Characterization of Thermohaline Convection in Porous Medium: Brinkman Model

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Abstract

The present paper prescribes upper bounds for oscillatory motions of neutral or growing amplitude in thermohaline configurations of G. Veronis [1] and M. E. Stern [2] types in porous medium in such a way that it also results in sufficient conditions of stability for an initially top heavy as well as initially bottom heavy configuration.

Keywords: Thermohaline convection, top heavy, bottom heavy, oscillatory motions, porous medium.

Introduction

The Thermohaline convection problem has been extensively studied in the recent past on account of its interesting complexities as a double diffusive phenomenon as well as its direct relevance in many problems of practical interest in the fields of oceanography, astrophysics, limnology and chemical engineering etc.[3] Two fundamental configurations have been studied in the context of thermohaline convection problem, one by Veronis [1], wherein the temperature gradient is destabilizing and the concentration gradient is stabilizing and another by Stern [2] wherein the temperature gradient is stabilizing and the concentration gradient is destabilizing. The main findings of Veronis and Stern for their respective configuration are that both allow the occurrence of a steady motion or an oscillatory motion of growing amplitude, provided the destabilizing temperature gradient or the concentration gradient is sufficiently large. In case of Veronis' configuration, oscillatory motions of growing amplitude are preferred mode of onset of instability whereas in case of Sterns' configuration, steady motion (stationary convection) is the preferred mode of onset of instability and these results are independent of the initially gravitationally stable or unstable character of the two configurations. Thus thermohaline configurations of Veronis and Stern type can further be classified into the following two classes:

(i) the first class, in which thermohaline instability manifests itself when the total density field is initially bottom heavy, and

(ii) the second class, in which thermohaline instability manifests itself when the total density field is initially top heavy.

In recent years, many researchers have shown their keen interest in analyzing the onset of convection in a fluid layer subjected to a vertical temperature gradient in a porous medium because of its importance in the prediction of ground water movement in aquifers, in the energy extraction process from the geothermal reservoirs, in assessing the effectiveness of fibrous insulations and in nuclear engineering [4,5].The stability of flow of a fluid through porous medium was studied by Lapwood [6], Wooding [7].Tountan and Lightfoot [8] characterized salt fingers in thermohaline convection in porous medium. The problem of double diffusive convection in porous medium has been extensibly investigated and the growing volume of work in this area is well documented by Ingham and Pop [9], Nield and Bejan [4] and K. Vafai [10]. Recently Jyoti Prakash and Vinod Kumar [11,12] derived characterization theorems for the nonexistence of oscillatory motions of growing amplitude in an initially bottom heavy configuration of Veronis type and Stern type.

The above researchers have studied double diffusive convection in porous medium by considering the Darcy flow model which is relevant to densely packed, low permeability porous medium. However, experiments conducted with several combinations of solids and fluids covering wide ranges of governing parameters indicate that most of the experimental data do not agree with the theoretical predictions based on the Darcy flow model. Hence, non-Darcy effects on double diffusive convection in porous media have received a great deal of attention in recent years. Poulikakos [13] has used the Brinkman extended Darcy flow model for the problem to investigate the linear stability analysis. Recently, Givler and Altobelli [14] have demonstrated that for high permeability porous media the effective viscosity is about ten times the fluid viscosity. Therefore, the effect of viscosities on the stability analysis is of practical interest. Thus in the present paper the Brinkman extended Darcy model has been used to investigate the thermohaline convection in porous medium and upper bounds for the oscillatory motions of neutral or growing amplitude in thermohaline configuration of Veronis and Stern types in porous medium are obtained in such a manner that it also results in sufficient conditions for stability for an initially bottom heavy or an initially top heavy configurations.

Mathematical formulation and analysis:

An infinite horizontal porous layer filled with a viscous fluid is statically confined between two horizontal boundaries z = 0 and z = d maintained at constant temperatures T_0 and T_1 and solute concentrations S_0 and S_1 at the lower and upper boundaries respectively, where $T_1 < T_0$ and $S_1 < S_0$. It is further assumed that the saturating fluid and the porous layer are incompressible and that the porous medium is a constant porosity medium. Let the origin be taken on the lower boundary z = 0 with z-axis perpendicular to it.

The basic hydrodynamic equations that govern the problem are given by:

The continuity equation for an incompressible fluid is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (1)$$

Equations of Motion is

$$\frac{1}{\epsilon}\frac{\partial u}{\partial t}+\frac{1}{\epsilon^2}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho_0}\right)-\frac{\nu}{k_1}u+\frac{\mu_e}{\rho_0}\nabla^2 u, \qquad (2)$$

$$\frac{1}{\epsilon}\frac{\partial v}{\partial t}+\frac{1}{\epsilon^2}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial}{\partial y}\left(\frac{p}{\rho_0}\right)-\frac{\nu}{k_1}v+\frac{\mu_e}{\rho_0}\nabla^2 v, \qquad (3)$$

$$\frac{1}{\epsilon}\frac{\partial w}{\partial t}+\frac{1}{\epsilon^2}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial}{\partial z}\left(\frac{p}{\rho_0}\right)-\frac{\nu}{k_1}w+\frac{\mu_e}{\rho_0}\nabla^2 w-\frac{\rho}{\rho_0}g, \qquad (4)$$

Equation of Heat Conduction

$$E\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\kappa_T\nabla^2 T, \qquad (5)$$

Equation of Mass Diffusion

$$E'\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y}+w\frac{\partial S}{\partial z}=\kappa_S\nabla^2 S, \qquad (6)$$

Equation of State

$$\rho=\rho_0[1+\alpha(T_0-T)-\gamma(S_0-S)], \qquad (7)$$

where u, v, w are the components of velocity in the x, y, z-directions respectively and $\frac{p}{\rho_0}$ is the modified hydrodynamic pressure. Further, t, ρ, T, S, ϵ, k_1, μ_e, ν, κ_T and κ_S are, respectively, the time, the density, the temperature, the concentration, the porosity of the porous medium, the permeability of the porous medium, the effective viscosity, the kinematic viscosity, the thermal diffusivity and the mass diffusivity; and α and γ are respectively the coefficients of volume expansion due to temperature and concentration variation. Here $E=\epsilon+(1-\epsilon)\frac{\rho_s C_s}{\rho_0 C_f}$ is a constant and E′ is also a constant analogous to E but corresponding to concentration rather than heat, where ρ_s, C_s and ρ_0, C_f stand for density and heat capacity of the solid (porous matrix) material and fluid respectively. The suffix '0' denotes the values of the various parameters at some suitably chosen reference temperature T_0 and concentration S_0.

The basic state is assumed to be quiescent state and is given by

$$\left.\begin{aligned}(u,v,w)&\equiv(0,0,0)\\ p&\equiv p(z)\\ T&\equiv T(z)\\ S&\equiv S(z)\\ \rho&\equiv\rho(z)\end{aligned}\right\} \qquad (8)$$

Thus the initial stationary state solutions are given by:

$$\left.\begin{aligned}(u,v,w)&=(0,0,0)\\ \frac{p}{\rho_0}&=P=P_0-g\rho_0\left(z+\frac{\alpha\beta z^2}{2}-\frac{\gamma\delta z^2}{2}\right)\\ T&=T_0-\beta z\\ S&=S_0-\delta z\\ \rho&=\rho_0[1+\alpha(T_0-T)-\gamma(S_0-S)]\\ \rho&=\rho_0[1+\alpha\beta z-\gamma\delta z]\end{aligned}\right\} \qquad (9)$$

Here P_0 represents the pressure at the lower boundary z = 0, $\beta=\frac{T_0-T_1}{d}$ and $\delta=\frac{S_0-S_1}{d}$, are the respective maintained temperature and concentration gradients.

The initial stationary state is now slightly perturbed so that the perturbed state is given by:

$$\left.\begin{aligned}(u,v,w)_{PS}&=(0+u',0+v',0+w')\\ (p)_{PS}&=P+P'\\ (T)_{PS}&=T_0-\beta z+\theta'=T+\theta'\\ (S)_{PS}&=S_0-\delta z+\phi'=S+\phi'\\ (\rho)_{PS}&=\rho_0[1+\alpha(T_0-T-\theta')-\gamma(S_0-S-\phi')]\end{aligned}\right\} \qquad (10)$$

where u′, v′, w′, P′, θ′, ϕ′ denote, respectively, the perturbations in three components of velocity, pressure, temperature and concentration and are assumed to be small around the basic state. Then the linearized perturbation equations are given by

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0, \qquad (11)$$

$$\frac{1}{\epsilon}\frac{\partial u'}{\partial t}=-\frac{\partial P'}{\partial x}-\frac{\nu}{k_1}u'+\frac{\mu_e}{\rho_0}\nabla^2 u', \qquad (12)$$

$$\frac{1}{\epsilon}\frac{\partial v'}{\partial t}=-\frac{\partial P'}{\partial y}-\frac{\nu}{k_1}v'+\frac{\mu_e}{\rho_0}\nabla^2 v', \qquad (13)$$

$$\frac{1}{\epsilon}\frac{\partial w'}{\partial t}=-\frac{\partial P'}{\partial z}-\frac{\nu}{k_1}w'+\frac{\mu_e}{\rho_0}\nabla^2 w'+g\alpha\theta'-g\gamma\phi',$$

$$E\frac{\partial\theta'}{\partial t}-\beta w'=\kappa_T\nabla^2\theta', \qquad (15)$$

$$E'\frac{\partial\phi'}{\partial t}-\delta w'=\kappa_S\nabla^2\phi', \qquad (16)$$

For the system of equations (11) – (16) the analysis can be made in terms of two dimensional periodic waves of assigned wave numbers. Thus we ascribe to all quantities describing the perturbations a dependence on x, y and t of the form

$$[i(k_x x + k_y y) + nt] \quad (17)$$

where $k = \sqrt{(k_x^2 + k_y^2)}$. Here k_x and k_y are the wave numbers along the x- and y- directions, respectively, and k is the resultant wave number. The above consideration allows us to suppose that the perturbations u′, v′, w′, P′, θ′ and ϕ′ have the form

$$F'(x, y, z, t) = F''(z)\exp[i(k_x x + k_y y) + nt], \quad (18)$$

For functions with these dependencies on x, y and t we have

$$\frac{\partial}{\partial t} = n, \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2, \quad \text{and} \quad \nabla^2 = \frac{d^2}{dz^2} - k^2 \quad (19)$$

Following the normal mode analysis, equations (11) – (16), thus, becomes

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0, \quad (20)$$

$$\frac{1}{\epsilon} nu'' = -ik_x P'' - \frac{\nu}{k_1} u'' + \frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right) u'', \quad (21)$$

$$\frac{1}{\epsilon} nv'' = -ik_y P'' - \frac{\nu}{k_1} v'' + \frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right) v'', \quad (22)$$

$$\frac{1}{\epsilon} nw'' = -\frac{dP''}{dz} - \frac{\nu}{k_1} w'' + \frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right) w'' + g\alpha\theta'' - g\gamma\phi'', \quad (23)$$

$$En\theta'' - \beta w'' = \kappa_T \left(\frac{d^2}{dz^2} - k^2\right)\theta'', \quad (24)$$

$$E'n\phi'' - \delta w'' = \kappa_S \left(\frac{d^2}{dz^2} - k^2\right)\phi''. \quad (25)$$

Multiplying equations (21) and (22) by ik_x and ik_y respectively; adding the resulting equations and using equations (17) and (20), we obtain

$$-\frac{n}{\epsilon}\frac{dw''}{dz} = k^2 P'' + \frac{\nu}{k_1}\frac{dw''}{dz} - \frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)\frac{dw''}{dz}. \quad (26)$$

Now eliminating P'' between equations (23) and (26), we get the resulting equation in the form

$$\frac{\mu_e}{\rho_0}\left(\frac{d^2}{dz^2} - k^2\right)^2 w'' - \left(\frac{n}{\epsilon} + \frac{\nu}{k_1}\right)\left(\frac{d^2}{dz^2} - k^2\right) w'' = k^2(g\alpha\theta'' - g\gamma\phi'') \quad (27)$$

Also equations (24) and (25) can be written as

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{En}{\kappa_T}\right)\theta'' = -\frac{\beta w''}{\kappa_T} \quad (28)$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{E'n}{\kappa_S}\right)\phi'' = -\frac{\delta w''}{\kappa_S} \quad (29)$$

Now using the following non-dimensional parameters

$a_* = kd$, $z_* = \frac{z}{d}$, $\tau_* = \frac{\kappa_S}{\kappa_T}$, $p_1 = \frac{\nu}{\kappa_T}$, $p_l = \frac{k_1}{d^2}$, $D_* = d\frac{d}{dz}$, $\sigma_* = \frac{nd^2}{\nu}$, $w_* = \frac{d}{\nu} w''$, $R_* = \frac{g\alpha\beta d^4}{\kappa_T \nu}$, $R_{S*} = \frac{g\gamma\delta d^4}{\kappa_T \nu}$, $\theta_* = \frac{\kappa_T}{\beta\nu d}\theta''$, $\phi_* = \frac{\kappa_T}{\delta\nu d}\phi''$, $\Lambda = \frac{\mu_e}{\mu}$.

we can write equations (27 – (29) in the following non-dimensional form (dropping the asterisks for simplicity)

$$\Lambda(D^2 - a^2)^2 w - \left(\frac{\sigma}{\epsilon} + \frac{1}{P_l}\right)(D^2 - a^2)w = R a^2\theta - R_S a^2\phi, \quad (30)$$

$$(D^2 - a^2 - E\sigma p_1)\theta = -w, \quad (31)$$

$$\left(D^2 - a^2 - \frac{E'\sigma p_1}{\tau}\right)\phi = -\frac{w}{\tau}, \quad (32)$$

Equations (30) – (32) are to be solved using the following boundary conditions:

$w = \theta = \phi = Dw = 0$ at z = 0 and at z = 1, (when both the boundaries are rigid) (33)

$w = \theta = \phi = D^2w = 0$ at z = 0 and at z = 1, (when both the boundaries are free) (34)

where z is the vertical co-ordinate such that $0 \le z \le 1$, D is the differentiation w.r.t. z, a^2 is square of the wave number, $p_1 > 0$ the Prandtl number, $\tau > 0$ is the Lewis number, $R > 0$ is the Rayleigh number, $R_S > 0$ is the thermohaline Rayleigh number, $\sigma = \sigma_r + i\sigma_i$ is the complex growth rate which is complex constant in general and as a consequence the dependent variables $w(z) = w_r(z) + iw_i(z)$, $\theta(z) = \theta_r(z) + i\theta_i(z)$ and $\phi(z) = \phi_r(z) + i\phi_i(z)$ are complex valued functions of the real variable z such that $w_r(z)$, $w_i(z)$, $\theta_r(z)$, $\theta_i(z)$, $\phi_r(z)$ and $\phi_i(z)$ are real valued functions of the real variable z.

The system of equations (30) – (34) describes an eigen value problem for σ, for a given values of other parameters, namely a^2, p_1, R, R_S, P_l, E, E′, ϵ and τ and govern thermohaline instability in a porous medium with constant porosity. It is also assumed that the saturating fluid and the porous layer are incompressible and a given state of the system is stable, neutral or unstable depending on whether σ_r is negative, zero or positive. Furthermore,

(a) $\sigma_r \ge 0$ and $\sigma_i \ne 0$ describe oscillatory motions of neutral or growing amplitude;

(b) $R > 0$ and $R_S > 0$ describe Veronis thermohaline configuration;

(c) $R < 0$ and $R_S < 0$ describe Stern's thermohaline configuration;

(d) $\lambda = \frac{R}{R_S} \leq 1$ describes an initially bottom heavy configuration;

and (e) $\lambda \geq 1$ describes an initially top heavy configuration.

we prove the following theorem:

Theorem 1. If $(w, \theta, \phi, \sigma)$, $\sigma_r \geq 0$ and $\sigma_i \neq 0$ is a nontrivial solution of equations (30) – (32) together with the boundary condition (33) or (34) and R > 0, $R_S > 0$ then,

$$|\sigma| < \frac{\lambda R_S}{Ep_1\left[4\pi^2\left(\Lambda + \frac{\tau}{\epsilon E' p_1}\right) + \frac{1}{P_l}\right]}\sqrt{\Omega^2 - 1}$$

where

$$\Omega = \frac{\lambda R_S}{\frac{4\pi^2}{P_l} + \frac{27}{4}\pi^4\left(\Lambda + \frac{\tau}{\epsilon E' p_1}\right)} \text{ and } \lambda = \frac{R}{R_S}.$$

Proof: Multiplying equation (30) by $\frac{w^*}{R_S}$ throughout, integrating the resulting equation over the vertical range of z, and using equations (31) and (32), we get

$$\frac{\Lambda}{R_S}\int_0^1 w^*(D^2 - a^2)^2 w\, dz - \frac{1}{R_S}\left(\frac{\sigma}{\epsilon} + \frac{1}{P_l}\right)\int_0^1 w^*(D^2 - a^2)w\, dz$$
$$= -\lambda a^2 \int_0^1 \theta\,(D^2 - a^2 - E\sigma^* p_1)\theta^*\, dz + \tau a^2 \int_0^1 \phi\left(D^2 - a^2 - E'\sigma* p1\tau\phi* dz \qquad (35)$$

Integrating various terms of equation (35) by parts for an appropriate number of times and using the boundary conditions (33) or (34), we obtain

$$\frac{\Lambda}{R_S}\int_0^1(|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\, dz + \frac{1}{R_S}\left(\frac{\sigma}{\epsilon} + \frac{1}{P_l}\right)\int_0^1(|Dw|^2 + a^2|w|^2)\, dz$$
$$-\lambda a^2 \int_0^1(|D\theta|^2 + a^2|\theta|^2 + E\sigma^* p_1|\theta|^2)\, dz + \tau a^2 \int_0^1\left(|D\phi|^2 + a^2|\phi|^2 + \frac{E'\sigma^* p_1}{\tau}|\phi|^2\right) dz = 0 \qquad (36)$$

Equating the real and imaginary parts of equation (36) to zero and cancelling σ_i ($\neq 0$) throughout from the imaginary part, we have

$$\frac{\Lambda}{R_S}\int_0^1(|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\, dz + \frac{1}{R_S}\left(\frac{\sigma_r}{\epsilon} + \frac{1}{P_l}\right)\int_0^1(|Dw|^2 + a^2|w|^2)\, dz$$
$$-\lambda a^2 \int_0^1(|D\theta|^2 + a^2|\theta|^2 + E\sigma_r p_1|\theta|^2)\, dz + \tau a^2 \int_0^1\left(|D\phi|^2 + a^2|\phi|^2 + \frac{E'\sigma_r p_1}{\tau}|\phi|^2\right) dz = 0 \qquad (37)$$

and

$$\frac{1}{\epsilon R_S}\int_0^1(|Dw|^2 + a^2|w|^2)\, dz + a^2\left\{Ep_1\lambda\int_0^1|\theta|^2 dz - E'p_1\int_0^1|\phi|^2 dz\right\} = 0 \qquad (38)$$

Equation (37) can be written as

$$\frac{\Lambda}{R_S}\int_0^1(|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\, dz + \frac{1}{R_S}\left(\frac{\sigma_r}{\epsilon} + \frac{1}{P_l}\right)\int_0^1(|Dw|^2 + a^2|w|^2)\, dz$$
$$-\lambda a^2 \int_0^1(|D\theta|^2 + a^2|\theta|^2)\, dz + \tau a^2 \int_0^1(|D\phi|^2 + a2\phi 2dz$$
$$-\sigma_r a^2\left\{Ep_1\lambda\int_0^1|\theta|^2 dz - E'p_1\int_0^1|\phi|^2 dz\right\} = 0 \qquad (39)$$

Multiplying equation (38) by σ_r and adding the resulting equation to equation (39), we get

$$\frac{\Lambda}{R_S}\int_0^1(|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\, dz + \frac{1}{R_S}\left(\frac{2\sigma_r}{\epsilon} + \frac{1}{P_l}\right)\int_0^1(|Dw|^2 + a^2|w|^2)\, dz$$
$$-\lambda a^2 \int_0^1(|D\theta|^2 + a^2|\theta|^2)\, dz + \tau a^2 \int_0^1(|D\phi|^2 + a2\phi 2dz=0 \qquad (40)$$

Equation (38) implies that

$$a^2\int_0^1|\phi|^2 dz \geq \frac{1}{\epsilon R_S E' p_1}\int_0^1(|Dw|^2 + a^2|w|^2)\, dz \qquad (41)$$

Also, since w, θ and ϕ vanish at z = 0 and z = 1,the Rayleigh-Ritz inequality [15] gives

$$\int_0^1|Dw|^2 dz \geq \pi^2\int_0^1|w|^2 dz \qquad (42)$$
$$\int_0^1|D\theta|^2 dz \geq \pi^2\int_0^1|\theta|^2 dz \qquad (43)$$
$$\int_0^1|D\phi|^2 dz \geq \pi^2\int_0^1|\phi|^2 dz \qquad (44)$$

Combining inequalities (41) and (42), we obtain

$$a^2\int_0^1|\phi|^2 dz \geq \frac{(\pi^2 + a^2)}{\epsilon R_S E' p_1}\int_0^1|w|^2 dz \qquad (45)$$

which in particular also implies that

$$\int_0^1|\phi|^2 dz \geq \frac{1}{\epsilon R_S E' p_1}\int_0^1|w|^2 dz \qquad (46)$$

and

$$a^2\int_0^1|\phi|^2 dz \geq \frac{\pi^2}{\epsilon R_S E' p_1}\int_0^1|w|^2 dz \qquad (47)$$

Also, by utilizing the inequality (42), we can write

$$\int_0^1(|Dw|^2 + a^2|w|^2)\, dz \geq (\pi^2 + a^2)\int_0^1|w|^2 dz \qquad (48)$$

Further, multiplying equation (31) by its complex conjugate and integrating the resulting equation appropriate number of times and utilizing boundary conditions (33) or (34), we obtain

$$\int_0^1[(D^2 - a^2 - E\,\sigma p_1)\theta]\,[(D^2 - a^2 - E\,\sigma^* p_1)\theta^*]dz = \int_0^1 ww^* dz$$

which gives

$$\int_0^1|(D^2 - a^2)\theta|^2 dz + 2E\,p_1\sigma_r\int_0^1(|D\theta|^2 + a^2|\theta|^2)\, dz + E^2p_1^2|\sigma|^2\int_0^1|\theta|^2 dz = \int_0^1|w|^2 dz, \qquad (49)$$

since, $\sigma_r \geq 0$, it follows from equation (49) that

$$\int_0^1|w|^2 dz \geq \int_0^1|(D^2 - a^2)\theta|^2 dz + E^2p_1^2|\sigma|^2\int_0^1|\theta|^2 dz \qquad (50)$$

and

$$\int_0^1|w|^2 dz \geq \int_0^1|(D^2 - a^2)\theta|^2 dz \qquad (51)$$

Furthermore, utilizing the Schwartz's inequality, we have

$$\left[\int_0^1|\theta|^2 dz\right]^{\frac{1}{2}}\left[\int_0^1|D^2\theta|^2 dz\right]^{\frac{1}{2}} \geq \left|-\int_0^1\theta^* D^2\theta\, dz\right| = \int_0^1|D\theta|^2 dz$$

$\geq \pi^2 \int_0^1 |\theta|^2 dz$ (using (43))

Consequently,

$$\int_0^1 |D^2\theta|^2 dz \geq \pi^4 \int_0^1 |\theta|^2 dz \quad (52)$$

using the similar logic we can show that

$$\int_0^1 |D^2 w|^2 dz \geq \pi^4 \int_0^1 |w|^2 dz \quad (53)$$

utilizing inequalities (42) and (53), we get

$$\int_0^1 (|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\, dz \geq (\pi^2 + a^2)^2 \int_0^1 |w|^2 dz \quad (54)$$

Thus, utilizing (43) and (52), we can write

$$\int_0^1 |(D^2 - a^2)\theta|^2\, dz \geq (\pi^2 + a^2)^2 \int_0^1 |\theta|^2 dz \quad (55)$$

Combining inequalities (50) and (55), we obtain

$$\int_0^1 |w|^2\, dz \geq [(\pi^2 + a^2)^2 + E^2 p_1^2 |\sigma|^2] \int_0^1 |\theta|^2\, dz \quad (56)$$

Further,

$$\int_0^1 |w|^2\, dz = \left[\int_0^1 |w|^2\, dz\right]^{\frac{1}{2}} \left[\int_0^1 |w|^2\, dz\right]^{\frac{1}{2}}$$

$> (\pi^2 + a^2)\{1 +$

E2p12σ2π2+a221201D2−a2θ2dz120102dz12

(utilizing (51) and (56))

$$\geq (\pi^2 + a^2)\left\{1 + \frac{E^2 p_1^2 |\sigma|^2}{(\pi^2 + a^2)^2}\right\}^{\frac{1}{2}} \left| -\int_0^1 \theta^* (D^2 - a^2)\theta\, dz \right|$$

(using Schwartz inequality)

$$= (\pi^2 + a^2)\left\{1 + \frac{E^2 p_1^2 |\sigma|^2}{(\pi^2+a^2)^2}\right\}^{\frac{1}{2}} \int_0^1 (|D\theta|^2 + a^2|\theta|^2) dz \quad (57)$$

Using inequality (54) in the first integral, inequality (48) in the second integral, inequality (57) in the third integral and inequalities (44) and (45) in the last integral of (40) and utilizing the fact that $\sigma_r \geq 0$, we obtain

$$\frac{\Lambda}{R_S}(\pi^2 + a^2)^2 \int_0^1 |w|^2\, dz + \frac{(\pi^2+a^2)}{P_l R_S} \int_0^1 |w|^2\, dz + \frac{\tau(\pi^2+a^2)^2}{\epsilon R_S E' p_1} \int_0^1 |w|^2\, dz$$

$$- \frac{\lambda a^2}{(\pi^2+a^2)\left\{1 + \frac{E^2 p_1^2 |\sigma|^2}{(\pi^2+a^2)^2}\right\}^{\frac{1}{2}}} \int_0^1 |w|^2\, dz < 0$$

or

$$\left[\frac{\Lambda}{R_S}(\pi^2 + a^2)^2 + \frac{(\pi^2+a^2)}{P_l R_S} + \frac{\tau(\pi^2+a^2)^2}{\epsilon R_S E' p_1} - \right.$$

λa2π2+a21+E2p12σ2π2+a221201w2dz<0

which can be rearranged as

$$\left[\left\{\left(\frac{\Lambda}{R_S} + \frac{\tau}{\epsilon R_S E' p_1}\right)\frac{(\pi^2+a^2)^3}{a^2} + \frac{(\pi^2+a^2)^2}{a^2 P_l R_S}\right\}\left\{1 + \frac{E^2 p_1^2 |\sigma|^2}{(\pi^2+a^2)^2}\right\}^{\frac{1}{2}} - \lambda\right] \int_0^1 |w|^2\, dz < 0 \quad (58)$$

Inequality (58) clearly implies that we must have

$$\frac{1}{R_S}\left\{\left(\Lambda + \frac{\tau}{\epsilon E' p_1}\right)\frac{(\pi^2+a^2)^3}{a^2} + \frac{(\pi^2+a^2)^2}{a^2 P_l}\right\}\left\{1 + \frac{E^2 p_1^2 |\sigma|^2}{(\pi^2+a^2)^2}\right\}^{\frac{1}{2}} < \lambda \quad (59)$$

Since minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ w.r.t. a^2 is $\frac{27}{4}\pi^4$ (for $a^2 = \frac{\pi^2}{2}$) and minimum value of $\frac{(\pi^2+a^2)^2}{a^2}$ w.r.t. a^2 is $4\pi^2$ (for $a^2 = \pi^2$), we obtain, from inequality (59) that

$$\left\{1 + \frac{E^2 p_1^2 |\sigma|^2}{(\pi^2 + a^2)^2}\right\}^{\frac{1}{2}} < \frac{\lambda R_S}{\frac{27}{4}\pi^4\left(\Lambda + \frac{\tau}{\epsilon E' p_1}\right) + \frac{4\pi^2}{P_l}}$$

or

$$|\sigma| < \frac{(\pi^2+a^2)}{E\, p_1}\sqrt{\Omega^2 - 1} \quad (60)$$

where

$$\Omega = \frac{\lambda R_S}{\frac{27}{4}\pi^4\left(\Lambda + \frac{\tau}{\epsilon E' p_1}\right) + \frac{4\pi^2}{P_l}}$$

Since, $\sigma_r \geq 0$, it follows from equation (40) that

$$\frac{\Lambda}{R_S}\int_0^1 \left(|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2\right) dz + \frac{1}{R_S P_l}\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz$$

$$+ \tau a^2 \int_0^1 (|D\phi|^2 + a^2|\phi|^2)\, dz \leq \lambda\, a^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2)\, dz \quad (61)$$

which upon using inequalities (44),(45),(48),(54) and (57) gives

$$(\pi^2 + a^2)\left\{\left(\Lambda + \frac{\tau}{\epsilon E' p_1}\right)\frac{(\pi^2+a^2)^2}{a^2} + \frac{(\pi^2+a^2)}{a^2 P_l}\right\} < \lambda R_S \quad (62)$$

Since, $\frac{(\pi^2+a^2)}{a^2 P_l} > \frac{1}{P_l}$ and minimum value of $\frac{(\pi^2+a^2)^2}{a^2}$ w.r.t. a^2 is $4\pi^2$ (at $a^2 = \pi^2$), it follows from inequality (62) that

$$\leq \frac{\lambda R_S}{4\pi^2\left(\Lambda + \frac{\tau}{\epsilon E' p_1}\right) + \frac{1}{P_l}} \quad (63)$$

Combining inequalities (60) and (63), we obtain

$$|\sigma| < \frac{\lambda R_S}{E\, p_1\left[4\pi^2\left(\Lambda + \frac{\tau}{\epsilon E' p_1}\right) + \frac{1}{P_l}\right]}\sqrt{\Omega^2 - 1}.$$

which completes the proof of the theorem.

Theorem 1, from the physical point of view may be stated as follows: the complex growth rate $\sigma = \sigma_r + i\sigma_i$ of an arbitrary oscillatory $(\sigma_i \neq 0)$ perturbation of growing amplitude $(\sigma_r \geq 0)$ in thermohaline instability of Veronis type in a porous medium lies inside a semicircle in the right half of the $p_r p_i$ - plane whose centre is at the origin and radius is

$$\frac{\lambda R_S}{E\, p_1\left[4\pi^2\left(\Lambda+\frac{\tau}{\epsilon E' p_1}\right)+\frac{1}{P_l}\right]}\sqrt{\Omega^2-1} = \frac{R}{E\, p_1\left[4\pi^2\left(\Lambda+\frac{\tau}{\epsilon E' p_1}\right)+\frac{1}{P_l}\right]}\sqrt{\Omega^2-1}.$$

It may further be noted that the above result is uniformly valid for an initially top-heavy ($\lambda \geq 1$) as well as an initially bottom-heavy ($\lambda \leq 1$) configuration.

Corollary 1: If $(w, \theta, \phi, \sigma)$, $\sigma = \sigma_r + i\sigma_i$, $(\sigma_i \neq 0)$, is a nontrivial solution of equations (30) – (32) together with boundary conditions (33) or (34) and $R > 0$, $R_S > 0$ and

$$\lambda < \frac{\frac{27}{4}\pi^4\left(\Lambda+\frac{\tau}{\epsilon E' p_1}\right)+\frac{4\pi^2}{P_l}}{R_S},$$

then $\sigma_r < 0$

Proof: Corollary follows from theorem 1.

Corollary 1 implies that oscillatory motions of neutral or growing amplitude are not allowed in thermohaline instability of Veronis type if the initial stability parameter λ does not exceed the value $\frac{\frac{27}{4}\pi^4\left(\Lambda+\frac{\tau}{\epsilon E' p_1}\right)+\frac{4\pi^2}{P_l}}{R_S}$. Further this result is uniformly valid for an initially top-heavy ($\lambda \geq 1$) as well as an initially bottom-heavy ($\lambda \leq 1$) configuration.

Theorem 2: If $(w, \theta, \phi, \sigma)$, $\sigma = \sigma_r + i\sigma_i$, $\sigma_r \geq 0$, $\sigma_i \neq 0$, is a nontrivial solution of equations (30) – (32), together with boundary conditions (33) or (34) and $R < 0$, $R_S < 0$ and

$$|\sigma| < \frac{\bar{\lambda}\,|R|\tau^2}{E'\, p_1\left[4\pi^2\left(\Lambda+\frac{1}{\epsilon E p_1}\right)+\frac{1}{P_l}\right]}\sqrt{\bar{\Omega}^2-1} \quad (65)$$

where

$$\bar{\Omega} = \frac{\bar{\lambda}\,|R|\tau}{\frac{27}{4}\pi^4\left(\Lambda+\frac{1}{\epsilon E p_1}\right)+\frac{4\pi^2}{P_l}} \quad \text{and} \quad \bar{\lambda} = \frac{|R_S|}{|R|}.$$

Proof: Replacing R with $-|R|$ and R_S with $-|R_S|$ in equations (30) – (32) and proceeding exactly as in Theorem 1, we obtain the desired result.

Corollary 2: If $(w, \theta, \varphi, \sigma)$, $\sigma = \sigma_r + i\sigma_i$, $\sigma_i \neq 0$, is a nontrivial solution of equations (30) – (32), together with the boundary conditions (33) or (34) and $R < 0$, $R_S < 0$ and

$$\bar{\lambda} \leq \frac{1}{|R|\tau}\left[\frac{27}{4}\pi^4\left(\Lambda+\frac{1}{\epsilon E p_1}\right)+\frac{4\pi^2}{P_l}\right], \text{ then } \sigma_r < 0.$$

Proof: Corollary follows from Theorem 2. The essential contents of theorem 2 and corollary 2 from the point of view of hydrodynamic stability are similar to those of Theorem 1 and corollary 1, but in this case they pertain to thermohaline instability of the Stern type.

References

1. G. Veronis, On finite amplitude instability in thermohaline convection, J. Mar. Res., **23**, 1 – 17 (1965).
2. M. E. Stern, The salt fountain and thermohaline convection, Tellus, **12**, 171 – 175 (1960).
3. J.S. Turner, Double-Diffusive phenomenon, Ann. Rev. Fluid Mechanics, **6**, 37 – 54 (1974).
4. D. A. Nield and A. Bejan, Convection in Porous Media, Springer (2006).
5. B. Straughan, Stability and wave motion in porous media, Springer (2008).
6. E. R. Lapwood, Convection of a fluid in a porous medium, Proc. Camb. Phil. Soc. **44**, 508 – 521 (1948).
7. R. A. Wooding, Rayleigh instability of a thermal boundary layer in flow through a porous medium, J. Fluid Mech. **9**, 183 – 192 (1960).
8. J. W. Tounton, E. N. Lightfoot, and T. Green, Thermohaline instability and salt fingers in a porous medium, The Phys. Fluids, **15(5)**, 748 – 753 (1972).
9. D. B. Ingham, and I. Pop, Transport phenomenon in porous media, Vol. III, Elsevier (2005).
10. K. Vafai, Handbook of Porous media, Taylor and Francis (2006).
11. Jyoti Prakash and Vinod Kumar, On nonexistence of oscillatory motions in thermohaline convection of Stern type in porous medium, J. Rajasthan Acad. Phy. Sci., **10(4)**, 331 – 338 (2011).
12. Jyoti Prakash and Vinod Kumar, Recent Advances in Continuum Mechanics and Algebra, Regal Publications, New Delhi (2010).
13. D. Poulikakos, Double-diffusive convection in a horizontally sparsely packed porous layer, Int. Comm. Heat Mass Transfer, **13**, 587 – 598 (1986).
14. R. C. Givler and S. A. Altobelli, A determination of the effective viscosity for the Brinkman-Forchheimer flow model, J. Fluid Mech, **258**, 355 – 370 (1994).
15. M. H. Schultz, Spline Analysis, Prentice Hall, Englewood Cliffs, NJ, (1973).