FINITE ELEMENT ANALYSIS USING THE TANGENT STIFFNESS MATRIX FOR TRANSIENT NON-LINEAR HEAT TRANSFER IN A BODY

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Abstract— Reduction of computational complexity is one of the active research areas in engineering fields. Several efforts have been made in this direction. Solution of non-linear transient heat conduction problems is not only complex but also a time consuming process. In the present study, effort is made to reduce the complexity in the solution of non-linear heat conduction transient problems, by using the tangent stiffness matrix and the results obtained has been compared with ANSYS results.

Index Terms—Ansys, tangent stiffness matrix, Matlab.

1. INTRODUCTION

Solutions of large nonlinear transient systems are expensive both with respect to storage and computational time and costs. Hence it is difficult though not impossible to deal with situations containing a large number of parameters and bodies meshed into a large number of nodes with multiple degrees of freedom. Though computational methods and devices have improved tremendously but still reducing the computational time and complexity remains a very active field of research and analysis.

To reduce the computational time and complexity of transient non-linear problems several efforts are made. One of the methods used here is to derive the tangent stiffness matrix and the use the matrix after linearising the equation about the previous time step.

2. PROBLEM DEFINITION

The one dimensional body has cold gas (Tc) flowing at one surface and hot gas (T_H) flowing at the other surface. The temperatures at the various locations are to be determined. Heat transfer is at both the faces of the body, the upper and the lower surfaces of the body are insulated as shown. The complexities involved in solving the problem are:

1. The body has temperature varying coefficients of thermal conductivity (K) and heat capacity(C).
2. Time varying heat transfer coefficients.
3. The convection and radiations as boundary conditions. The radiation loss is always non linear in nature.

Fig. 1 shows the cross sectional view of the body used for the analysis. Fig 2a and 2b represents the variation of the heat transfer coefficients and the temperature at both ends with time.

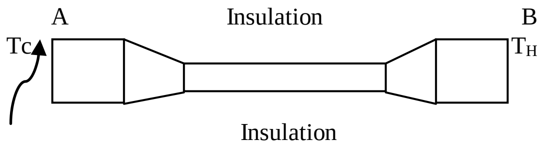

Figure 1: Cross sectional area of an one dimensional body

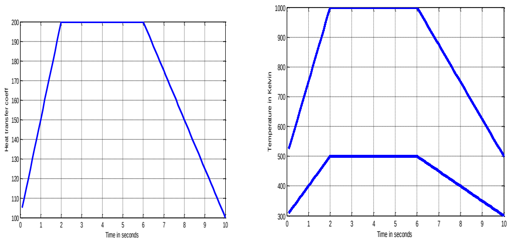

Fig.2a Variation of Heat transfer coefficient with time

Fig 2bVariation of Bulk Temperature with time

3. LITERATURE SURVEY

One of the ways to reduce the computational time in transient non-linear problems is to use the novel time integration schemes. The governing equation of Heat transfer is first linearized. Secondly the temperature at the first time step is determined by one of the two level time integration schemes like Euler's forward difference, Euler's backward difference or the Crank-Nicolson scheme. Each step requires the calculation of the tangent stiffness matrix.
Akrivis et al [1] has tried to solve non-linear parabolic equations by using linear multistep scheme. In this scheme part of the equation is discretized implicitly and another part explicitly. Fairweather and Johnson [2] used the extrapolation in conjunction with discrete time. Galerkin methods for the approximate solution of non-linear equations.

4. FINITE ELEMENT METHOD FOR TRANSIENT NON-LINEAR HEAT TRANSFER PROBLEMS

4.1 GOVERNING EQUATION

The basic idea of Finite element [3] is to subdivide a given domain Ω into an assembly of simple geometric shapes called finite elements, for which it is possible to systematically generate the approximation function needed in the solution of the differential equations by weighted – residual method. The approximation functions are constructed using the interpolation theory and hence are also called interpolation functions.

The major steps in Finite element analysis (FEA) are:

1. Discretization of the domain into a set of finite elements.
2. Weak formulation of the differential equation to be analyzed.
3. Development of the finite element model using its weak form.
4. Assembly of the finite elements.
5. Imposition of the boundary conditions.
6. Solution of the equation.

The governing equation for heat transfer analysis of a body idealized by a system of finite elements can be:

$$\rho C\left(\frac{\partial T}{\partial t}\right)= \frac{\partial}{\partial x}\left(K_{11}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{22}\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{33}\frac{\partial T}{\partial z}\right)]+Q \quad \text{in } \Omega \qquad (1)$$

where ρ = density, C = specific heat capacity,
T(x,y,z,t)= temperature,
t= time, Q= heat source in the body
K_{11}, K_{22} and K_{33} = coefficient of heat conduction in x,y and z directions,
Ω = orthotropic medium with boundary Γ

Equation (1) is solved in conjunction with the specified boundary conditions

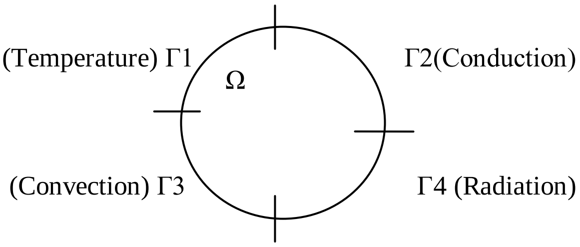

Figure 3. Boundary condition

T = T(s,t) on $\Gamma 1$
$\Gamma 2$, $\Gamma 3$ and $\Gamma 4$ are disjoint portions of the boundary and represent heat transfer due to conduction, convection and radiation respectively.

$$\left(K_{11}\frac{\partial T}{\partial x}\right)n_x+\left(K_{22}\frac{\partial T}{\partial y}\right)n_y+\left(K_{33}\frac{\partial T}{\partial z}\right)n_z + qc + qr = q(s,t) = qn \quad \text{on } \Gamma 2, \Gamma 3, \Gamma 4$$

Where $\Gamma 1$, $\Gamma 2$, $\Gamma 3$ and $\Gamma 4$ are disjoint portions of the boundary (fig. 3) Γ such that
$\Gamma = \Gamma 1 \cup \Gamma 2 \cup \Gamma 3 \cup \Gamma 4$

qc = heat transfer due to convection = hc(s,T,t) $(T_H - Tc)$
qr = heat transfer due to radiation
= hr(s,T,t) $(T_H - Tc) = \sigma \epsilon\, (T_H{}^2 + Tc^2)(T_H + Tc)(T_H - Tc)$
hc = convective heat transfer coefficient
hr = radiative heat transfer coefficient
n_x, n_y, n_z = direction cosines of the unit normal vectors on the boundary.

4.2 FINITE ELEMENT APPROXIMATION

Here domain of a typical element will be denoted by Ω^e and its boundary by Γ^e. The element Ω^e can be a triangle or quadrilateral in shape and the degree of interpolation over it can be linear, quadratic and so on. The Temperature can be approximated as

$$T(x,y,z,t) \approx T^e(x,y,z,t) \approx \sum_{j=1}^{n} T_j^e(t)\psi_j^e(x,y,z) \qquad (2)$$

where $T^e(x,y,z,t)$ represents an approximation of T(x,y,z) over the element Ω^e
T_j^e represents the value of function T(x,y,z) (at node j in the element Ω^e)
ψ_j^e represents the approximation function associated with the element.
n= number of nodes considered.

4.3 WEIGHTED – RESIDUAL FORMULATION

Transient problems are solved in two stages. The first stage is spatial discretization which involves the development of the weak form of the equation over the element. The second stage consists of the time approximation i.e. numerical integration of the equation by a suitable scheme.The weak form is developed in three steps

1. Multiply the residual obtained due to the temperature approximation by a weighted

function φ and integrate the resulting equation over the element domain.

$$\int_{\Omega^e} \varphi\left[\left(\frac{\partial}{\partial x} K_{11}\frac{\partial T}{\partial x}\right)+\left(\frac{\partial}{\partial y} K_{22}\frac{\partial T}{\partial y}\right)+\left(\frac{\partial}{\partial z} K_{33}\frac{\partial T}{\partial z}\right) - \left(\rho C\left(\frac{\partial T}{\partial t}\right) + Q\right)\right] dx\, dy\, dz = 0 \quad (3)$$

2. Distribute the differentiation between T and φ equally.

3. Formulate the finite element model using the boundary conditions.

Using Galerkin's method and Integration by parts of Eq. (1)

$$\int_{\Omega^e} \left[\frac{\partial\varphi}{\partial x}\left(K_{11}\frac{\partial T}{\partial x}\right)+\frac{\partial\varphi}{\partial y}\left(K_{22}\frac{\partial T}{\partial y}\right)+\frac{\partial\varphi}{\partial z}\left(K_{33}\frac{\partial T}{\partial z}\right) + \varphi\left(\rho C\left(\frac{\partial T}{\partial t}\right) - Q\right)\right] dx\, dy\, dz = \oint_{\Gamma^e} \left(\varphi\left(K_{11}\frac{\partial T}{\partial x}\right) n_x + \left(K_{22}\frac{\partial T}{\partial y}\right) n_y + \left(K_{33}\frac{\partial T}{\partial z}\right) n_z\right) ds \quad (4)$$

4.4 TEMPORAL APPROXIMATION

For the time dependent problem no integration by parts is used for the time function and the weight function is not a function of time. In selecting the approximation for T (temperature) the time dependence is separated from the spatial variable as follows.

$$T(x,y,z,t) \approx \sum_{j=1}^{n} T_j^e(t)\,\psi_j^e(x,y,z)$$

The i-th differential equation in time of the finite element model is obtained by substituting

$\varphi = \psi_j^e$ and replacing T_n and is given as,

$$0 = \sum_{j=1}^{n}\left(M_{ij}^e \frac{\partial T_j^e}{\partial t} + K_{ij}^e T_j^e\right) - Q_i^e - q_j^e \quad (5)$$

In Matrix formulation

$$[M^e]\{\dot{T}^e\} + [K^e]\{T^e\} = \{Q^e\} + \{q^e\} \quad (6)$$

$$[M^e] = \int_{\Omega^e} \rho\, C\, \varphi_i\, \varphi_j\, dx\, dy\, dz$$

$$[K^e] = \int_{\Omega^e}\left[\frac{\partial\varphi_i}{\partial x}\left(K_{11}\frac{\partial\varphi_j}{\partial x}\right)+\frac{\partial\varphi_i}{\partial y}\left(K_{22}\frac{\partial\varphi_j}{\partial y}\right)+\frac{\partial\varphi_i}{\partial z}\left(K_{33}\frac{\partial\varphi_j}{\partial z}\right)\right] dv$$

$$\{Q^e\} = \int_{\Omega^e} \varphi_i\, Q\, dv$$

$$\{q^e\} = \oint_{\Gamma^e} \varphi_i\,(qn - qc - qr)\, ds$$

Equation (6) can be rewritten as

$$[M^e]\{\dot{T}^e\} + [\widehat{K}^e]\{T^e\} = \{\widehat{F}^e\} \quad (7)$$

Where $[\widehat{K}^e]$ = Stiffness matrix + matrix due to convective and radiation losses

$[\widehat{K}^e] = [K^e] + [C^e] + [R^e]$

And $\{\widehat{F}^e\} = Q - q_o + Fc + Fr$ = Heat source + convective and radiation loads.

Equation (7) is the elemental equation. All the elemental equations are then assembled to obtain an equation of the form

$$[M]\{\dot{T}\} + [\widehat{K}]\{T\} = \{\widehat{F}\} \quad (8)$$

This equation (8) is then integrated using a time integration scheme. The time integration is done by dividing the time period into a number of discrete steps. The time steps are denoted by $\Delta t = t^{n+1} - t^{n}$, where t = time and n = the number of time step. Different time integration schemes are available like the Euler's Forward method, Euler's backward method, Crank-Nicolson method etc. For the study Euler's Backward method is used.

The Euler's formulation of Equation (8) gives:

$$\frac{1}{\Delta t}[M(T^{n+1}) + \widehat{K}(T^{n+1})]T^{n+1} = \frac{1}{\Delta t} M(T^{n+1})T^{n} + \widehat{F}(T^{n+1})$$

Linearising the above equation at T^n

$$\Delta T^{n+1} = \mathrm{inv}(\Delta t\, K_t)(T^n)\,[-\Delta t\, K(T^n)(T^n) + \Delta t\, \widehat{F}(T^n)] \quad (9)$$

Where K_t is the Tangent stiffness matrix

Δt = time step,

ΔT^{n+1} = temperature difference at time n+1

T^n = Temperature at time n

4.5 IMPLEMENTATION OF FINITE ELEMENT METHOD TO ONE DIMENSIONAL PROBLEM

4.5.1 STEADY STATE

To gain familiarity with FEA, code is developed in MATLAB for a steady state condition. Code is developed for a one dimensional rod (figure 4) of uniform cross section and having constant coefficient of thermal conductivity (K). The rod is insulated longitudinally and has a constant temperature of 300°K at end A and a gas with ambient temperature of

1000°K flowing at the end B with heat transfer due to convection and radiation at end B.

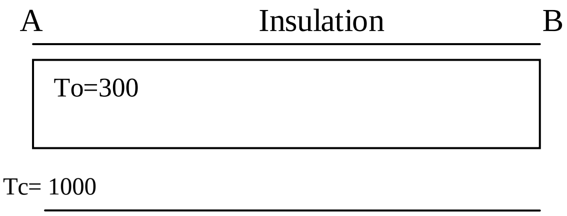

Figure 4. One Dimensional rod

The heat flux is equal to both convection and radiation losses at end B.

$$- K\frac{dT}{dx} = hc\,(T_L - Tc) + hr\,(T_L - Tc)$$

Where hr = $\sigma€\,(T_L^{\,2} + Ts^2)\,(T_L + Ts)$ = Heat transfer coefficient by radiation
hc = Heat transfer coefficient by convection,
K = Coefficient of thermal conductivity
T_L = Temperature at end B, Tc = Bulk temperature and
Ts = Surrounding temperature.

To solve this Picards iterative method is used with initial value of radiation loss (hr = 0)
as zero. The convergence is achieved with a tolerance value of $\|T^n - T^{n+1}\|_2 \leq 10^{-8}\|T^n\|_2$

4.5.2 TRANSIENT STATE

For the Transient state a variable coefficient of thermal conductivity (K) is used; here K is a function of temperature. Euler's Backward difference scheme is used for the time integration. The problem is solved using Newton Raphson's iterative method. To solve by this scheme the Tangent stiffness matrix is derived for the one dimensional problem with variable K and having heat transfer by both conduction and radiation.

Here we assume that there is no heat source in the body. Note that the stiffness matrix includes both the heat transfer due to convection and radiation. Hence equation (7) can be rewritten as

$$[M^e]\{\dot{T}\} + [Ko + h + hr]\{T\} = \{q\} = \{F + h + hr\}$$

And from equation (9)

$$\Delta T^{n+1} = inv(\Delta t\, K_t)(T^n)\,[\,-\Delta t\, K(T^n)(\,T^n) + \Delta t\,\hat{F}(T^n)]$$

The tangent stiffness matrix used for the computation is as given below.

$$K_t\,(T^n) = \frac{Ko}{le}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{a}{2le}[T_1^n + T_2^n]\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{a}{2le}[T_1^n - T_2^n]\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$$

where K = Ko + aT with T as the temperature (subscript denotes the node number) and le denotes the element length.

The convergence was achieved with a tolerance value of $\|T^n - T^{n+1}\|_2 \leq 10^{-8}\|T^n\|_2$

5.0 RESULTS AND ANALYSIS

For this particular case the total time period is assumed to be 10sec and the time interval 1sec. The results obtained are compared with ANSYS solution and are shown in Table 1.
Also the boundary conditions which are the bulk temperature (Tc and T_H) (Figure 2a and 2b) and the convective heat transfer coefficient (h) vary with time and are given in Table2.
Since the boundary conditions are time variant the results at 7th second has only been tabled below. The results at all the other time intervals have not been tabled. Instead the results at all the other time intervals have been displaced in the form of a graph (Figure5).

Table 1: Comparison of Ansys results and the Matlab results

No of Elements	Ansys solution(°K) at 7sec	Matlab solution(°K) at 7sec
1	515	515
2	563	563
3	612	612
4	665	665
5	720	721
6	780	781
7	844	844
8	907	908
9	970	971

Table 2: The Time varying parameters

Time (sec)	h (W/ m^2 °K)	T_H (°K)	Tc (°K)
0	100	500	300
2	200	1000	400
6	200	1000	400

10	100	500	300
10.6	100	500	300

The following values are taken for the analysis:

1) K = Ko + aT where Ko = 6 and a = 0.1 = Thermal conductivity (W/m ^{0}K)
2) Ts = 300 ^{0} K (radiation) = Surrounding Temperature
3) C = 50 = Specific heat (J/Kg ^{0}K)
4) ρ = 1 = density (Kg/m^3)

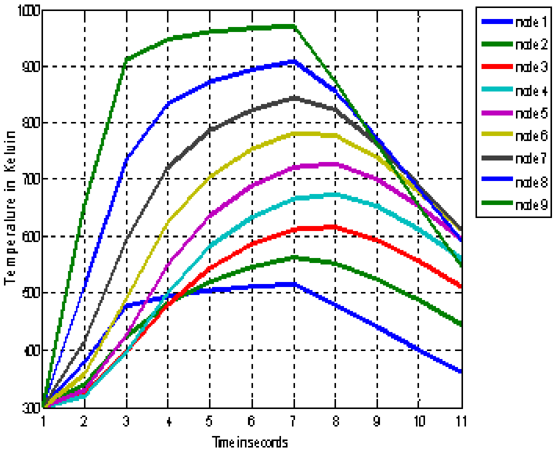

Figure 5: Variation of Temperature at each node with time

6.0 CONCLUSION AND FUTURE SCOPE

From the above analysis it can be seen that the results obtained by using the tangent stiffness matrix is comparable with the Ansys results. Similar efforts can be made to derive the Tangent stiffness matrix for Axisymmetric body also. Also for the present study a two level time integration scheme i.e. Euler Backward method is used, instead a three level time integration scheme can be used to reduce the computational time further.

7.0 REFERENCES

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