 < \gamma_0'' < 1;
\end{aligned} \tag{4}$$

$$A_1 = -B_1 = (C-A)/A, \quad A = B, \quad a = b = A/C, \quad c^2 = M\,g\,l/C,$$
$$\varepsilon = c\sqrt{\gamma_0''}/r_0, \quad x_0 = y_0 = 0, \quad z_0 \neq 0, \quad N = 3g/R, \quad g = \lambda/R^2, \tag{5}$$

A, B and C are the principal moments of inertia; x_0, y_0 and z_0 are the coordinates of the centre of mass in the moving coordinate system $(Oxyz)$; γ, γ' and γ'' are the direction cosines of the downwards fixed Z–axis of the fixed frame in space $(OXYZ)$; p, q and r are the projections of the angular velocity vector of the body on the principal axes of inertia; R is the distance from the fixed point O to the centre of attraction O_1; λ is the coefficient of attraction of such centre; ℓ_1, ℓ_2 and ℓ_3 are the components of the gyrostatic moment vector $\underline{\ell}$; and p_0, q_0, r_0, γ_0, γ_0' and γ_0'' are the initial values of the corresponding variables.

III. Reduction of the Equations of Motion to a Quasilinear Autonomous System

From the first two equations of (2), one can express the variables r_1 and γ_1'' as

$$\begin{aligned} r_1 &= 1 + \frac{1}{2}\varepsilon^2 [S_1 + 2z_0'(1-\gamma_1'') \\ &\quad - k(1-\gamma_1''^2)] + \cdots, \\ \gamma_1'' &= 1 + \varepsilon S_2 - \frac{1}{2}\varepsilon^2 [S_1 + 2z_0'(1-\gamma_1'') \\ &\quad - k(1-\gamma_1''^2)] + \cdots, \end{aligned} \tag{6}$$

Differentiate the first and the fourth equations of (1) and use (6) to reduce the four remaining equations to the following two second order differential equations

$$\begin{aligned} \ddot{p}_1 + \omega'^2 p_1 &= (cA\sqrt{\gamma_0''})^{-1} A_1 \ell_1 + \varepsilon\{A_1 (Cr_0)^{-1}(q_1\ell_1 - p_1\ell_2) q_1 + A_1(Ar_o)^{-1}\ell_1 S_1 \\ &\quad - (Ar_0)^{-1} C_1 p_1 q_1 \ell_2 + z_0 a^{-1}(1-A_1)\gamma_1 + k(\omega^2 - A_1)\gamma_1 + (Ar_0)^{-1} \\ &\quad \times [a^{-1} z_0' \gamma_1 + kA_1\gamma_1]\ell_3\} + \varepsilon^2\{[-\omega^2 S_1 p_1 - a^{-1} z_0' p_1] + kA_1[p_1(1-\gamma_1'^2) \\ &\quad + q_1(1-C_1)\gamma_1\gamma_1' - (1-A_1)S_2\gamma_1] - r_0^{-1}\ell_3 p_1 A^{-1} A_1 [S_1 + 2z_0'(1-\gamma_0'') \\ &\quad - k(1-\gamma_1''^2)] + (Ar_0)^{-1} kA_1\gamma_1 S_2 \ell_3\} + \varepsilon^3\{\frac{1}{2} z_0'\gamma_1 a^{-1}(1-A_1)[S_1 \\ &\quad + 2z_0'(1-\gamma_0'') - k(1-\gamma_1''^2)] - \frac{1}{2}(Ar_0)^{-1} kA_1\gamma_1[S_1 + 2z_0'(1-\gamma_1'') \\ &\quad - k(1-\gamma_1''^2)]\ell_3 + (2kA_1 - a^{-1} z_0') S_2 p_1\} + \cdots, \end{aligned} \tag{7}$$

$$\begin{aligned} \ddot{\gamma}_1 + \gamma_1 &= (Ar_0)^{-1}\ell_1 + \varepsilon[(1-A_1)p_1 + (Ar_0)^{-1}(\ell_1 S_2 - \ell_3 p_1) + (Cr_0)^{-1}(q_1\ell_1 \\ &\quad - p_1\ell_2)\gamma_1'] + \varepsilon^2[-S_1\gamma_1 + (1-A_1)p_1 S_2 + p_1 q_1 \gamma_1' - \gamma_1(z_0' a^{-1} + q_1^2) \\ &\quad + kA_1\gamma_1] + \varepsilon^3(a^{-1} z_0' - 2kA_1)\gamma_1 S_2 + \cdots, \end{aligned} \tag{8}$$

where ω' is a new frequency called Ismail and Amer's frequency [23] and takes the form

$$\omega'^2 = \omega^2 - 2A^{-1}A_1 r_0^{-1}\ell_3, \qquad \omega = A_1.$$

Here r_0 is large, so r_0^{-2}, r_0^{-3}, $\cdots$ are neglected. Solving the first and the fourth equations of system (1), and using (6), we obtain q_1 and γ_1' in the form

$$\begin{aligned} q_1 &= (A_1 r_1)^{-1}[1 - (AA_1 r_0 r_1)^{-1}\ell_3 + \cdots][(cA\sqrt{\gamma_0''})^{-1} r_1 \ell_2 - \dot{p}_1 \\ &\quad - \varepsilon a^{-1}(z_0'\gamma_1' - kaA_1\gamma_1'\gamma_1'')], \\ \gamma_1' &= r_1^{-1}(\dot{\gamma}_1 - \varepsilon q_1 \gamma_1''). \end{aligned} \tag{9}$$

Let us introduce new variables p_2 and γ_2 such that

$$p_2 = p_1 - \varepsilon\,\chi - \varepsilon\,\chi_1\gamma_2, \qquad \gamma_2 = \gamma_1 - \varepsilon\,\nu\,p_2, \tag{10}$$

where

$$\chi = (C r_0 A_1 A^2 \omega'^2)^{-1} y_2^2 \ell_1,$$
$$\chi_1 = (1-\omega'^2)^{-1}[-z_0' a^{-1}(1-A_1) + k(A_1 - \omega^2) + (Ar_0)^{-1}(a^{-1} z_0' - k A_1)\ell_3],$$
$$\nu = (1-\omega'^2)^{-1}[1 - A_1 - (Ar_0)^{-1}\ell_3], \qquad y_i = (cC\sqrt{\gamma_0''})^{-1}\ell_i \qquad i = 1,2,3.$$

In terms of the new variables p_2 and γ_2, the variables q_1 and γ_1' take the form

$$\begin{aligned} q_1 &= -X(\dot p_2 - C A^{-1} y_2) - \varepsilon X[\chi_2 \dot\gamma_2 - (Ar_0)^{-1}\ell_2 S_{11}] + \varepsilon^2\{X[(kA_1 \\ &\quad - a^{-1} z_0')\nu + S_{11}]\dot p_2 - \frac{1}{2} A_1^{-1} S_{11}\dot p_2 + X(kA_1\dot\gamma_2)S_{21}\} + \cdots, \\ \gamma_1' &= \dot\gamma_2 + X(Ar_0)^{-1}\ell_2 + \varepsilon[\nu_2\,\dot p_2 + X(Ar_0)^{-1}\ell_2 S_{21}] + \varepsilon^2\{X[(Ar_0)^{-1}\ell_2 S_{11} \\ &\quad - \chi_2\dot\gamma_2 - S_{21}\dot p_2] - \frac{1}{2} S_{11}\dot\gamma_2\} + \cdots, \end{aligned} \tag{11}$$

where

$$X = A_1^{-1}[1 - (AA_1 r_0)^{-1}\ell_3], \quad \chi_2 = \chi_1 + a^{-1} z_0' - kA_1, \quad \nu_2 = \nu - X.$$

Making use of (11) and (10) into (3), we obtain the following expressions for S_1 and S_2 in terms of power series in ε

$$S_i = S_{i1} + 2^{2-i}\varepsilon S_{i2} + \cdots, \qquad (i = 1,2) \tag{12}$$

where

$$\begin{aligned} S_{11} &= a\{(p_{20}^2 - p_2^2) + X^2(\dot p_{20}^2 - \dot p_2^2) + k[(\gamma_{20}^2 - \gamma_2^2) + (\dot\gamma_{20}^2 - \dot\gamma_2^2) \\ &\quad + 2X(Ar_0)^{-1}(\dot\gamma_{20} - \dot\gamma_2)\ell_2] - 2CX^2A^{-1}y_2(\dot p_{20} - \dot p_2)\}, \\ S_{12} &= a[\chi(p_{20} - p_2) + \chi_1(p_{20}\gamma_{20} - p_2\gamma_2)] - aX^2[(Ar_0)^{-1}\ell_2 S_{11} \\ &\quad \times(\dot p_{20} - \dot p_2) - \chi_2(\dot p_{20}\dot\gamma_{20} - \dot p_2\dot\gamma_2)] + (z_0' - k)S_{21} + k\{\nu a(p_{20}\gamma_{20} \\ &\quad - p_2\gamma_2) + a\nu_2(\dot p_{20}\dot\gamma_{20} - \dot p_2\dot\gamma_2) + X(Ar_0)^{-1} a\ell_2[(\dot\gamma_{20} - \dot\gamma_2)S_{21} \\ &\quad + (\dot p_{20} - \dot p_2)\nu_2]\} - CX^2A^{-1}\chi_2 y_2(\dot\gamma_{20} - \dot\gamma_2), \\ S_{21} &= a\{(p_{20}\gamma_{20} - p_2\gamma_2) - X[(\dot p_{20}\dot\gamma_{20} - \dot p_2\dot\gamma_2) - X(Ar_0)^{-1}\ell_2(\dot p_{20} - \dot p_2)]\} \\ &\quad + y_1(\gamma_{20} - \gamma_2) + y_2(1 - aXCA^{-1})(\dot\gamma_{20} - \dot\gamma_2), \\ S_{22} &= a[\nu(p_{20}^2 - p_2^2) + \chi(\gamma_{20} - \gamma_2) + \chi_1(\gamma_{20}^2 - \gamma_2^2)] + aX\{-\nu_2(\dot p_{20}^2 - \dot p_2^2) \\ &\quad + A^{-1}(Cy_2\nu_2 - Xr_0^{-1}\ell_2 S_{21})(\dot p_{20} - \dot p_2) + \ell_2(Ar_0)^{-1}(S_{11} - X\chi_2) \\ &\quad \times(\dot\gamma_{20} - \dot\gamma_2) - \chi_2(\dot\gamma_{20}^2 - \dot\gamma_2^2)\} + y_1\nu(p_{20} - p_2) + y_2(\dot p_{20} - \dot p_2) - y_3 S_{21}. \end{aligned} \tag{13}$$

From (12), (13) and (6), we get

$$
\begin{aligned}
r_1 &= 1+\frac{1}{2}\,\varepsilon^2 S_{11}+\varepsilon^3[\,S_{12}+(k-z_0')\,S_{21}\,]+\cdots,\\
\gamma_1'' &= 1+\varepsilon\,S_{21}+\varepsilon^2(S_{22}-\frac{1}{2}\,S_{11})-\varepsilon^3[\,S_{12}+(k-z_0')\,S_{21}\,]+\cdots.
\end{aligned} \tag{14}
$$

Substituting (10), (11), (12), (13) and (14) into (7) and (8), we obtain the following quasilinear autonomous system of two degrees of freedom

$$
\begin{aligned}
&\ddot p_2+\omega'^2 p_2 = C\,A_1 A^{-1} y_1+\varepsilon\,F(\,p_2,\,\dot p_2,\gamma_2,\dot\gamma_2,\varepsilon\,),\\
&\ddot\gamma_2+\gamma_2=(A r_0)^{-1}\ell_1+\varepsilon\,\Phi(\,p_2,\,\dot p_2,\gamma_2,\dot\gamma_2,\varepsilon\,),
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
&F=F_1+\varepsilon F_2+\varepsilon^2 F_3+\cdots, \qquad\qquad \Phi=\Phi_1+\varepsilon\Phi_2+\varepsilon^2\,\Phi_3+\cdots,\\
&F_1=-(A r_0)^{-1}\chi_1\,\ell_1+(A_1 r_0)^{-1}\ell_2\{\,C^{-1}[\,(\,\dot p_2^2-2\,A^{-1}y_2\,\dot p_2)+2\,\dot p_2\,\chi_2\,\dot\gamma_2]\\
&\qquad -C^{-1}A_1\,p_2(A^{-1}\,y_2-\dot p_2\,)\,\}-(A r_0)^{-1}k\,C_1\,\gamma_2\,\dot\gamma_2\,\ell_2,\\
&\Phi_1=(A r_0)^{-1}S_{21}\,\ell_1+(C r_0)^{-1}\dot\gamma_2[\,A^{-1}\ell_1(A^{-1}y_2-\dot p_2\,)-\ell_2\,p_2\,]\\
&F_2=f_2-\nu\,\chi_1(1-\omega'^2\,)p_2, \qquad\qquad \Phi_2=\varphi_2+\nu\,(1-\omega'^2)(\,\chi+\chi_1\gamma_2\,),\\
&F_3=f_3-\chi_1\varphi_2-\nu\,\chi_1(\,1-\omega'^2\,)(\,\chi+\chi_1\gamma_2\,), \qquad \Phi_3=\varphi_3-\nu\,f_2+\nu^2\chi_1(1-\omega'^2\,)p_2,
\end{aligned}
$$

$$
\begin{aligned}
f_2 &= (A_1\,r_0)^{-1}\ell_2\{-[\,2\,y_2\,(AC)^{-1}-p_2 A_1\,C^{-1}]\chi_2\,\dot\gamma_2-C^{-1}A_1(\,\chi+\chi_1\,\gamma_2\,)(\,A^{-1}\,y_2-\dot p_2\,)\}\\
&\quad +(A r_0)^{-1}\nu\,\ell_2\,p_2\,\dot\gamma_2+2\,(A r_0)^{-1}A_1\ell_1 S_{12}-\omega^2 S_{11}\,p_2-a^{-1}\,p_2\,z_0'+A_1\,k\,p_2\{1-\dot\gamma_2^2\\
&\quad -2\,(A\,A_1\,r_0)^{-1}\ell_2\,\dot\gamma_2-2[\,\dot\gamma_2+(A\,A_1\,r_0)^{-1}\ell_2][\,\nu_2\,\dot p_2+(A\,A_1\,r_0)^{-1}\ell_2\,S_{21}\,]\,\}+k A_1 X\,\gamma_2\\
&\quad \times[\,\dot\gamma_2+(A\,A_1\,r_0)^{-1}\ell_2](A^{-1}y_2-\dot p_2)-k A_1 S_{21}(1-A_1)\gamma_2-(A\,r_0)^{-1}A_1\ell_3 S_{11}\,p_2\\
&\quad +\chi_1\{C A_1 A^{-1}\nu\,y_1+r_0^{-1}[\,A^{-1}\ell_1 S_{21}+C^{-1}\dot\gamma_2[\,\ell_1(A^{-1}y_2-\dot p_2)-\ell_2\,p_2]]\},
\end{aligned}
$$

$$
\begin{aligned}
\varphi_2 &= r_0^{-1}\ell_1\{A^{-1}S_{22}+(C\,A_1)^{-1}[\,\nu_2\,\dot p_2(\,A^{-1}y_2-\dot p_2\,)-\chi_2\dot\gamma_2^2\,]\}-(C\,r_0)^{-1}\ell_2[\,\nu_2\,p_2\,\dot p_2\\
&\quad +\dot\gamma_2(\,\chi+\chi_1\gamma_2\,)]-\gamma_2\,S_{11}+(1-A_1\,)\,p_2\,S_{21}+(1-C_1\,)A_1^{-1}[\,\dot\gamma_2+\ell_2(A\,A_1\,r_0)^{-1}\\
&\quad \times(A^{-1}y_2-\dot p_2)]-\gamma_2\{z_0'\,a^{-1}-k A_1+X^2[\dot p_2^2+A^{-1}\,y_2(\,A^{-1}\,y_2-2\dot p_2\,)]\}\\
&\quad -S_{21}(a^{-1}z_0'\,\gamma_2-2\,k\,A_1\,\gamma_2),
\end{aligned}
$$

$$\begin{aligned} f_3 &= \ell_2 (A_1 r_0)^{-1}\{ C^{-1}(\chi_2\dot{\gamma}_2)^2 + [2C^{-1}(A^{-1}y_2 - \dot{p}_2) - C^{-1}A_1 p_2] \\ &\times\{[(kA_1 - a^{-1}z_0')\nu + \frac{1}{2}S_{11}]\dot{p}_2 + kA_1\dot{\gamma}_2 S_{21}\} + \chi_2\dot{\gamma}_2(C^{-1}A_1) + \nu^2 p_2\dot{p}_2\} \\ &-\omega^2[2p_2 S_{12} + S_{11}(\chi+\chi_1\gamma_2)] - (\chi+\chi_1\gamma_2)\{a^{-1}(z_0') - kA_1[1-\dot{\gamma}_2^2 \\ &-2(AA_1 r_0)^{-1}\ell_2\dot{\gamma}_2 - 2(\dot{\gamma}_2 + (AA_1 r_0)^{-1}\ell_2)(\nu_2\dot{p}_2 + (AA_1 r_0)^{-1}\ell_2 S_{21})]\} \\ &+kA_1\{\gamma_2[\dot{\gamma}_2 + (AA_1 r_0)^{-1}\ell_2][(AA_1 r_0)^{-1}\ell_2 S_{11} - X\chi_2\dot{\gamma}_2] + X[\gamma_2(\nu\dot{p}_2 \\ &+(AA_1 r_0)^{-1}\ell_2 S_{21}) + \nu p_2(\dot{\gamma}_2 + (AA_1 r_0)^{-1}\ell_2)](A^{-1}y_2 - \dot{p}_2)\} - kA_1(1-A_1) \\ &\times(\gamma_2 S_{22} + \nu p_2) - (r_0 A)^{-1}A_1\ell_3\{S_{11}(\chi+\chi_1\gamma_2) + 2p_2[S_{12} + S_{21}(k - z_0')]\} \\ &+\frac{1}{2}S_{11}[a^{-1}z_0'\gamma_2(1-A_1) - \ell_3(Ar_0)^{-1}kA_1\gamma_2] + p_2 S_{21}(2kA_1 - a^{-1}z_0') \\ &+\nu\chi_1\ell_2 r_0^{-1}\{\dot{p}_2(CA_1)^{-1}[\dot{p}_2 - 2(A^{-1}y_2 - \chi_2\dot{\gamma}_2)] - p_2 C^{-1}(A^{-1}y_2 - \dot{p}_2)\}, \end{aligned} \tag{16}$$

$$\begin{aligned} \varphi_3 &= \ell_1(A_1 C r_0)^{-1}\{-A_1^{-1}(A^{-1}y_2 - \dot{p}_2)(\chi_2\dot{\gamma}_2 + S_{21}\dot{p}_2) + \dot{\gamma}_2[\dot{p}_2(kA_1 - a^{-1}z_0' + \frac{1}{2}S_{11}) \\ &+kA_1\dot{\gamma}_2 S_{21}] + \nu_2\dot{p}_2[S_{11}\ell_2(Ar_0)^{-1} - \chi_2\dot{\gamma}_2]\} - \ell_2(Cr_0)^{-1}[-A_1^{-1}p_2(\chi_2\dot{\gamma}_2 + S_{21}\dot{p}_2) \\ &+\nu_2\dot{p}_2(\chi+\chi_1\gamma_2)] - (\nu p_2 S_{11} + 2\gamma_2 S_{12}) + (1-A_1)[p_2 S_{22} + (\chi+\chi_1\gamma_2)S_{21}] \\ &+X\{p_2[S_{11}\ell_2(Ar_0)^{-1} - \chi_2\dot{\gamma}_2][\dot{\gamma}_2 + \ell_2(AA_1 r_0)^{-1}] + (A^{-1}y_2 - \dot{p}_2)[p_2(\nu_2\dot{p}_2 \\ &+\ell_2 S_{21}(AA_1 r_0)^{-1}) + (\chi+\chi_1\gamma_2)(\dot{\gamma}_2 + \ell_2(AA_1 r_0)^{-1})]\} - \nu p_2\{a^{-1}z_0' + X^2[\dot{p}_2^2 \\ &+A^{-1}y_2(A^{-1}y_2 - 2\dot{p}_2)]\} - 2X^2\gamma_2(A^{-1}y_2 - \dot{p}_2)[S_{11}\ell_2(Ar_0)^{-1} - \chi_2\dot{\gamma}_2] + k\nu A_1 p_2 \\ &-S_{21}(a^{-1}z_0'\gamma_2 - 2kA_1\gamma_2). \end{aligned}$$

The first integral of system (15) can be obtained from (2) in the form

$$\begin{aligned} &\gamma_2^2 + \dot{\gamma}_2[\dot{\gamma}_2 + 2\ell_2(AA_1 r_0)^{-1}] + 2\varepsilon\{\nu\gamma_2 p_2 + [\dot{\gamma}_2 + \ell_2(AA_1 r_0)^{-1}][\nu_2\dot{p}_2 \\ &+\ell_2 S_{21}(AA_1 r_0)^{-1}] + S_{21}\} + \varepsilon^2\{\nu^2 p_2^2 + \nu_2\dot{p}_2[\nu_2\dot{p}_2 + 2\ell_2 S_{21}(AA_1 r_0)^{-1}] \\ &+2X[\dot{\gamma}_2 + \ell_2(AA_1 r_0)^{-1}][\ell_2 S_{11}(Ar_0)^{-1} - \chi_2\dot{\gamma}_2 - S_{21}\dot{p}_2] + S_{21}^2 + 2(S_{22} \\ &-\frac{1}{2}S_{11})\} = (\gamma_0'')^{-2} - 1. \end{aligned} \tag{17}$$

Our aim is to find the periodic solutions of system (15) under the condition $A = B > C$ or $A = B < C$ (ω'^2 is positive). This means that, the body is set in a fast initial spin r_0 about the major axis of the ellipsoid of inertia or about the minor axis of the ellipsoid of inertia.

IV. Formal Construction of the Periodic Solutions

Since the system (15) is autonomous, the following conditions

$$\begin{aligned} &p_2(0,0) = CA_1 B^{-1} y_1, \\ &\dot{p}_2(0,0) = 0, \\ &\gamma_2(0,\varepsilon) = 0, \end{aligned} \tag{18}$$

do not affect the generality of the solutions [23]. The generating system of (15) is

$$\ddot{p}_2^{(0)} + \omega'^2 p_2^{(0)} = 0, \qquad \ddot{\gamma}_2^{(0)} + \gamma_2^{(0)} = 0, \tag{19}$$

which admits periodic solutions with period $T_0 = 2\pi n$ in the form

$$p_2^{(0)} = M_1\cos\omega'\tau + M_2\sin\omega'\tau, \quad \gamma_2^{(0)} = M_3\cos\tau, \tag{20}$$

where $M_i,\ i = (1,2,3)$ are constants to be determined. So, we suppose the required periodic solutions of the initial autonomous system in the form

$$p_2(\tau,\varepsilon)=(M_1+\beta_1)\cos\omega'\tau+(M_2+\beta_2)\sin\omega'\tau+\sum_{k=1}^{\infty}\varepsilon^k G_k(\tau),$$
$$\gamma_2(\tau,\varepsilon)=(M_3+\beta_3)\cos\tau+\sum_{k=1}^{\infty}\varepsilon^k H_k(\tau), \tag{21}$$

with period $T(\varepsilon)=T_0+\alpha(\varepsilon)$. The quantities β_1, $\omega'\beta_2$ and β_3 represent the deviations of the initial values of p_2, $\dot p_2$ and γ_2 of system (15) from their initial values of system (19); these deviations are functions of ε and vanish when $\varepsilon=0$. The initial conditions of (21) can be expressed as

$$p_2(0,\varepsilon)=M_1+\beta_1,\quad \dot p_2(0,\varepsilon)=\omega'(M_2+\beta_2),\quad \gamma_2(0,\varepsilon)=M_3+\beta_3,\quad \dot\gamma_2(0,\varepsilon)=0. \tag{22}$$

Let us define the functions $G_k(\tau)$ and $H_k(\tau)$ $(k=1,2,3,\cdots)$ by the operator [23]

$$U=u+\frac{\partial u}{\partial M_1}\beta_1+\frac{\partial u}{\partial M_2}\beta_2+\frac{\partial u}{\partial M_3}\beta_3+\frac{1}{2}\frac{\partial^2 u}{\partial M_1^2}\beta_1^2+\cdots,\quad \begin{pmatrix} U=G_k,H_k\\ u\ =g_k,h_k\end{pmatrix} \tag{23}$$

where

$$g_k(\tau)=\frac{1}{\omega'}\int_0^{\tau}F_k^{(0)}(t_1)\sin\omega'(\tau-t_1)\,dt_1,$$
$$h_k(\tau)=\int_0^{\tau}\Phi_k^{(0)}(t_1)\sin\omega'(\tau-t_1)\,dt_1\quad (k=1,2,3). \tag{24}$$

Now, we try to find the expressions of the functions $F_1^{(0)}$, $\Phi_1^{(0)}$, $F_2^{(0)}$ and $\Phi_2^{(0)}$. The periodic solutions (20) can be rewritten as

$$p_2^{(0)}=E\cos(\omega'\tau-\eta),\qquad \gamma_2^{(0)}=M_3\cos\tau, \tag{25}$$

where

$$E=\sqrt{M_1^2+M_2^2},\qquad \eta=\tan^{-1}M_2/M_1.$$

Making use of (25) and (13), we obtain

$$S_{11}^{(0)}=aE^2\{\cos^2\eta+X^2\omega'^2\sin^2\eta+\frac{1}{2}(X^2\omega'^2-1)[1-\cos 2(\omega'\tau-\eta)]\}$$
$$+2\,k\,a\,M_3\ell_3(AA_1r_0)^{-1}\sin\tau-2\,a\,E\,\omega'X^2\ell_2(cA\sqrt{\gamma_0''})^{-1}[\sin\eta+\sin(\omega'\tau-\eta)],$$

$$S_{21}^{(0)}=M_3\,E\,a\{\cos\eta+\frac{1}{2}(\omega'X-1)\cos[(\omega'-1)\tau-\eta]-\frac{1}{2}(\omega'X+1)\cos[(\omega'+1)\tau-\eta]\}$$
$$+aX\{CA^{-1}M_3y_2\sin\tau-E\omega'\ell_2(AA_1r_0)^{-1}[\sin\eta+\sin(\omega'\tau-\eta)]\}$$
$$+M_3[y_1(1-\cos\tau)+y_2\sin\tau],$$

$$\begin{aligned}S_{12}^{(0)} &= a\,E\,\{\,X\,[\cos\eta-\cos\,(\omega'\tau-\eta\,)]+\chi_1\,M_3[\cos\eta-\cos\tau\cos\,(\omega'\tau-\eta\,)]\}-a\,X^2\,E\,\omega' \\ &\times\{\,\ell_2\,S_{11}^{(0)}\,(Ar_0)^{-1}[\sin\eta+\sin(\omega'\tau-\eta)]+\chi_2\,M_3\sin\tau\sin(\omega'\tau-\eta\,)\}+k\,a\{\,\nu\,E\,M_3 \\ &\times[\,\cos\eta-\cos\tau\cos(\omega'\tau-\eta\,)\,]-[\,\nu_2\,\omega' E\,M_3\,\sin\tau\,\sin(\omega'\tau-\eta\,)-\ell_2\,X\,(Ar_0)^{-1} \\ &\times[\,M_3\,S_{21}^{(0)}\sin\tau+\nu_2\,\omega' E\,(\sin\eta+\sin(\omega'\tau-\eta\,))\,]\,]\}-X^2 M_3\,\chi_2\,\ell_2(cA\sqrt{\gamma_0''}\,)^{-1}\sin\tau \\ &+(z_0'-k\,)S_{21}^{(0)},\end{aligned} \tag{26}$$

$$\begin{aligned}S_{22}^{(0)} &= a\{\nu E^2[\cos^2\eta-\cos^2(\omega'\tau-\eta\,)]+\chi\,M_3(1-\cos\tau\,)+\chi_1 M_3^2\sin^2\tau\}+a\{-E^2 \\ &\times X\,\nu_2\,\omega'^2[\,\sin^2\eta-\sin^2(\omega'\tau-\eta)\,]+E\,\omega'[\,\nu_2\,y_2\,C\,X\,A^{-1}-\ell_2\,S_{21}^{(0)}\,(AA_1^2\,r_0)^{-1}\,] \\ &\times[\,\sin\eta+\sin(\omega'\tau-\eta\,)\,]+M_3\sin\tau[\,X\,(\,\chi_2 M_3\sin\tau+(Ar_0)^{-1}\ell_2\,S_{11}^{(0)}) \\ &-\ell_2\,\chi_2(AA_1^2\,r_0)^{-1}\,]\,\}+E\{\,\nu\,y_1[\,\cos\eta-\cos(\omega'\tau-\eta\,)\,]+\omega'\,y_2\,[\,\sin\eta \\ &+\sin(\omega'\tau-\eta\,)\,]\,\}-y_3\,S_{21}^{(0)}\,.\end{aligned}$$

Substitution of (25) and (26) into formulas (16), to get

$$\begin{aligned}F_1^{(0)} &= [\,2(\,AA_1\,C\,r_0\,)^{-1}\omega'\,\ell_2 y_2\,][\,M_1\sin\omega'\tau-M_2\cos\omega'\tau\,]+\cdots, \\ \Phi_1^{(0)} &= y_2\{1+A^{-1}[\,a\,X\,C-(\,A_1\,C\,r_0\,)^{-1}\ell_1\,]M_3\sin\tau+\cdots, \\ F_2^{(0)} &= L(\omega')[\,M_1\cos\omega'\tau+M_2\sin\omega'\tau\,]+\cdots, \\ \Phi_2^{(0)} &= M_3\,N(\omega')\cos\omega'\tau+\cdots,\end{aligned} \tag{27}$$

where

$$\begin{aligned}L(\,\omega'\,) &= -[\,a^{-1}\,z_0'+\nu\,\chi_1\,(1-\omega'^2\,)\,]+A_1[\,k-2\,\ell_1\,a\,\chi\,(Ar_0)^{-1}]+(\,r_0^{-1}\ell_3\,A^{-1}A_1-\omega^2) \\ &\times\{\,a[(M_1^2+\omega'^2 X^2 M_2^2\,)-\frac{1}{2}\,(1+\omega'^2 X^2\,)]-\frac{1}{2}[\,k\,M_3^2\,C_1\,+\acute{}a(\,\omega'^2\,X^2-1) \\ &\times(M_1^2+M_2^2\,)]-2\,\ell_2\,a\,\omega' M_2\,X^2(cA\sqrt{\gamma_0''}\,)^{-1}\}+\cdots, \\ N(\,\omega'\,) &= -a\,(M_1^2+X^2\,\omega'^2\,M_2^2\,)+X^2\,\omega'^2\,(1+a)(M_1^2+M_2^2\,)-[\,z_0'\,b^{-1} \\ &-\nu\,\chi_1\,(1-\omega'^2\,)]+k\,A_1-(A^{-1}X\,y_2)^2-r_0^{-1}\,\ell_1\,\chi\,a\,A^{-1} \\ &+2\,\ell_2\,a\,\omega' M_2\,X^2(c\,A\sqrt{\gamma_0''}\,)^{-1}-(\,z_0'\,a^{-1}-2\,k\,A_1)[M_3\,(aM_1 \\ &+y_1\,)-\ell_2\,a\,\omega' M_2\,X\,(A_1\,r_0)^{-1}\,]+\cdots.\end{aligned} \tag{28}$$

Form (24), (27) and (28), the following results are obtained

$$\begin{aligned}&g_1(T_0)=2\pi\,n\,M_1(AA_1C\,r_0\,)^{-1}\,\ell_2\,y_2, \\ &\dot g_1(T_0)=-2\pi\,n\,\omega' M_2(AA_1\,C\,r_0\,)^{-1}\,\ell_2\,y_2, \\ &g_2(T_0)=-\pi\,n(\,\omega'\,)^{-1}M_2\,L(\omega'),\quad \dot g_2(T_0)=\pi\,n\,M_1\,L(\omega'), \\ &h_1(T_0)=0,\qquad \dot h_1(T_0)=\pi\,n\,M_3\,y_2\{1+A^{-1}[\,a\,X\,C-(A_1\,C\,r_0\,)^{-1}\,\ell_1\,]\}, \\ &h_2(T_0)=0,\qquad \dot h_2(T_0)=\pi\,n\,M_3\,N(\omega').\end{aligned} \tag{29}$$

Substituting (22) into (17) for $\tau=0$, we obtain

$$M_3^2+2M_3\beta_3+\beta_3^2+2\varepsilon[\nu(M_1+\beta_1)(M_3+\beta_3)+\ell_2\nu_2\omega'(AA_1r_0)^{-1}$$
$$(M_2+\beta_2)]+\cdots=(\gamma_0'')^{-2}-1.$$

Supposing that γ_0'' is independent of ε, we get M_3 and β_3 as

$$M_3=(1-\gamma_0''^2)^{\frac{1}{2}}(\gamma_0'')^{-1} \qquad 0<M_3<\infty, \qquad \beta_3=-\varepsilon\nu(M_1+\beta_1)+\cdots. \tag{30}$$

The independent conditions for the periodicity are [21]

$$\begin{aligned}&-\pi n\beta_2(\omega')^{-1}\{L_1(\omega')-\omega'^2N_1(\omega')[1+\ell_1(Ar_0)^{-1}(M_3+\beta_3-\ell_1(Ar_0)^{-1})^{-1}]\}\\&+\varepsilon[G_2(T_0)+\cdots]=0,\\&\pi n\beta_1\{L_1(\omega')+N_1(\omega')(CA_1A^{-1}y_1-\omega'\beta_1)[1+\ell_1(Ar_0)^{-1}(M_3+\beta_3\\&-\ell_1(Br_0)^{-1})^{-1}]\}+\varepsilon[\dot{G}_2(T_0)+\cdots]=0,\\&\varepsilon[M_3+\beta_3-\ell_1(Ar_0)^{-1}]^{-1}[\dot{H}_1(T_0)+\varepsilon\dot{H}_2(T_0)+\varepsilon^2\dot{H}_3(T_0)+\cdots]=\alpha(\varepsilon),\end{aligned} \tag{31}$$

where $L_1(\omega')$ and $N_1(\omega')$ can be obtained from (28) by replacing M_1, M_2 and M_3 by β_1, β_2 and $M_3+\beta_3$ respectively. Then, we have

$$L_1(\omega')-\omega'^2N_1(\omega')=(\beta_1^2+\beta_2^2)W_1(\omega')+z_0'W_2(\omega')+kW_3(\omega')+W_4(\omega'), \tag{32}$$

where

$$\begin{aligned}W_1(\omega')&=\frac{1}{2}a[(\omega'X)^2-1][\omega'^2-(r_0A)^{-1}A_1\ell_3]-(1+a)(\omega'^2X)^2,\\W_2(\omega')&=-a^{-1}\{1-a^{-1}\omega'^2[1+(a\beta_1+y_1)(M_3+\beta_3)-\ell_2a\omega'\beta_2(AA_1^2r_0)^{-1}]\},\\W_3(\omega')&=A_1-\omega'^2A_1[1+2(a\beta_1+y_1)(M_3+\beta_3)-2a\omega'\beta_2(AA_1^2r_0)^{-1}\ell_2],\\W_4(\omega')&=a\chi\ell_1(Ar_0)^{-1}(\omega^2-2A_1)+(Xy_2A^{-1}\omega')^2-\nu\chi_1(1-\omega'^4)\\&\quad+\frac{1}{4}ay_1\{1+X^2\omega'^2+4\ell_2X^2\omega'\beta_2(cA\sqrt{\gamma_0''})^{-1}+2[\beta_1^2+(X\omega'\beta_2)^2]\}.\end{aligned}$$

From the condition that the $z-$ axis has to be directed along the major or the minor axis of the ellipsoid of inertia of the body, it follows that $W_1(\omega')>0$ for all ω' under consideration. So, let us assume

$$z_0'W_2(\omega')+kW_3(\omega')+W_4(\omega')\neq 0.$$

Using (31), the expression of β_1 and β_2 are obtained in the form of a power series of integral powers of ε. These expansions begin with terms of order higher than ε^2. Consequently, the first terms in the expansions of the periodic solutions and the quantity $\alpha(\varepsilon)$ can be expressed as

$$\begin{aligned}p_1&=\varepsilon[\ell_1y_2^2(CA_1A^2r_0\omega'^2)^{-1}+\chi_1M_3\cos\tau]\\&\quad+\cdots,\\q_1&=CXA^{-1}y_2+\varepsilon[2ak(AA_1r_0)^{-2}\ell_2\ell_3\\&\quad+X\chi_2]M_3\sin\tau+\cdots,\\r_1&=1-\varepsilon^2kM_3a\ell_3(AA_1r_0)^{-1}\sin\tau+\cdots,\\\gamma_1&=M_3\cos\tau+\cdots,\end{aligned}$$

$$\gamma_1' = -M_3 \sin\tau + \ell_2 (AA_1 r_0)^{-1}\{1+\varepsilon M_3[y_1(1-\cos\tau)+y_2\sin\tau]\}+\varepsilon^2\{X\chi_2 M_3\sin\tau$$
$$+kM_3 b\ell_3(AA_1 r_0)^{-1}\sin\tau[2\ell_2(AA_1 r_0)^{-1}+XM_3\sin\tau]\}+\cdots,$$
$$\gamma_1'' = 1+\varepsilon M_3[y_1(1-\cos\tau)+y_2\sin\tau - aC\ell_2 y_2(A^2A_1^2 r_0)^{-1}\sin\tau]$$
$$+\varepsilon^2 M_3\{a\chi(1-\cos\tau)+\frac{1}{2}aM_3(\chi_1+X\chi_2)(1-\cos 2\tau)-a\ell_2\chi_2$$
$$\times(AA_1^2 r_0)^{-1}\sin\tau - y_3[y_1(1-\cos\tau)+y_2\sin\tau-aC\ell_2 y_2(A^2A_1^2 r_0)^{-1}\sin\tau]$$
$$-ka\ell_3(AA_1 r_0)^{-1}\sin\tau\}+\cdots.$$
$$\alpha(\varepsilon) = -\varepsilon\pi n\{1+\ell_1 A^{-1}[r_0(M_3+\beta_3)-\ell_3]^{-1}\}[z_0' a^{-1}-\nu\chi_1(1-\omega'^2)$$
$$+(y_2XA^{-1})^2-kA_1+a\chi\ell_1(Ar_0)^{-1}+y_1M_3(z_0'a^{-1}-2kA_1)]+\cdots. \quad (33)$$

The solutions obtained in [22], [34] and [35] have singular points when $\omega = 1, 2, 3, 1/2, 1/3, \cdots$. These singularities are solved separately in [34], [36], [37] and [38]. In our problem when we used Ismail and Amer's frequency ω' [23] instead of ω, there are no singular points at all. Moreover, the obtained solutions are valid for all rational values of ω' and are considered as a general case of [21], [22] and [23].

V. Geometric Interpretation of Motion

In this section, the motion of the rigid body is investigated by introducing Euler's angles θ, ψ and φ, which can be determined through the obtained periodic solutions. Since the initial system is autonomous, the periodic solutions are still periodic if t is replaced by $(t+t_0)$, where t_0 is an arbitrary interval of time. Euler's angles, in terms of time t, take the forms [27,28]

$$\cos\theta = \gamma'', \qquad \frac{d\psi}{dt} = \frac{p\gamma+q\gamma'}{1-\gamma''^2}, \qquad \tan\varphi_0 = \frac{\gamma_0}{\gamma_0'}, \qquad \frac{d\varphi}{dt} = r - \frac{d\psi}{dt}\cos\theta. \quad (34)$$

Substituting (33) into (34), in which t has been replaced by $t+t_0$, and using relations (4), the following expressions for the angles θ, ψ and φ are obtained as

$$\varphi_0 = (\pi/2)+r_0h+\cdots, \qquad \theta_0 = \tan^{-1} M_3,$$
$$\theta = \theta_0 - \varepsilon[\theta_1(t+h)-\theta_1(h)]-\varepsilon^2[\theta_2(t+h)-\theta_2(h)],$$
$$\psi = \psi_0 + c\,\mathrm{cosec}\,\theta_0\sqrt{\cos\theta_0}\,\{[\psi_1(t+h)-\psi_1(h)]+\varepsilon[\psi_2(t+h)-\psi_2(h)]$$
$$+\varepsilon^2[\psi_3(t+h)-\psi_3(h)]\},$$
$$\varphi = \varphi_0 + r_0 t - c\cot\theta_0\sqrt{\cos\theta_0}\,\{[\varphi_1(t+h)-\varphi_1(h)]+\varepsilon[\varphi_2(t+h)-\varphi_2(h)]\}$$
$$-\varepsilon^2\{\tan\theta_0[\varphi_3(t+h)-\varphi_3(h)]+c\cot\theta_0\sqrt{\cos\theta_0}[\varphi_4(t+h)-\varphi_4(h)]\},$$

where

$$\theta_1(t) = -y_1\cos r_0t+[1-ca\ell_2(A_1^2A^2r_0)^{-1}]y_2\sin r_0t,$$
$$\theta_2(t) = (y_1y_3-a\chi)\cos r_0t+\{-y_2y_3-a(A_1Ar_0)^{-1}[\ell_2(A_1^{-1}\chi_2)+k\ell_3]\}\sin r_0t$$
$$-\frac{1}{2}a\tan\theta_0(\chi_1+X\chi_2)\cos 2r_0t,$$
$$\psi_1(t) = Cy_2(A_1Ar_0)^{-1}[\ell_2\cos\theta_0(A_1A)^{-1}t+\cos r_0t],$$
$$\psi_2(t) = [(A_1^2A^2r_0)^{-1}\ell_2Cy_1y_2-\frac{1}{2}X\chi_2\tan\theta_0]t+\frac{1}{4}(r_0A_1)^{-1}\chi_2\tan\theta_0\sin 2r_0t,$$

$$\psi_3(t) = \frac{1}{2}[\chi_1 \tan\theta_0 + k\,C\,X^2\,y_2\,a\,\ell_3\,(A_1 A^2 r_0)^{-1}\tan\theta_0 + \chi_2\,y_2\,\ell_2\,(A_1^2 A r_0)^{-1}\tan\theta_0]\,t$$
$$+\frac{1}{4}r_0^{-1}\chi_1\tan\theta_0\sin 2r_0 t - C\,y_2\,\chi_2\,(A_1^2 A r_0)^{-1}\cos r_0 t,$$
$$\varphi_1(t) = \psi_1(t), \qquad \varphi_2(t) = \psi_2(t), \qquad \varphi_4(t) = \psi_3(t),$$
$$\varphi_3(t) = \frac{1}{8}k\,a\,\ell_3(A_1 A r_0)^{-1}\cos r_0 t.$$

It is evident that the Eulerian angles $\theta,\ \psi$ and φ depend on some arbitrary constants θ_0 , ψ_0 , φ_0 and r_0 (r_0 is large). For $\varepsilon = 0,$ we have $\dot\theta = 0,\ \dot\psi = 0$ and $\dot\varphi = r_0$. This permits permanent rotation of the body with spin r_0 (sufficiently large) about the $z-$ axis.

VI. Numerical Solutions Matching of Analytical Solutions

This section is devoted to ascertain accuracy of the obtained solutions.

(***i***) We introduce the analytical solutions through computer program. So, let us consider the following data that determine the motion of the body

$$A = B = 22.49\,kg.m^2 < C = 35.6\,kg.m^2, \quad A = B = 50.32\,kg.m^2 > C = 33.4\,kg.m^2$$
$$r_0 = 1000\,m, R = 2000\,m, \quad \lambda = 0.6, \quad M = 30\,kg, \quad z_0 = 5\,m, \quad \gamma_0'' = 0.352,$$
$$\ell_1 = \ell_2 = \ell_3 = (0, 10, 20, 30, 40, 50)\,kg.m^2.s^{-1}, \quad T = 12.566371.$$

Consider p_{2a}, γ_{2a} denoting the analytical solutions p_2, γ_2. The graphical representations for these solutions are given in figures (1)-(5) for the case $A = B < C$ and figures (6)-(10) for the case $A = B > C$.

(***ii***) The quasilinear autonomous system (15) is solved numerically using the fourth order Runge-Kutta method through another program with the same previous data and the initial values of the analytical solutions. Consider p_{2n}, γ_{2n} to denote the numerical solutions p_2, γ_2. The numerical graphical representations are given in figures (11)-(15) for the case $A = B < C$ and figures (16)-(20) for the case $A = B > C$.

The comparison between the analytical and the numerical solutions shows quite agreement between them, see the corresponding figures (1)-(5), (11)-(15) and (6)-(10), (16)-(20) for the cases $A = B < C$ and $A = B > C$ respectively. This agreement gives powerful ascertain for the analytical technique. The corresponding phase plane diagrams for some of these solutions describing the stability of the solutions are given in figures (4), (5), (9), (10) for the analytical solutions and (14), (15), (19), (20) for the numerical solutions.

Here, the concerned plots represent the functional time dependence of the amplitude of the waves revealing when $\ell \equiv |\underline{\ell}|$ increases. We conclude that when ℓ increases the amplitude of the wave increases also and the number of the waves remain unchanged, see figures (1), (2), (11) and (12) for the case $A = B < C$ but for the case $A = B > C$, we can see from figures (6), (7), (16) and (17) that the amplitude of the wave decreases. Also, the solutions γ_{2a} and γ_{2n} remain unchanged for different values of ℓ because these solutions do not include the variables $\ell_1,\ \ell_2,\ \ell_3,\ A,\ B$ and C.

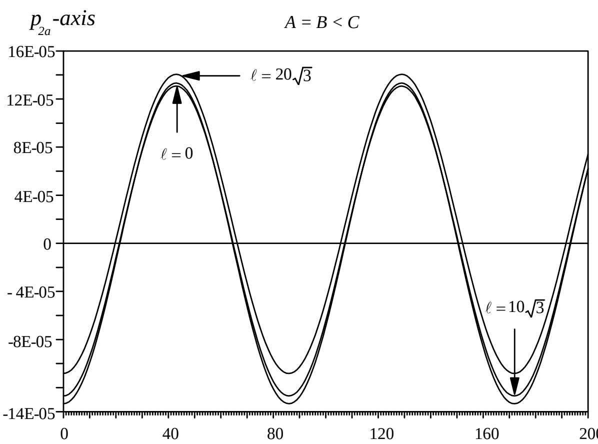
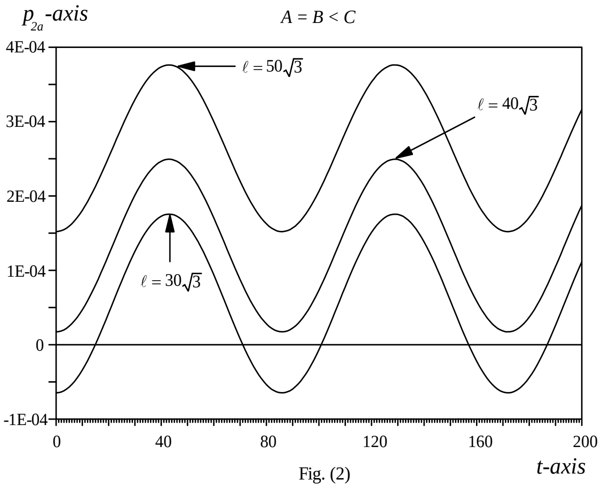


Fig. (1)



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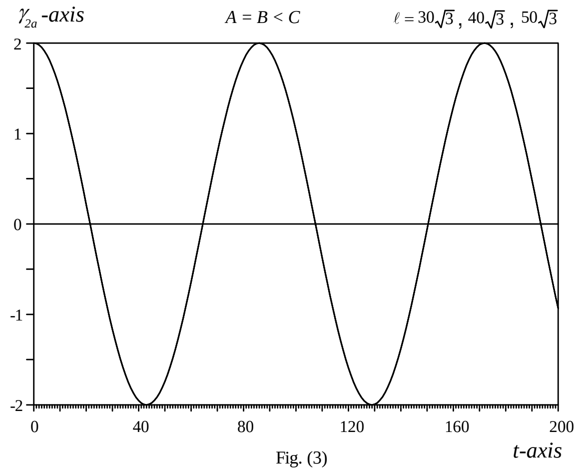
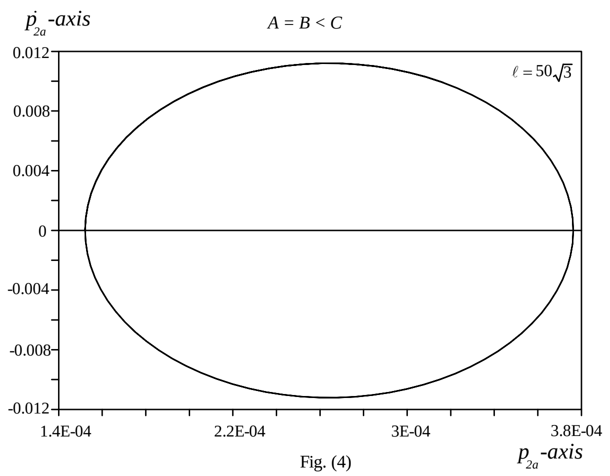


Fig. (3)



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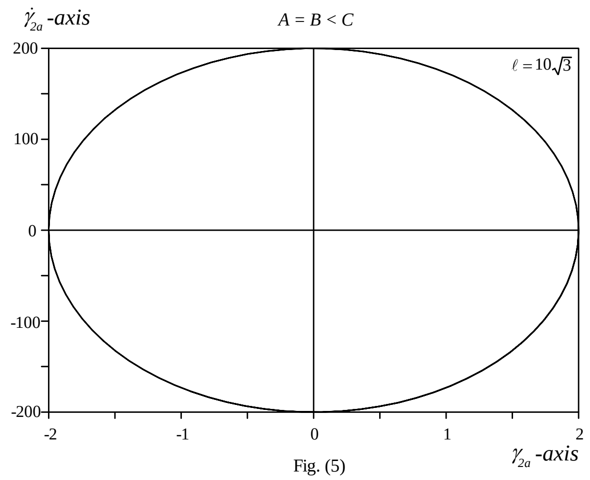

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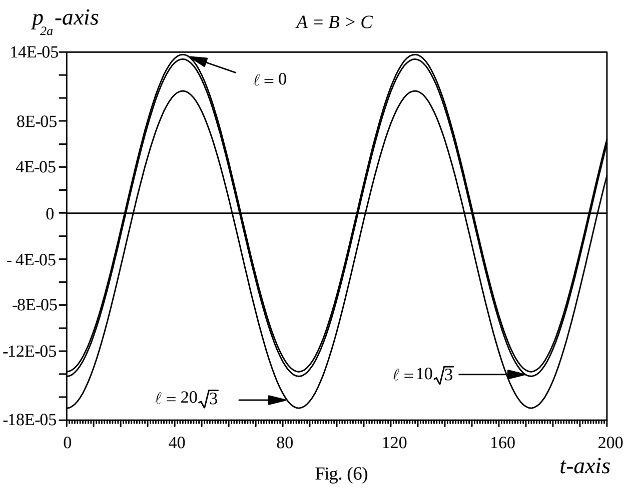

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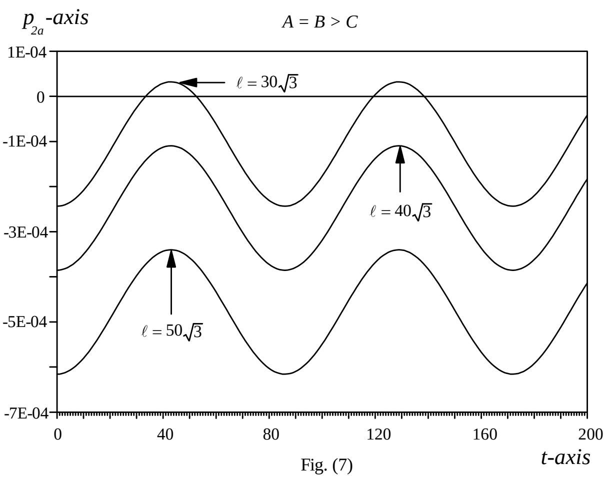

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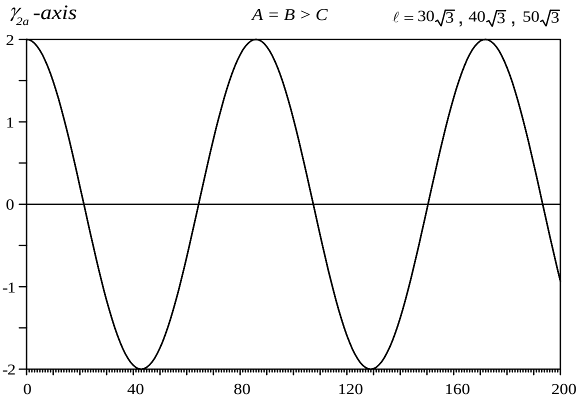

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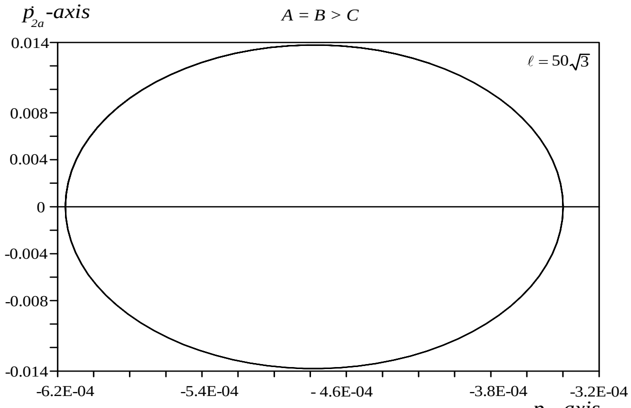

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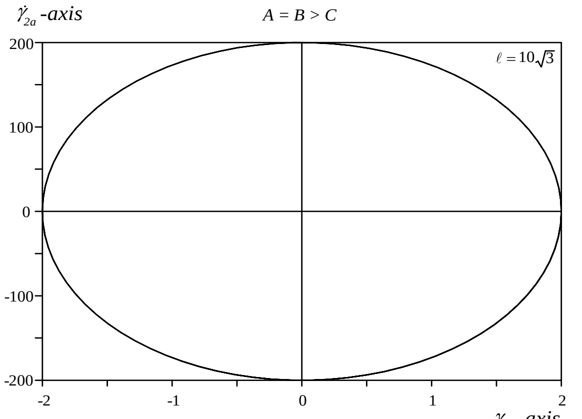

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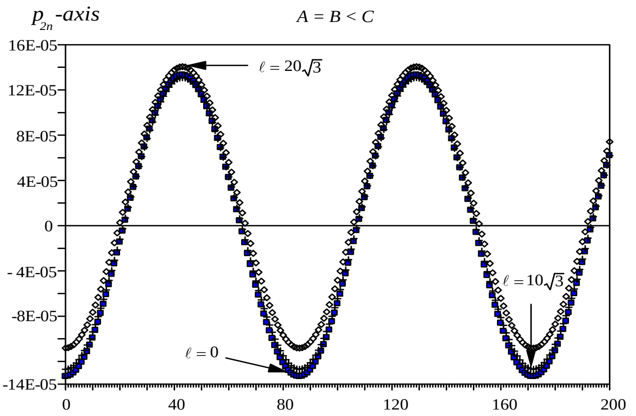

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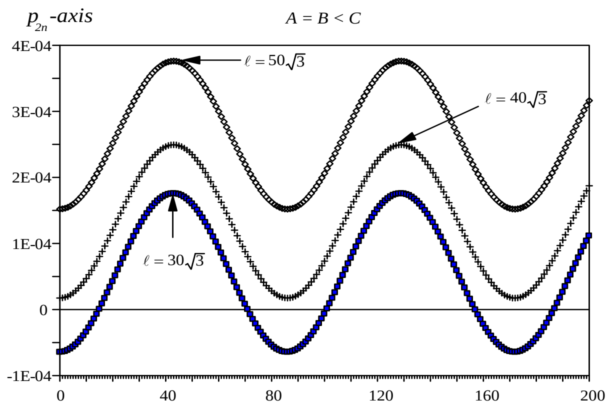

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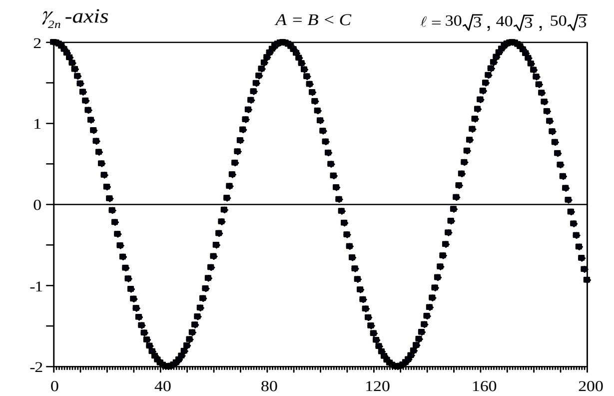

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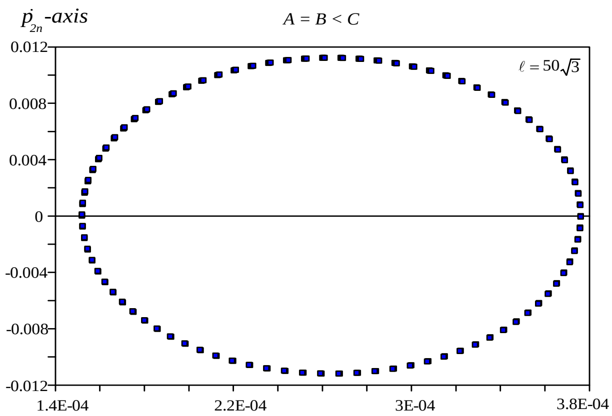

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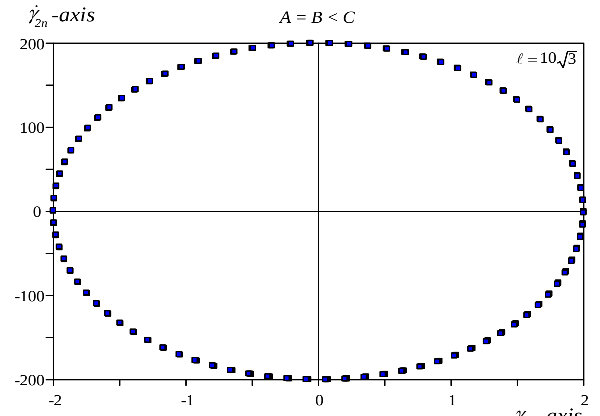

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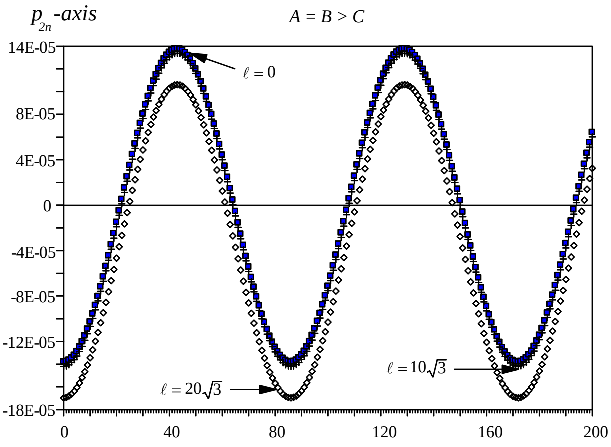

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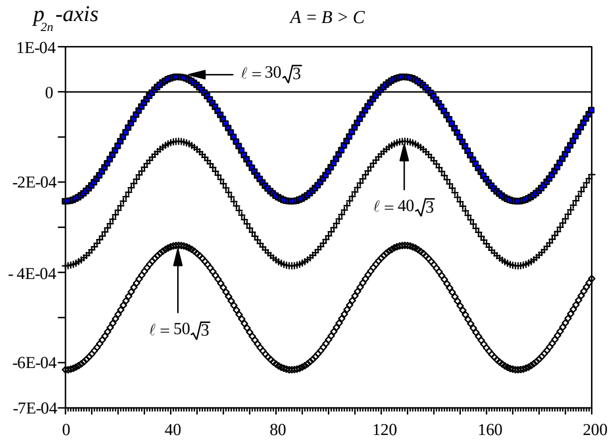

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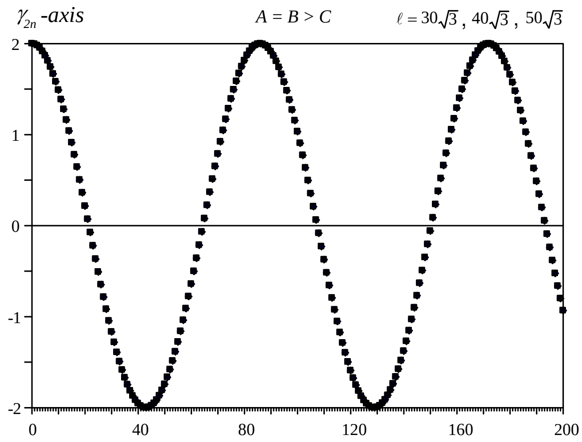

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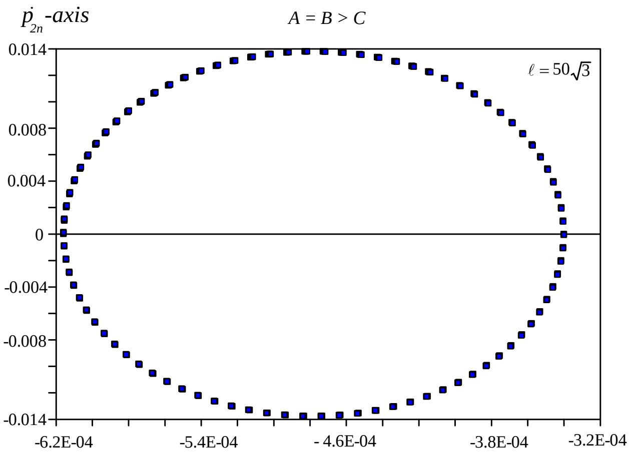

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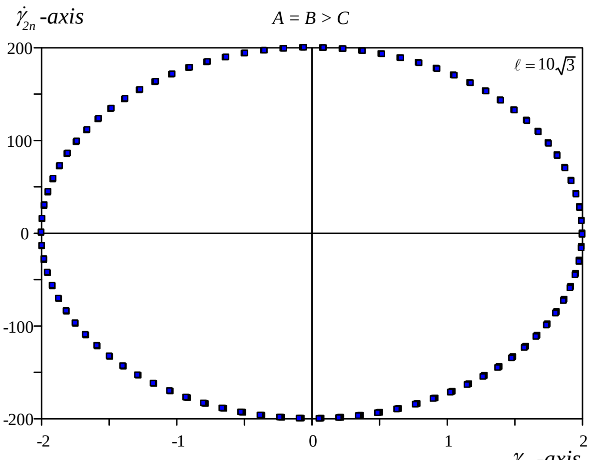

Fig. (20)

VII. Conclusion

The problem of the three-dimensional motion of a gyrostat in the Newtonian force field with a gyrostatic moment about one of the principal axes of the ellipsoid of inertia, is investigated by reducing the six first-order non-linear differential equations of motion and their first three integrals into a quasilinear autonomous system with two degrees of freedom and one first integral. Poincaré's small parameter method is used to investigate the periodic solutions of the present problem up to the first order approximation in terms of the small parameter ε . The periodic solutions (33) are considered as a generalization of those obtained in [21] (in the case of the uniform force field), [22] (in the case of the Newtonian force field) and [23] (in the case of presence ℓ_3 only). The solutions and the correction of the period for the latter problems can be deduced from the obtained solutions in this work as limiting cases by reducing the Newtonian terms and the gyrostatic moment. The introduction of an alternative frequency ω' instead of ω avoids the singularities traditionally appearing in the solutions of other treatments. The analytical solutions are analysed geometrically using Euler's angles to describe the orientation of the body at any instant of time. These solutions are performed by computer program to get their graphical representations. The fourth order Runge-Kutta method is applied through another computer program to solve the autonomous system and represent the obtained numerical solutions. The comparison between both the analytical and the numerical solutions is considered to show the difference between them. These deviations are very small, that is the numerical solutions are in full agreement with the analytical ones. The great effect of the gyrostatic moment ℓ is shown obviously from the graphical representations.

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Captions of Figures

Fig. 1
The graphical representation of the analytical solution p_2 via t when $\ell = (0, 10\sqrt{3}, \text{ and } 20\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 2
The graphical representation of the analytical solution p_2 via t when $\ell = (30\sqrt{3}, 40\sqrt{3}, \text{ and } 50\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 3
The graphical representation of the analytical solution γ_2 via t when $\ell = (30\sqrt{3}, 40\sqrt{3}, \text{ and } 50\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 4
The phase plane diagram of the analytical solution p_2 when $\ell = 50\sqrt{3}\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 5
The phase plane diagram of the analytical solution γ_2 when $\ell = 10\sqrt{3}\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 6
The graphical representation of the analytical solution p_2 via t when $\ell = (0, 10\sqrt{3}, \text{ and } 20\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 7
The graphical representation of the analytical solution p_2 via t when $\ell = (30\sqrt{3}, 40\sqrt{3}, \text{ and } 50\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 8
The graphical representation of the analytical solution γ_2 via t when $\ell = (30\sqrt{3}, 40\sqrt{3}, \text{ and } 50\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 9
The phase plane diagram of the analytical solution p_2 when $\ell = 50\sqrt{3}\ kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 10
The phase plane diagram of the analytical solution γ_2 when $\ell = 10\sqrt{3}\ kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 11
The graphical representation of the numerical solution p_2 via t when $\ell = (0, 10\sqrt{3}, \text{ and } 20\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 12
The graphical representation of the numerical solution p_2 via t when $\ell = (30\sqrt{3}, 40\sqrt{3}, \text{ and } 50\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 13
The graphical representation of the numerical solution γ_2 via t when $\ell = (30\sqrt{3}, 40\sqrt{3}, \text{ and } 50\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 14
The phase plane diagram of the numerical solution p_2 when $\ell = 50\sqrt{3}\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 15
The phase plane diagram of the numerical solution γ_2 when $\ell = 10\sqrt{3}\ kg.m^2.s^{-1}$ for the case $A = B < C.$

Fig. 16
The graphical representation of the numerical solution p_2 via t when $\ell = (0, 10\sqrt{3},\ and\ 20\sqrt{3})\ kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 17
The graphical representation of the numerical solution p_2 via t when $\ell = (30\sqrt{3}, 40\sqrt{3},\ and\ 50\sqrt{3})\, kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 18
The graphical representation of the numerical solution γ_2 via t when $\ell = (30\sqrt{3}, 40\sqrt{3},\ and\ 50\sqrt{3})\, kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 19
The phase plane diagram of the numerical solution p_2 when $\ell = 50\sqrt{3}\ kg.m^2.s^{-1}$ for the case $A = B > C.$

Fig. 20
The phase plane diagram of the numerical solution γ_2 when $\ell = 10\sqrt{3}\ kg.m^2.s^{-1}$ for the case $A = B > C.$