A Common Fixed Point Theorem through Weak and Semi-Compatibility in Fuzzy Metric Space

Deepak Singh, M.S. Rathore, Krishnapal Singh Sisodia
*NITTTR, Bhopal (M.P.), India
**Department of Mathematics, Govt. P. G. College Sehore (M.P.), India
***Department of Mathematics, SISTec-E, Ratibad, Bhopal (M.P.), India

Abstract
In this paper, a common fixed point theorem for six self mappings has been established using the concept of semi-compatibility and weak compatibility in Fuzzy metric space, which generalizes the result of Singh B.S., Jain A. and Masoodi A.A. [6].

Keywords: Fuzzy metric space, common fixed point, t-norm, compatible map, semi-compatible, weak compatible map.
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Introduction: The concept of Fuzzy sets was introduced by Zadeh [9]. Following the concept of Fuzzy sets, Fuzzy metric spaces have been introduced by Kramosil and Michalek [4] and George and Veeramani [2] modified the notion of Fuzzy metric spaces with the help of continuous t-norms. Vasuki [8] investigated some fixed point theorems in Fuzzy metric spaces for R-weakly commuting mappings. Inspired by the results of B. Singh, A. Jain and A.A. Masoodi [6], in this paper, we prove a common fixed point theorem for six self maps under the condition of weak compatibility and semi-compatibility in Fuzzy metric spaces.

Preliminaries:

Definition 1: A binary operation $*$: $[0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a*1=a$ for all $a \in [0,1]$;
(d) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ and $a,b,c,d \in [0,1]$.

Definition 2 [2]: A 3-tuple $(X, M, *)$ is said to be a Fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions: for all $x,y,z \in X$, $s,t>0$,

(fm1) $M(x, y, t)>0$;
(fm2) $M(x, y, t) =1$ iff $x=y$;
(fm3) $M(x, y, t) = M(y, x, t)$;
(fm4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$;
(fm5) $M(x, y, .): (0, \infty) \to [0,1]$ is continuous.
(fm6) $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$.

Then M is called a Fuzzy metric on X. The function M (x, y, t) denote the degree of nearness between x and y with respect to t.

Example 1: Let (X, d) be a metric space. Denote $a*b=ab$ for $a, b \in [0, 1]$ and let M_d be Fuzzy set on $X^2 \times (0,\infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t+d(x,y)}$$

Then $(X, M_d, *)$ is a Fuzzy metric space, we call this Fuzzy metric induced by a metric d the standard intuitionistic Fuzzy metric.

Definition 3 [2]: Let $(X, M, *)$ be a Fuzzy metric space, then

(a) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for each $\epsilon>0$ and each $t>0$, there exists $n_0 \in N$ such that $M(x_n, x, t)>1-\epsilon$ for all $n \geq n_0$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon>0$ and each $t>0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t)>1-\epsilon$ for all $n, m \geq n_0$.
(c) A Fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Proposition 1: In a Fuzzy metric space $(X, M, *)$, if $a*a \geq a$ for $a \in [0, 1]$ then $a*b=\min\{a, b\}$ for all $a, b \in [0, 1]$.

Definition 4 [7]: Two self mappings A and S of a Fuzzy metric space $(X, M, *)$ are called compatible if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X.

Definition 5 [7]: Two self mappings A and S of a Fuzzy metric space $(X, M, *)$ are called semi-compatible if $\lim_{n\to\infty} M(ASx_n, Sx, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some x in X.

Definition 6 [7]: Two self maps A and S of a Fuzzy metric space (X, M,$*$) are called weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if Ax=Sx for some x ϵX then ASx=SAx.

Proposition 2 [7]: In a Fuzzy metric space (X, M,$*$), limit of a sequence is unique.

Proposition 3 [7]: In a Fuzzy metric space (X, M,$*$), If (A, S) is a semi-compatible pair of self maps and S is continuous, then (A, S) is compatible

Remark 1: If self maps A and S of a Fuzzy metric space (X, M,$*$) are compatible then they are weakly compatible.

Lemma 1 [3]: Let (X, M,$*$) be a Fuzzy metric space. Then for all x, y ϵX, $M(x, y, .)$ is a non decreasing function.

Lemma 2 [5]: Let (X, M,$*$) be a Fuzzy metric space. If there exists k ϵ[0, 1] such that $M(x, y, kt) \geq M(x, y, t)$ then x=y.

Lemma 3 [1]: let $\{y_n\}$ be a sequence in a Fuzzy metric space (X, M,$*$) with the condition (fm6). If there exists k ϵ[0, 1] such that $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all t>0 and n ϵN, then $\{y_n\}$ is a Cauchy sequence in X.

Main Results:

In this result, we are proving a theorem to obtain a unique common fixed point for six self mappings using weak and semi-compatibility in a Fuzzy metric space.

Theorem 1: Let (X, M,$*$) be a complete Fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:

(i) P(X) $\subseteq$ ST(X), Q(X) $\subseteq$ AB(X);
(ii) AB=BA, ST=TS, PB=BP, QT=TQ
(iii) Either AB or P is continuous;
(iv) The pair (P, AB) is semi-compatible and (Q, ST) is weakly compatible.
(v) There exists $q \in (0,1)$ such that for every $x, y \in X$ and t>0

$$M(Px, Qy, qt) \geq \min\{M(ABx, STy, t), M(Px, ABx, t), M(Qy, STy, t), M(Px, STy, t)\}$$

.

Then A, B, S, T, P and Q have a unique common fixed point.

Proof: Let $x_0 \in X$. By (i), there exists x_1, x_2 ϵX such that $Px_0=STx_1= y_0$ and $Qx_1=ABx_2=y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Px_{2n}=STx_{2n+1}=y_{2n}$ and $Qx_{2n+1}=ABx_{2n+2}=y_{2n+1}$ for n=0, 1, 2, ----.

Step-1: putting $x=x_{2n}$ and y= x_{2n+1} in (v), we have

$$M(Px_{2n}, Qx_{2n+1}, qt)$$
$$\geq \min\{M(ABx_{2n}, STx_{2n+1}, t), M(Px_{2n}, ABx_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, t), M(Px_{2n}, STx_{2n+1}, t)\}$$

$$M(y_{2n}, y_{2n+1}, qt) \geq$$
$$\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n}, t)\}$$

By lemma 1 and lemma 2; we get

$$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t)$$

Similarly; we have

$$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t)$$

Thus we have

$$M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t)$$

Using lemma 3; we have

$\{y_n\}$ is a Cauchy sequence in X.

Since (X, M,$*$) is complete, $\{y_n\}$ converges to some point $z \epsilon$X. Also its subsequences converges to the same point i.e. $z \epsilon$X.

i.e. $\{Qx_{2n+1}\}$ →z and $\{STx_{2n+1}\}$ →z ---------- (1)

$\{Px_{2n}\}$ →z and $\{ABx_{2n}\}$ →z --------------- (2)

Case I: suppose P is continuous.

Step-2: since P is continuous and (P, AB) is semi-compatible pair, we have $P(AB)x_{2n}$→Pz and $P(AB)x_{2n}$→ABz.

Since the limit in Fuzzy metric space is unique, we get

$Pz = ABz$. ---------------- (3)

Step-3: putting $x=Px_{2n}$ and $y=x_{2n+1}$ in condition (v); we have

$$M(PPx_{2n}, Qx_{2n+1}, qt)$$
$$\geq \min\{M(ABPx_{2n}, STx_{2n+1}, t), M(PPx_{2n}, ABPx_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, t), M(PPx_{2n}, STx_{2n+1}, t)\}$$

Taking n→∞ and using (1), (2) and (3); we get

$$M(Pz, z, qt) \geq \min\{M(ABz, z, t), M(Pz, ABz, t), M(z, z, t), M(Pz, z, t)\}$$
$$= \min\{M(Pz, z, t), M(Pz, Pz, t), M(z, z, t), M(Pz, z, t)\}$$

i.e. $M(Pz, z, qt) \geq M(Pz, z, t)$

Therefore by using lemma 2; we have

$z = Pz = ABz.$

Step-4: putting x=Bz and y= x_{2n+1} in condition (v); we get

$$M(PBz, Qx_{2n+1}, qt)$$
$$\geq \min\{M(ABBz, STx_{2n+1}, t), M(PBz, ABBz, t), M(Qx_{2n+1}, STx_{2n+1}, t), M(PBz, STx_{2n+1}, t)\}$$

BP=PB, AB=BA; so we have

P(Bz)=B(Pz)=Bz and AB(Bz)=B(AB)z=Bz

Taking n→∞ and using (1), we get

$$M(Bz, z, qt) \geq \min\{M(Bz, z, t), M(Bz, Bz, t), M(z, z, t), M(Bz, z, t)\}$$
$$M(Bz, z, qt) \geq M(Bz, z, t)$$

Therefore, by using lemma 2, we get

$Bz = z$.

And also we have $ABz = z$

$\Rightarrow Az = z$.

$\therefore Az = Bz = Pz = z$. ------------ (4)

Step-5: since P(X) $\subseteq$ ST(X), there exists $u \in X$ such that

$z = Pz = STu$ --------------- (5)

Putting $x=x_{2n}$ and y= u in (v), we have

$M(Px_{2n}, Qu, qt)$

$\geq \min\{M(ABx_{2n}, STu, t), M(Px_{2n}, ABx_{2n}, t), M(Qu, STu, t), M(Px_{2n}, STu, t)\}$

Taking n→∞ and using (2) and (5), we get

$M(z, Qu, qt) \geq \min\{M(z,z,t), M(z,z,t), M(Qu,z,t), M(z,z,t)\}$

$= M(Qu, z, t)$

$M(z, Qu, qt) \geq M(Qu, z, t)$

Therefore by using lemma 2; we get

$Qu = z$.

Hence $STu = Qu = z$

Since (Q, ST) is weakly compatible, therefore we have

$QSTu = STQu$

Thus $Qz = STz$ ----------- (6)

Step-6: Putting $x=x_{2n}$ and y= z in (v), we have

$M(Px_{2n}, Qz, qt)$

$\geq \min\{M(ABx_{2n}, STz, t), M(Px_{2n}, ABx_{2n}, t), M(Qz, STz, t), M(Px_{2n}, STz, t)\}$

Taking n→∞ and using (2) and (6); we get

$M(z, Qz, qt) \geq \min\{M(z,Qz,t), M(z,z,t), M(Qz,Qz,t), M(z,Qz,t)\}$

$= M(z, Qz, t)$

$M(z, Qz, qt) \geq M(z, Qz, t)$

Therefore by using lemma 2, we get

$Qz = z$.

Step-7: putting $x=x_{2n}$ and y= Tz in (v), we have

$M(Px_{2n}, QTz, qt)$

$\geq \min\{M(ABx_{2n}, STTz, t), M(Px_{2n}, ABx_{2n}, t), M(QTz, STTz, t), M(Px_{2n}, STTz, t)\}$

As QT=TQ and ST=TS, we have

QTz=TQz=Tz and STTz=TSTz=T(STz)=T(Qz)=Tz

Taking n→∞ and using (2); we get

$M(z, Tz, qt) \geq \min\{M(z,Tz,t), M(z,z,t), M(Tz,Tz,t), M(z,Tz,t)\}$

$= M(z, Tz, t)$

$M(z, Tz, qt) \geq M(z, Tz, t)$

Therefore by using lemma 2; we get

$Tz = z$.

Now $STz = Tz = z$ implies $Sz = z$.

Hence $Sz = Tz = Qz = z$. -------------- (7)

Combining (4) and (7); we get

$Az = Bz = Pz = Qz = Tz = Sz = z$.

Hence z is the common fixed point of A, B, S, T, P and Q.

Case II: suppose AB is continuous.

Since AB is continuous and (P, AB) is semi-compatible, we have

$ABPx_{2n} \to ABz$ ------------- (8)

$(AB)^2 x_{2n} \to ABz$ ----------- (9)

$P(AB)x_{2n} \to ABz$ ----------- (10)

Thus, $ABPx_{2n} = P(AB)x_{2n} = ABz$

Now, we prove $ABz = z$.

Step-8: putting $x = ABx_{2n}$ and $y = x_{2n+1}$ in (v); we get

$M(P(AB)x_{2n}, Qx_{2n+1}, qt)$

$\geq \min\{M(AB(AB)x_{2n}, STx_{2n+1}, t), M(P(AB)x_{2n}, AB(AB)x_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, t),$

$M(P(AB)x_{2n}, STx_{2n+1}, t)\}$

$M(P(AB)x_{2n}, Qx_{2n+1}, qt)$

$\geq \min\{M((AB)^2 x_{2n}, STx_{2n+1}, t), M(P(AB)x_{2n}, (AB)^2 x_{2n}, t), M(Qx_{2n+1}, STx_{2n+1}, t),$

$M(P(AB)x_{2n}, STx_{2n+1}, t)\}$

Taking n→∞ and using (1), (9) and (10); we get

$M(ABz, z, qt) \geq \min\{M(ABz,z,t), M(ABz,ABz,t), M(z,z,t), M(ABz,z,t)\}$

$M(ABz, z, qt) \geq M(ABz, z, t)$

Therefore, by using lemma 2, we get

$ABz = z$.

Step-9: putting x=z and y= x_{2n+1} in condition (v); we get

$M(Pz, Qx_{2n+1}, qt)$

$\geq \min\{M(ABz, STx_{2n+1}, t), M(Pz, ABz, t), M(Qx_{2n+1}, STx_{2n+1}, t), M(Pz, STx_{2n+1}, t)\}$

Taking n→∞ and using (1); we get

$M(Pz, z, qt) \geq \min\{M(ABz,z,t), M(Pz,ABz,t), M(z,z,t), M(Pz,z,t)\}$

$M(Pz, z, qt) \geq \min\{M(z,z,t), M(Pz,z,t), M(z,z,t), M(Pz,z,t)\}$

$M(Pz, z, qt) \geq M(Pz, z, t)$

Therefore, by using lemma 2, we get

$Pz = z$.

$\therefore ABz = Pz = z$.

Further, using step (4), we get

$Bz = z$.

Thus, $ABz = z$ gives $Az = z$

And so $Az = Bz = Pz = z$.

Also it follows from steps (5), (6) and (7) that

$Sz = Tz = Qz = z$.

Hence we get $Az = Bz = Pz = Sz = Tz = Qz = z$.

i.e. z is a common fixed point of A, B, P, Q, S and T in this case also.

Uniqueness : Let w be another common fixed point of A, B, P, Q, S and T,

Then

$Aw = Bw = Pw = Sw = Tw = Qw = w$.

Putting x=z and y= w in condition (v); we get

$M(Pz, Qw, qt) \geq \min\{M(ABz,STw,t), M(Pz,ABz,t), M(Qw,STw,t), M(Pz,STw,t)\}$

$M(Pz, z, qt) \geq \min\{M(ABz,z,t), M(Pz,ABz,t), M(z,z,t), M(Pz,z,t)\}$

$M(z, w, qt) \geq \min\{M(z,w,t), M(z,z,t), M(w,w,t), M(z,w,t)\}$

$M(z, w, qt) \geq M(z, w, t)$

Therefore, by using lemma 2, we get

$z = w$.

Therefore z is the unique common fixed point of self maps A, B, P, Q, S and T.

Remark 2: If we take B=T=I_X (the identity map on X) in the theorem 1, then condition (ii) is satisfied trivially and we get

Corollary 1: Let (X, M,$*$) be a complete Fuzzy metric space and let A, S, P and Q be mappings from X into itself such that the following conditions are satisfied:

(i) P(X) $\subseteq$ S(X), Q(X) $\subseteq$ A(X);
(ii) Either A or P is continuous;
(iii) The pair (P, A) is semi-compatible and (Q, S) is weakly compatible.
(iv) There exists $q \in (0,1)$ such that for every $x, y \in X$ and t>0

$$M(Px,Qy,qt) \geq \min\{M(Ax,Sy,t), M(Px,Ax,t), M(Qy,Sy,t), M(Px,Sy,t)\}.$$

Then A, S, P and Q have a unique common fixed point.

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