4-Point Block Method for Direct Integration of First-Order Ordinary Differential Equations

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ABSTRACT

This research paper examines the derivation and implementation of a new 4-point block method for direct integration of first-order ordinary differential equations using interpolation and collocation techniques. The approximate solution is a combination of power series and exponential function. The paper further investigates the properties of the new integrator and found it to be zero-stable, consistent and convergent. The new integrator was tested on some numerical examples and found to perform better than some existing ones.

Keywords: Approximate Solution, Block Method, Exponential Function, Order, Power Series
AMS Subject Classification: 65L05, 65L06, 65D30

1. INTRODUCTION

Nowadays, the integration of Ordinary Differential Equations (ODEs) is carried out using some kinds of block methods. Therefore, in this paper, we propose a new 4-point block method for the solution of first-order ODEs of the form:

$$y' = f(x,y),\ y(a) = \eta\ \forall\, a \le x \le b \qquad (1)$$

where f is continuous within the interval of integration $[a,b]$. We assume that f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1). The problem (1) occurs mainly in the study of dynamical systems and electrical networks. According to [1] and [2], equation (1) is used in simulating the growth of populations, trajectory of a particle, simple harmonic motion, deflection of a beam etc.

Development of Linear Multistep Methods (LMMs) for solving ODEs can be generated using methods such as Taylor's series, numerical integration and collocation method, which are restricted by an assumed order of convergence, [3]. In this work, we will follow suite from the previous paper of [4] by deriving a new 4-point block method in a multistep collocation technique introduced by [5].

Block methods for solving ODEs have initially been proposed by [6], who used them as starting values for predictor-corrector algorithm, [7] developed Milne's method in form of implicit methods, and [8] also contributed greatly to the development and application of block methods. More recently, authors like [9], [10], [11], [12], [13], [14], [15], [16], have all proposed LMMs to generate numerical solution to (1). These authors proposed methods in which the approximate solution ranges from power series, Chebychev's, Lagrange's and Laguerre's polynomials. The advantages of LMMs over single step methods have been extensively discussed by [17].

In this paper, we propose a new Continuous Linear Multistep Method (CLMM), in which the approximate solution is the combination of power series and exponential function. This work is an improvement on [4].

2. METHODOLOGY: CONSTRUCTION OF THE NEW BLOCK METHOD

Interpolation and collocation procedures are used by choosing interpolation point s at a grid point and collocation points r at all points giving rise to $\xi = s + r - 1$ system of equations whose coefficients are determined by using appropriate procedures. The approximate solution to (1) is taken to be a combination of power series and exponential function given by:

$$y(x) = \sum_{j=0}^{4} a_j x^j + a_5 \sum_{j=0}^{5} \frac{\alpha^j x^j}{j!} \qquad (3)$$

with the first derivative given by:

$$y'(x) = \sum_{j=0}^{4} j a_j x^{j-1} + a_5 \sum_{j=1}^{5} \frac{\alpha^j x^{j-1}}{(j-1)!} \qquad (4)$$

where $a_j, \alpha^j \in \mathbb{R}$ for $j = 0(1)5$ and $y(x)$ is continuously differentiable. Let the solution of (1) be sought on the partition $\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots < x_N = b$, of the integration interval $[a,b]$, with a constant step-size h, given by, $h = x_{n+1} - x_n$, $n = 0,1,\ldots,N$.

Then, substituting (4) in (1) gives:

$$f(x,y) = \sum_{j=0}^{4} j a_j x^{j-1} + a_5 \sum_{j=1}^{5} \frac{\alpha^j x^{j-1}}{(j-1)!} \qquad (5)$$

Now, interpolating (3) at point $x_{n+s}, s=0$ and collocating (5) at points $x_{n+r}, r=0(1)4$, leads to the following system of equations:

$$AX = U \qquad (6)$$

where

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \left(1+\alpha x_n + \frac{\alpha^2 x_n^2}{2} + \frac{\alpha^3 x_n^3}{6} + \frac{\alpha^4 x_n^4}{24} + \frac{\alpha^5 x_n^5}{120}\right) \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \left(\alpha + \alpha^2 x_n + \frac{\alpha^3 x_n^2}{2} + \frac{\alpha^4 x_n^3}{6} + \frac{\alpha^5 x_n^4}{24}\right) \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & \left(\alpha + \alpha^2 x_{n+1} + \frac{\alpha^3 x_{n+1}^2}{2} + \frac{\alpha^4 x_{n+1}^3}{6} + \frac{\alpha^5 x_{n+1}^4}{24}\right) \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & \left(\alpha + \alpha^2 x_{n+2} + \frac{\alpha^3 x_{n+2}^2}{2} + \frac{\alpha^4 x_{n+2}^3}{6} + \frac{\alpha^5 x_{n+2}^4}{24}\right) \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & \left(\alpha + \alpha^2 x_{n+3} + \frac{\alpha^3 x_{n+3}^2}{2} + \frac{\alpha^4 x_{n+3}^3}{6} + \frac{\alpha^5 x_{n+3}^4}{24}\right) \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & \left(\alpha + \alpha^2 x_{n+4} + \frac{\alpha^3 x_{n+4}^2}{2} + \frac{\alpha^4 x_{n+4}^3}{6} + \frac{\alpha^5 x_{n+4}^4}{24}\right) \end{bmatrix}$$

$$A = [a_0, a_1, a_2, a_3, a_4, a_5]^T$$

$$U = [y_n, f_n, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}]^T$$

and

Solving (6), for a_j's, $j=0(1)5$ and substituting back into (3) gives a continuous linear multistep method of the form:

$$y(x) = \alpha_0(x) y_n + h\sum_{j=0}^{4} \beta_j(x) f_{n+j} \qquad (7)$$

where $\alpha_0 = 1$ and the coefficients of f_{n+j} gives:

$$\left.\begin{aligned} \beta_0 &= \frac{1}{720}(6t^5 - 75t^4 + 350t^3 - 750t^2 + 720t) \\ \beta_1 &= -\frac{1}{360}(12t^5 - 135t^4 + 520t^3 - 720t^2) \\ \beta_2 &= \frac{1}{60}(3t^5 - 30t^4 + 95t^3 - 90t^2) \\ \beta_3 &= -\frac{1}{360}(12t^5 - 105t^4 + 280t^3 - 240t^2) \\ \beta_4 &= \frac{1}{720}(6t^5 - 45t^4 + 110t^3 - 90t^2) \end{aligned}\right\} \qquad (8)$$

where $t = (x - x_n)/h$. Evaluating (7) at $t = 1(1)4$ gives a block scheme of the form:

$$A^{(0)}\mathbf{Y}_m = \mathbf{E}\mathbf{y}_n + hd\mathbf{f}(\mathbf{y}_n) + hb\mathbf{F}(\mathbf{Y}_m) \qquad (9)$$

where

$$\mathbf{Y}_m = [y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}]^T$$

$$\mathbf{y}_n = [y_{n-3}, y_{n-2}, y_{n-1}, y_n]^T$$

$$\mathbf{F}(\mathbf{Y}_m) = [f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}]^T$$

$$\mathbf{f}(\mathbf{y}_n) = [f_{n-3}, f_{n-2}, f_{n-1}, f_n]^T$$

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$d = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{720} \\ 0 & 0 & 0 & \frac{29}{90} \\ 0 & 0 & 0 & \frac{27}{80} \\ 0 & 0 & 0 & \frac{14}{45} \end{bmatrix}$$

$$b=\begin{bmatrix} \frac{323}{360} & \frac{-11}{30} & \frac{53}{360} & \frac{-19}{720} \\ \frac{62}{45} & \frac{4}{15} & \frac{2}{45} & \frac{-1}{90} \\ \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & \frac{-3}{80} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \end{bmatrix}$$

3. ANALYSIS OF BASIC PROPERTIES OF THE NEW BLOCK METHOD

3.1 Order of the New Block Method

Let the linear operator $L\{y(x);h\}$ associated with the block (9) be defined as:

$$L\{y(x);h\}=A^{(0)}Y_m-Ey_n-hdf(y_n)-hbF(Y_m) \tag{10}$$

Expanding (10) using Taylor series and comparing the coefficients of h gives:

$$L\{y(x);h\}=c_0y(x)+c_1hy'(x)+c_2h^2y''(x)+\ldots+c_ph^py^p(x)+c_{p+1}h^{p+1}y^{p+1}(x)+\ldots \tag{11}$$

3.1.1 Definition

The linear operator L and the associated continuous linear multistep method (7) are said to be of order p if $c_0=c_1=c_2=\ldots=c_p$ and $c_{p+1}\neq 0$. c_{p+1} is called the error constant and the local truncation error is given by:

$$t_{n+k}=c_{p+1}h^{(p+1)}y^{(p+1)}(x_n)+O(h^{p+2}) \tag{12}$$

For our method:

$$L\{y(x);h\}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}[y_n]-h\begin{bmatrix} \frac{251}{720} & \frac{323}{360} & \frac{-11}{30} & \frac{53}{360} & \frac{-19}{720} \\ \frac{29}{90} & \frac{62}{45} & \frac{4}{15} & \frac{2}{45} & \frac{-1}{90} \\ \frac{27}{80} & \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & \frac{-3}{80} \\ \frac{14}{45} & \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\end{bmatrix}=0 \tag{13}$$

Expanding (13) in Taylor series gives:

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j-y_n-\frac{251h}{720}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{323}{360}(1)^j-\frac{11}{30}(2)^j+\frac{53}{360}(3)^j-\frac{19}{720}(4)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j-y_n-\frac{29h}{90}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{62}{45}(1)^j+\frac{4}{15}(2)^j+\frac{2}{45}(3)^j-\frac{1}{90}(4)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(3h)^j}{j!}y_n^j-y_n-\frac{27h}{80}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{51}{40}(1)^j+\frac{9}{10}(2)^j+\frac{21}{40}(3)^j-\frac{3}{80}(4)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(4h)^j}{j!}y_n^j-y_n-\frac{14h}{45}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{64}{45}(1)^j+\frac{8}{15}(2)^j+\frac{64}{45}(3)^j+\frac{14}{45}(4)^j\right\} \end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{14}$$

Hence, $c_0=c_1=c_2=c_3=c_4=c_5=0,\ c_6=[1.88(-02),\ 1.11(-02),\ 1.88(-02),\ -8.47(-03)]^T$.
Therefore, the new block method is of order five.

3.2 Zero Stability

3.2.1 Definition

The block method (9) is said to be zero-stable, if the roots $z_s, s=1,2,\ldots,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z)=\det(z\mathbf{A}^{(0)}-\mathbf{E})$ satisfies $|z_s|\le 1$ and every root satisfying $|z_s|\le 1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h\to 0$, $\rho(z)=z^{r-\mu}(z-1)^{\mu}$ where μ is the order of the differential equation, r is the

order of the matrices $\mathbf{A}^{(0)}$ and $\mathbf{E}$ (see [11] for details).
For our new integrator,

$$\rho(z)=\left| z\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\right|=0 \qquad (15)$$

$\rho(z)=z^3(z-1)=0, \Rightarrow z_1=z_2=z_3=0, z_4=1$. Hence, the new block method is zero-stable.

3.3 Convergence

The new block method is convergent by consequence of Dahlquist theorem below.

3.3.1 Theorem [18]

The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.

3.4 Region of Absolute Stability

3.4.1 Definition

The method (9) is said to be absolutely stable if for a given h, all the roots z_s of the characteristic polynomial $\pi(z,\bar{h})=\rho(z)+\bar{h}\sigma(z)=0$ satisfies $z_s<1,\ s=1,2,\ldots,n$ where $\bar{h}=\lambda h$ and $\lambda=\dfrac{\partial f}{\partial y}$.

We shall adopt the boundary locus method for the region of absolute stability of the block method. Substituting the test equation $y'=-\lambda y$ into the block formula gives,

$$\mathbf{A}^{(0)}\mathbf{Y}_m(r)=\mathbf{E}y_n(r)-h\lambda\mathbf{D}y_n(r)-h\lambda\mathbf{B}\mathbf{Y}_m(r) \qquad (16)$$

Thus:

$$\bar{h}(r,h)=-\left(\frac{\mathbf{A}^{(0)}Y_m(r)-\mathbf{E}y_n(r)}{\mathbf{D}y_n(r)+\mathbf{B}Y_m(r)}\right) \qquad (17)$$

Writing (17) in trigonometric ratios gives:

$$\bar{h}(\theta,h)=-\left(\frac{\mathbf{A}^{(0)}Y_m(\theta)-\mathbf{E}y_n(\theta)}{\mathbf{D}y_n(\theta)+\mathbf{B}Y_m(\theta)}\right) \qquad (18)$$

where $r=e^{i\theta}$. Equation (18) is our characteristic/stability polynomial. Applying (18) to our method, we have:

$$\bar{h}(\theta,h)=\frac{(\cos 2\theta)(\cos 3\theta)(\cos\theta)-(\cos 2\theta)(\cos 3\theta)(\cos 4\theta)(\cos\theta)}{\frac{1}{5}(\cos 2\theta)(\cos 3\theta)(\cos\theta)-\frac{1}{5}(\cos 2\theta)(\cos 3\theta)(\cos 4\theta)(\cos\theta)} \qquad (19)$$

which gives the stability region shown in fig. 1 below.

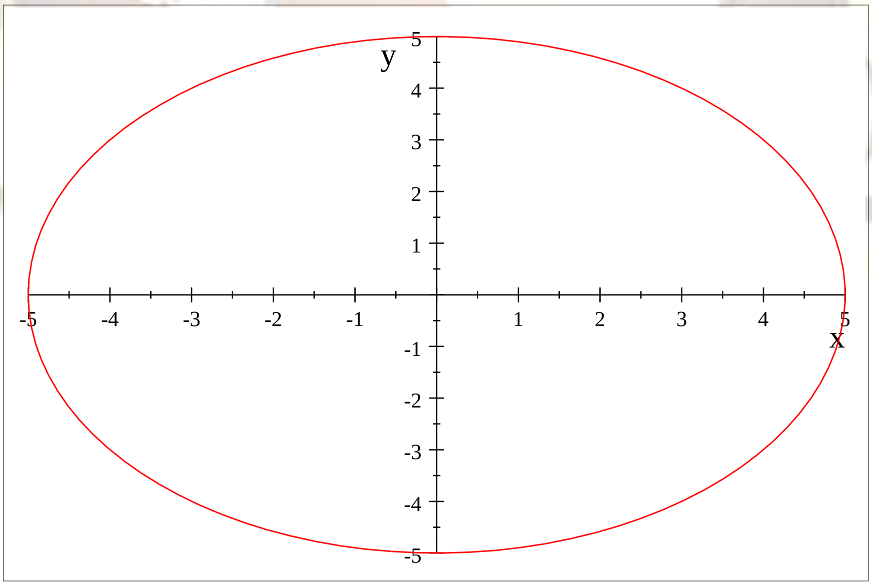

Fig. 1: Region of Absolute Stability of the 4-Point Block Method

4. NUMERICAL IMPLEMENTATIONS

We shall use the following notations in the tables below;

ERR- |Exact Solution-Computed Result|
EOAS- Error in [4]
EBM- Error in [13]
EMY- Error in [12]

Problem 1

Consider a linear first order IVP:

$$y' = -y,\ y(0) = 1,\ 0 \le x \le 2,\ h = 0.1 \quad (20)$$

with the exact solution:

$$y(x) = e^{-x} \quad (21)$$

This problem was solved by [12] and [4]. They adopted block methods of order four to solve the problem (20). We compare the result of our new block method (9) (which is of order five) with their results as shown in table 1 below.

Table 1: Performance of the New Block Method on $y' = -y,\ y(0) = 1,\ 0 \le x \le 1,\ h = 0.1$

x	*Exact solution*	*Computed Solution*	*ERR*	*EOAS*	*EMY*
0.1000	0.9048374180359595	0.9048374025462963	1.5490(-08)	2.3231(-07)	2.5292(-06)
0.2000	0.8187307530779818	0.8187307437037037	9.3743(-09)	1.0067(-07)	2.0937(-06)
0.3000	0.7408182206817178	0.7408182037500000	1.6932(-08)	3.2505(-07)	2.0079(-06)
0.4000	0.6703200460356393	0.6703200296296297	1.6406(-08)	4.6622(-07)	1.6198(-06)
0.5000	0.6065306597126334	0.6065306344848305	2.5228(-08)	3.4071(-07)	3.1608(-06)
0.6000	0.5488116360940264	0.5488116163781553	1.9716(-08)	4.8161(-07)	2.7294(-06)
0.7000	0.4965853037914095	0.4965852802878691	2.3504(-08)	5.6328(-07)	2.5457(-06)
0.8000	0.4493289641172216	0.4493289421226676	2.1995(-08)	4.4956(-07)	2.1713(-06)
0.9000	0.4065696597405991	0.4065696328791496	2.6862(-08)	5.3518(-07)	3.1008(-06)
1.0000	0.3678794411714423	0.3678794189516900	2.2220(-08)	5.7870(-07)	2.7182(-06)

Problem 2

Consider a linear first order IVP:

$$y' = xy,\ y(0) = 1,\ 0 \le x \le 2,\ h = 0.1 \quad (22)$$

with the exact solution:

$$y(x) = e^{\frac{x^2}{2}} \quad (23)$$

This problem was solved by [13] and [4]. They adopted a self-starting block method of order six and four respectively to solve the problem (22). We compare the result of our new block method (9) (which is of order five) with their results as shown in table 2.

Table 2: Performance of the New Block Method on $y' = xy,\ y(0) = 1,\ 0 \le x \le 1,\ h = 0.1$

x	*Exact solution*	*Computed Solution*	*ERR*	*EOAS*	*EBM*
0.1000	1.0050125208594010	1.0050122069965277	3.1386(-07)	5.2398(-07)	5.29(-07)
0.2000	1.0202013400267558	1.0202011563888889	1.3364(-07)	1.6913(-07)	1.77(-07)
0.3000	1.0460278599087169	1.0460275417187499	3.1819(-07)	8.7243(-07)	8.99(-07)
0.4000	1.0832870676749586	1.0832870577777778	9.8972(-09)	3.0098(-06)	3.09(-06)
0.5000	1.1331484530668263	1.1331477578536935	6.9521(-07)	1.7466(-06)	1.91(-06)
0.6000	1.1972173631218102	1.1972169551828156	4.0794(-07)	4.1710(-06)	4.48(-06)
0.7000	1.2776213132048868	1.2776205400192113	7.7319(-07)	9.6465(-06)	1.02(-05)
0.8000	1.3771277643359572	1.3771270561253053	7.0821(-07)	6.7989(-06)	7.74(-05)
0.9000	1.4993025000567668	1.4992996318515335	2.8682(-06)	1.2913(-05)	1.44(-05)
1.0000	1.6487212707001282	1.6487192043043850	2.0664(-06)	2.6575(-05)	2.93(-05)

5. CONCLUSION

In this paper, we have proposed a new 4-point block numerical method for the solution of first-order ordinary differential equations. The block integrator proposed was found to be zero-stable, consistent and convergent. The new method was also found to perform better than some existing methods.

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