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The Method of Separation of Variables of Nonlinear Partial Equations

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ABSTRACT
The paper examines separation of the methods of separation of variables of nonlinear partial equations. These equations have dependent and independent nonlinear variables. They are used in the analysis of physical and chemical processes that are complex with many variables, and these variables are obtained by derivatives. Two cases of the separation process were detailed for heat conduction using homogeneous and nonhomogeneous cases. Several other applications where the separation process is applied were discussed, and a general method for separation was presented. It is clear that this process of separation is used in a number of physical and chemical applications.

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I. SEPARATION OF VARIABLES

Whitham (1, p. 17) speak of nonlinear partial differential equation (PDE) as having nonlinear dependent and independent variables. It is used to model physical systems in fluid dynamics, variation of functions such as velocity and time that have many variables such as space, time, elasticity, chemical reactions, stock market modelling, climate variables, and many others. These physical and chemical systems are complex with multiple variables, and it is difficult to create models since they involve variables defined by derivatives. PDEs have a function withn ≥ 2 variables that have partial derivatives. The terminology used for PDEs is: order represents the highest derivative, so $u_{xt}$is the2^{nd} derivative. In nonlinear PDEs, the dependent variable along with the derivatives is nonlinear, while a homogeneous PDE has$u = 0$.

An example of PDE is given as follows: Bateman–Burgers equation used in fluid dynamics and traffic flow study is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}$$

Where,$u(x,t)$are variables for a field, v is the general Burgers equation form, and if$v = 0$, then it becomes the inviscid Burgers' equation (Whitham (1, p. 23).

The need for separation of variables arises when the boundary conditions or limiting condition and the PDE are homogeneous and linear, and it is essential to reduce the PDE to ordinary nonlinear differential equations (NDE). These are solved with exact methods such as Laplace transforms, Fourier series, Fourier transform and others that are simple and give the solutions (Debnath, 2, p. 56). Next section explains different methods of separation of variables of nonlinear partial equations.

II. APPLICATION

Separation of variables is employed to solve problems involving heat wave equation, biharmonic equations, Helmholtz equations and others. Some important applications are discussed in this section.

1. Homogeneous separation in heat equation:
This section explains the separation in homogeneous heat equation (Polyanin and Zaitsev, 3, p. 34-38).
Consider a heat equation that is one dimensions

$$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = 0 \quad (1)$$

The homogeneous boundary conditions with u as the temperature are:

$$u(0,t) = u(L,t) = 0 \quad (2)$$

To find a solution that does not 0 and satisfy the boundary condition with the property that u depends on x and t is separated:

$$u(x,t) = X(x)T(t) \quad (3)$$

When u is placed in the equation through the product rule, then:

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} \quad (4)$$

It is seen the left side has a variable t and the right side variable is x, and both will be constant at an Eigen value λ. Therefore:

$$T'(t) = -\lambda\alpha T(t) \quad (5)$$

$$X''(x) = -\lambda X(x) \quad (6)$$

λ the Eigen value is applied for both the differential operators while the corresponding eigen functions are $T(t)$ and $X(x)$.The solutions for $X(x)$ for $\lambda \leq 0$will not happen. Assuming that $\lambda = 0$, then real numbers B and C are available where:

$$X(x) = Be^{\sqrt{-\lambda x}} + Ce^{\sqrt{-\lambda x}} \quad (7)$$

From the boundary layer equation (2)
X(0) = 0 = X(L) (8)
and so B = 0 = C, indicating u is identical to 0
If $\lambda = 0$, then real numbers B and C are available so that X(x) = Bx + C
Therefore, with reference to the first equation, u = 0, and so $\lambda > 0$. This gives us real numbers A, B, C so that$T(t) = Ae^{-\lambda\alpha t}$
and$X(x) = B\sin(\sqrt{\lambda x}) + C\cos(\sqrt{\lambda x})$
When C = 0, then for a positive integer n:$\sqrt{\lambda x} = n\frac{\pi}{L}$
This provides the solution for the heat equation when u has a dependence on (3).
The sum of solution for equation (1) that follows the boundary conditions in equation (2) is seen to satisfy (1) and (3). The full solution is:

$$U(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(n\frac{\pi x}{L}\right) e^{-n^2\pi^2\frac{\alpha t}{L^2}}$$

WhereDn are coefficients set by the initial conditions.
For the initial boundary conditions of
$u(0,t) = f(x)$, obtaining$f(x) = \sum_{n=1}^{\infty} D_n \sin\left(n\frac{\pi x}{L}\right)$
This is the sine series expansion for$f(x)$. When both are multiplied by $\sin\left(n\frac{\pi x}{L}\right)$

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

As per the Sturm-Liouville theory, the Eigen functions of x $\{\sin n\pi x/L\}_{n=1}^{\infty}$
Are orthogonal and complete.

2. Nonhomogeneous separation in heat equation:
In a nonhomogeneous equation of Ax = b, b is not equal to zero. If A, b are an m× n matrix with an m vector, then solutions to the nonhomogeneous are calculated by translating the calculations of the homogeneous equation $Ax_h = b$ where x_h is the solution. Consider the equation (Myint-U and Lokenath, 4, p. 67-68):

$$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = h(x,t) \quad (8)$$

Boundary conditions are the same as (2).
The functions h(x,t), u (x, t), and f(x) are expanded as:

$$h(x,t) = \sum_{n=1}^{\infty} h_n \sin\left(n\frac{\pi x}{L}\right) \quad (9)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n \sin\left(n\frac{\pi x}{L}\right) \quad (10)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi x}{L}\right) \quad (11)$$

When values of (9), (10) are placed in 8 and with orthogonality of Sine functions, the values obtained are:

$$u_t + \alpha\left(\frac{n^2\pi^2}{L^2}\right)u_n = h_n$$

This is the sequence of linear differential equation that can be calculated with integrating factor or Laplace transform. The result is:

$$u_n(t) = e^{-n^2\pi^2\frac{\alpha t}{L^2}} b_n + \int_0^L h_n(s)e^{-n^2\pi^2\frac{\alpha t}{L^2}} ds$$

With nonhomogeneous boundary conditions, then the expansion of equations (9) and (10) is not valid. It is essential to obtain a function that meets the boundary condition and subtract it from the value of u.

III. OTHER APPLICATIONS

Some other applications where the process of separation is used include 1D wave equation that represents harmonic waves in a pipe, strings, surface water waves in a channel and other areas. Another application is the study of 2D/ 3D wave equation where the vibrations of a membrane and ripples in a pond are studied. Separation of variables is also done for 1D/ 2D/ 3D diffusion and heat equation where the conduction of heat in a tube, rod and method of dispersion of chemicals in a tube is studied. Study of the conservation law where the principle that mass cannot be created or destroyed is studied, and for the diffusion/ heat equation. Laplace's equation for the potential flow of electrostatic potential flow in fluids and soap films is studied (Galaktionov 5, p. 16-17; Farlow 6, p. 23).

IV. GENERAL METHOD

The steps given above are for specific heat transfer problems. The general method for other applications is indicated as follows. Obtain the linear and homogeneous PDE with linear and homogeneous BC and ignore the non-zero value. Separate the variables by finding the differential equations that are included in the product solutions and bring in a separate constant. Find the separation constant as the eigen values for the boundary problem. Complete calculating other differential equations and write down the product solutions. Use the principle of superposition for a linear combination of the product solutions and satisfy the initial conditions. Calculate the coefficients with the orthogonality of the eigenfunctions (Logan 7, p. 36).

V. CONCLUSION

The paper examined the methods of separation of variables of nonlinear partial equations. This process of separation is used in a large number of complex physical and chemical processes, stock market variations, and other processes, where the variables are nonlinear dependent and independent variables. The calculations and processes used for homogeneous and nonhomogeneous heat equations were studied. Several other applications of variables separation were briefly examined and these include 1D/ 2D/ 3D wave equation and heat equation, conduction of heat in tubes, and the conservation law. The general method for separation of variables was also detailed. It is clear that the process of separation of variables of nonlinear partial equations is used in a number of physical and chemical applications.

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