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Analytic Solution Of Burger's Equations By Variational Iteration Method

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Abstract

By means of variational iteration method the solutions of (1+1), (1+2) and (1+3) dimensional Burger equations are exactly obtained. In this paper, He's variational iteration method is introduced to overcome the difficulty arising in calculating Adomian polynomials.

Key words: Burger's equation, Nonlinear time dependent partial differential equations, Variational iteration method and Lagrange multiplier.

1 Introduction

We often come with non linear partial differential equations obtained through mathematical models of scientific phenomena. There are some methods to obtain approximate solution of this kind of equation. Some of them are numerical methods, homotopy analysis, Exp-function method, and linearization of the equation [1, 6, 7]. In 1999, the variational iteration method was developed by mathematician "He". This method is used for solving linear and non linear differential equations. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications [1, 2, 4, 5,]. It is based on Lagrange multiplier and it has the merits of simplicity and easy execution. This method avoids linearization of the problem.

In this paper exact solution of (1+1), (1+2) and (1+3) dimensional Burger equation [3] has been obtained by variational iteration method. The mentioned problem has been solved by N.Taghizadeh etc. [3] by Homotopy Perturbation Method and Reduced Differential Transformation Method here in the present paper we have solved the same problem by variational iteration mehod..

2 VARIATIONAL ITERATION METHOD:

Consider the following differential equation:

$$Lu + Nu = h(x,t) \tag{2.1}$$

where L is a linear operator, N a nonlinear operator, and an $h(x,t)$ is the source inhomogeneous term. The VIM was proposed by "He", where a correctional functional for equation (1.4.1) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\tau) + N\tilde{u}_n(\tau) - h(\tau))\,d\tau \quad , n \geq 0 \tag{2.2}$$

Where λ is a general Lagrange multiplier which can be identified optimally via the variational theory. The subscript n indicates the n^{th} approximation and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$. It is clear that the successive approximations $u_n, n \geq 0$ can be established by determining λ, so we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(x,t), n \geq 0$ of the solution $u(x,t)$ will be readily obtained using the Lagrange multiplier obtained and by using any selective function u_0. The initial values $u(x,0)$ and $u_t(x,0)$ are usually used for selecting the zeroth approximation u_0. With λ determined, then several approximation $u_n(x,t), n \geq 0$, follow immediately. Consequently the exact solution may be obtained by using

$$u = \lim_{n\to\infty} u_n \tag{2.3}$$

3. APPLICATION OF VIM FOR BURGER'S EQUATION:

3.1 (1+1)-Dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \alpha\left(u\frac{\partial u}{\partial x}\right) - \beta\left(\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{3.1.1}$$

I.C.: $u(x,0)=x$ (3.1.2)

Now following the variational iteration method given in the above section we get the following functional

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left(\frac{\partial u_n}{\partial t}+\alpha\left(u_n\frac{\partial u_n}{\partial x}\right)-\beta\left(\frac{\partial^2 u_n}{\partial x^2}\right)\right)d\tau \quad (3.1.3)$$

Stationary conditions can be obtained as follows:

$$\lambda'(\tau)=0$$
$$1+\lambda(\tau)\big|_{\tau=t} \quad (3.1.4)$$

The Lagrange multiplier can therefore be simply identified as $\lambda=-1$, and substituting this value of Lagrange multiplier into the functional (3.1.3) gives the following iteration equation.

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left(\frac{\partial u_n}{\partial t}+\alpha\left(u_n\frac{\partial u_n}{\partial x}\right)-\beta\left(\frac{\partial^2 u_n}{\partial x^2}\right)\right)d\tau \quad (3.1.5)$$

As stated before, we can select Initial condition given in the equation (3.1.2) and using this selection in (3.1.5) we obtain the following successive approximations:

$$u_1=\{x-tx\alpha\} \quad (3.1.6)$$

$$u_2=\{\{x-\alpha(tx-t^2x\alpha+\frac{1}{3}t^3x\alpha^2)\}\} \quad (3.1.7)$$

$$u_3=\{\{\{x+tx\alpha-t^2x\alpha^2+\frac{1}{3}t^3x\alpha^3 -\alpha(tx-t^2x\alpha+\frac{1}{3}t^3x\alpha^2)- \alpha(tx-t^2x\alpha+t^3x\alpha^2-\frac{2}{3}t^4x\alpha^3 +\frac{1}{3}t^5x\alpha^4-\frac{1}{9}t^6x\alpha^5+\frac{1}{63}t^7x\alpha^6)\}\}\} \quad (3.1.8)$$

$$u_4=\{\{\{\{x(1-t\alpha+t^2\alpha^2-t^3\alpha^3+t^4\alpha^4 -\frac{13t^5\alpha^5}{15}+\frac{2t^6\alpha^6}{3}-\frac{29t^7\alpha^7}{63}+\frac{71t^8\alpha^8}{252} -\frac{86t^9\alpha^9}{567}+\frac{22t^{10}\alpha^{10}}{315}-\frac{5t^{11}\alpha^{11}}{189}+\frac{t^{12}\alpha^{12}}{126} -\frac{t^{13}\alpha^{13}}{567}+\frac{t^{14}\alpha^{14}}{3969}-\frac{t^{15}\alpha^{15}}{59535})\}\}\}\} \quad (3.1.9)$$

$$u_5=\{\{\{\{\{x(1-t\alpha+t^2\alpha^2-t^3\alpha^3+t^4\alpha^4 -t^5\alpha^5+\frac{43t^6\alpha^6}{45}-\frac{13t^7\alpha^7}{15}+\frac{943t^8\alpha^8}{1260}- \frac{3497t^9\alpha^9}{5670}+\frac{27523t^{10}\alpha^{10}}{56700}-\frac{1477t^{11}\alpha^{11}}{4050} +\frac{17779t^{12}\alpha^{12}}{68040}-\frac{13141t^{13}\alpha^{13}}{73710}+\frac{1019t^{14}\alpha^{14}}{8820} -\frac{63283t^{15}\alpha^{15}}{893025}+\frac{43363t^{16}\alpha^{16}}{1058400}-\frac{1080013t^{17}\alpha^{17}}{48580560} +\frac{2588t^{18}\alpha^{18}}{229635}-\frac{162179t^{19}\alpha^{19}}{30541455}+\frac{16511t^{20}\alpha^{20}}{7144200} -\frac{207509t^{21}\alpha^{21}}{225042300}+\frac{557t^{22}\alpha^{22}}{1666980}-\frac{2447t^{23}\alpha^{23}}{22504230}+ \frac{16927t^{24}\alpha^{24}}{540101520}-\frac{5309t^{25}\alpha^{25}}{675126900}+\frac{t^{26}\alpha^{26}}{595350} -\frac{2t^{27}\alpha^{27}}{6751269}+\frac{13t^{28}\alpha^{28}}{315059220}-\frac{t^{29}\alpha^{29}}{236294415} +\frac{t^{30}\alpha^{30}}{3544416225}-\frac{t^{31}\alpha^{31}}{109876902975})\}\}\}\}\} \quad (3.1.10)$$

$$u(x,t)=u_1+u_2+u_3+.... \quad (3.1.11)$$

$$u(x,t)=\frac{x}{1+\alpha t} \quad (3.1.12)$$

3.2 (2+1)-Dimensional Burger's equation

$$\frac{\partial u}{\partial t}+\alpha\left(u\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}\right)- \beta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0 \quad (3.2.1)$$

I.C.: $u(x,y,0)=x+y$ (3.2.2)

Now following the variational iteration method, we get the following functional

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left(\begin{array}{l}\frac{\partial u_n}{\partial t}+\alpha\left(u_n\frac{\partial u_n}{\partial x}+u_n\frac{\partial u_n}{\partial y}\right)\\ -\beta\left(\frac{\partial^2 u_n}{\partial x^2}+\frac{\partial^2 u_n}{\partial y^2}\right)\end{array}\right)d\tau \quad (3.2.3)$$

Stationary conditions can be obtained as follows:

$$\lambda'(\tau)=0$$
$$1+\lambda(\tau)\Big|_{\tau=t} \quad (3.2.4)$$

The Lagrange multiplier can therefore be simply identified as $\lambda=-1$, and substituting this value of Lagrange multiplier into the functional (3.2.3) gives the following iteration equation.

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t \left(\begin{array}{l}\frac{\partial u_n}{\partial t}+\alpha\left(u_n\frac{\partial u_n}{\partial x}+u_n\frac{\partial u_n}{\partial y}\right)\\ -\beta\left(\frac{\partial^2 u_n}{\partial x^2}+\frac{\partial^2 u_n}{\partial y^2}\right)\end{array}\right)d\tau \quad (3.2.5)$$

As stated before, we can select Initial condition given in the equation (3.2.2) and using this selection in (3.2.5) we obtain the following successive approximations:

$$u_1=\{x+y-2t(x+y)\alpha\} \quad (3.2.6)$$

$$u_2=\{\{x+y-2tx\alpha-2ty\alpha+4t^2x\alpha^2+4t^2y\alpha^2-\frac{8}{3}t^3x\alpha^3-\frac{8}{3}t^3y\alpha^3\}\} \quad (3.2.7)$$

$$u_3=\{\{\{x+y-2tx\alpha-2ty\alpha+4t^2x\alpha^2+4t^2y\alpha^2-8t^3x\alpha^3-8t^3y\alpha^3+\frac{32}{3}t^4x\alpha^4+\frac{32}{3}t^4y\alpha^4-\frac{32}{3}t^5x\alpha^5-\frac{32}{3}t^5y\alpha^5+\frac{64}{9}t^6x\alpha^6+\frac{64}{9}t^6y\alpha^6-\frac{128}{63}t^7x\alpha^7-\frac{128}{63}t^7y\alpha^7\}\}\} \quad (3.2.8)$$

$$\begin{aligned}u_4=\{\{\{\{&x+y-2tx\alpha-2ty\alpha+4t^2x\alpha^2+4t^2y\alpha^2-8t^3x\alpha^3-8t^3y\alpha^3+16t^4x\alpha^4+16t^4y\alpha^4-\frac{416}{15}t^5x\alpha^5\\&-\frac{416}{15}t^5y\alpha^5+\frac{128}{3}t^6x\alpha^6+\frac{128}{3}t^6y\alpha^6-\frac{3712}{63}t^7x\alpha^7-\frac{3712}{63}t^7y\alpha^7+\frac{4544}{63}t^8x\alpha^8\\&+\frac{4544}{63}t^8y\alpha^8-\frac{44032}{567}t^9x\alpha^9-\frac{44032}{567}t^9y\alpha^9+\frac{22528}{315}t^{10}x\alpha^{10}+\frac{22528}{315}t^{10}y\alpha^{10}-\frac{10240}{189}t^{11}x\alpha^{11}\\&-\frac{10240}{189}t^{11}y\alpha^{11}+\frac{2048}{63}t^{12}x\alpha^{12}+\frac{2048}{63}t^{12}y\alpha^{12}-\frac{8192}{567}t^{13}x\alpha^{13}-\frac{8192}{567}t^{13}y\alpha^{13}+\frac{16384t^{14}x\alpha^{14}}{3969}\\&+\frac{16384t^{14}y\alpha^{14}}{3969}-\frac{32768t^{15}x\alpha^{15}}{59535}-\frac{32768t^{15}y\alpha^{15}}{59535}\}\}\}\}\end{aligned} \quad (3.2.9)$$

$$u_5 = \{\{\{\{\{x+y-2tx\alpha-2ty\alpha +4t^2x\alpha^2+4t^2y\alpha^2-8t^3x\alpha^3 -8t^3y\alpha^3+16t^4x\alpha^4+16t^4y\alpha^4 -32t^5x\alpha^5-32t^5y\alpha^5+\frac{2752}{45}t^6x\alpha^6 +\frac{2752}{45}t^6y\alpha^6-\frac{1664}{15}t^7x\alpha^7 -\frac{1664}{15}t^7y\alpha^7+\frac{60352}{315}t^8x\alpha^8 +\frac{60352}{315}t^8y\alpha^8-\frac{895232t^9x\alpha^9}{2835} -\frac{895232t^9y\alpha^9}{2835}+\frac{7045888t^{10}x\alpha^{10}}{14175} +\frac{7045888t^{10}y\alpha^{10}}{14175}-\frac{1512448t^{11}x\alpha^{11}}{2025} -\frac{1512448t^{11}y\alpha^{11}}{2025}+\frac{9102848t^{12}x\alpha^{12}}{8505} +\frac{9102848t^{12}y\alpha^{12}}{8505}-\frac{53825536t^{13}x\alpha^{13}}{36855} \frac{53825536t^{13}y\alpha^{13}}{36855}+\frac{4173824t^{14}x\alpha^{14}}{2205} +\frac{4173824t^{14}y\alpha^{14}}{2205}-\frac{2073657344t^{15}x\alpha^{15}}{893025} -\frac{2073657344t^{15}y\alpha^{15}}{893025} +\frac{88807424t^{16}x\alpha^{16}}{33075}+.......\}\}\}\}\} \quad (3.2.10)$$

$$u(x,t)=u_1+u_2+u_3+.... \quad (3.2.11)$$

$$u(x,y,t)=\frac{x+y}{1+2\alpha t} \quad (3.2.12)$$

3.3 (3+1)-Dimensional Burger's equation

$$\frac{\partial u}{\partial t}+\alpha\left(u\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}+u\frac{\partial u}{\partial z}\right) -\beta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=0 \quad (3.3.1)$$

I.C.: $u(x,y,z,0)=x+y+z$ (3.3.2)

Now following the variational iteration method, we get the following functional

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left(\begin{array}{l}\frac{\partial u_n}{\partial t}+\alpha\left(u_n\frac{\partial u_n}{\partial x}+u_n\frac{\partial u_n}{\partial y}+u_n\frac{\partial u_n}{\partial z}\right)\\ -\beta\left(\frac{\partial^2 u_n}{\partial x^2}+\frac{\partial^2 u_n}{\partial y^2}+\frac{\partial^2 u_n}{\partial z^2}\right)\end{array}\right)d\tau \quad (3.3.3)$$

Stationary conditions can be obtained as follows:

$$\lambda'(\tau)=0$$
$$1+\lambda(\tau)\Big|_{\tau=t} \quad (3.3.4)$$

The Lagrange multiplier can therefore be simply identified as $\lambda=-1$, and substituting this value of Lagrange multiplier into the functional (3.3.3) gives the following iteration equation.

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t \left(\begin{array}{l}\frac{\partial u_n}{\partial t}+\alpha\left(u_n\frac{\partial u_n}{\partial x}+u_n\frac{\partial u_n}{\partial y}+u_n\frac{\partial u_n}{\partial z}\right)\\ -\beta\left(\frac{\partial^2 u_n}{\partial x^2}+\frac{\partial^2 u_n}{\partial y^2}+\frac{\partial^2 u_n}{\partial z^2}\right)\end{array}\right)d\tau \quad (3.3.5)$$

As stated before, we can select Initial condition given in the equation (3.3.2) and using this selection in (3.3.5) we obtain the following successive approximations:

$$u_1=\{x+y+z-3t(x+y+z)\alpha\} \quad (3.3.6)$$

$$u_2=\{\{x+y+z-3tx\alpha-3ty\alpha -3tz\alpha+9t^2x\alpha^2+9t^2y\alpha^2 +9t^2z\alpha^2-9t^3x\alpha^3 -9t^3y\alpha^3-9t^3z\alpha^3\}\} \quad (3.3.7)$$

$$u_3=\{\{\{x+y+z-3tx\alpha-3ty\alpha -3tz\alpha+9t^2x\alpha^2+9t^2y\alpha^2 +9t^2z\alpha^2-27t^3x\alpha^3-27t^3y\alpha^3 -27t^3z\alpha^3+54t^4x\alpha^4+54t^4y\alpha^4 +54t^4z\alpha^4-81t^5x\alpha^5-81t^5y\alpha^5 -81t^5z\alpha^5+81t^6x\alpha^6+81t^6y\alpha^6 +81t^6z\alpha^6-\frac{243}{7}t^7x\alpha^7-\frac{243}{7}t^7y\alpha^7 -\frac{243}{7}t^7z\alpha^7\}\}\} \quad (3.3.8)$$

$$u_4=\{\{\{\{x+y+z-3tx\alpha-3ty\alpha$$
$$-3tz\alpha+9t^2x\alpha^2+9t^2y\alpha^2+9t^2z\alpha^2$$
$$-27t^3x\alpha^3-27t^3y\alpha^3-27t^3z\alpha^3$$
$$+81t^4x\alpha^4+81t^4y\alpha^4+81t^4z\alpha^4$$
$$-\frac{1053}{5}t^5x\alpha^5-\frac{1053}{5}t^5y\alpha^5$$
$$-\frac{1053}{5}t^5z\alpha^5+486t^6x\alpha^6$$
$$-\frac{1053}{5}t^5z\alpha^5+486t^6x\alpha^6$$
$$+486t^6y\alpha^6+486t^6z\alpha^6$$
$$-\frac{7047}{7}t^7x\alpha^7-\frac{7047}{7}t^7y\alpha^7$$
$$-\frac{7047}{7}t^7z\alpha^7+\frac{51759}{28}t^8x\alpha^8$$
$$+...........\}\}\}\} \quad (3.3.9)$$

$$u(x,y,z,t)=u_1+u_2+u_3+... \quad (3.3.10)$$

$$u(x,y,z,t)=\frac{x+y+z}{1+3\alpha t} \quad (3.3.11)$$

3.4 (n+1)-Dimensional Burger's equation

$$\frac{\partial u}{\partial t}+\alpha\left(u\frac{\partial u}{\partial x_1}+u\frac{\partial u}{\partial x_2}+u\frac{\partial u}{\partial x_3}+......+u\frac{\partial u}{\partial x_n}\right)$$
$$-\beta\left(\frac{\partial^2 u}{\partial x_1{}^2}+\frac{\partial^2 u}{\partial x_2{}^2}+\frac{\partial^2 u}{\partial x_3{}^2}+....+\frac{\partial^2 u}{\partial x_n{}^2}\right)=0$$
(3.4.1)

I.C.: $$u(x_1,x_2,x_3,..........,x_n,0)=x_1+x_2+......+x_n \quad (3.4.2)$$

Similarly, we apply VIM on the equation (3.4.1) we get the following exact solution.

$$u(x_1,x_2,x_3,.............,x_n,t)=\frac{(x_1+x_2+x_3+x_4+...........+x_n)}{1+n\alpha t} \quad (3.4.3)$$

4. Conclusion

In this paper, the variational iteration method has been successfully applied to finding the solution of (1+1), (1+2) and (1+3) dimensional Burger's equations. The solution obtained by the variational iteration method is an infinite power series for appropriate initial condition, which can, in turn, be expressed in a closed form, the exact solution. The results show that the variational iteration method is a powerful mathematical tool to solving Burger's equations. In our work, we use the Mathematica Package to calculate the series obtained from the variational iteration method.

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