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Using Differential Transform Method To Solve Non-Linear Partial Differential Equations

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ABSTRACT
Following the early application in electrical circuits, differential transform method for solving non-linear partial differential equations is gaining recognition based on the effectiveness of the solutions. This study shows the mathematical derivation of the technique and provides evidence for the effectiveness of the approach with two examples.

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I. INTRODUCTION

In academia and the industry, many applications include distributed-parameter systems, which are described by partial differential equations. An important characteristic is the involvement of non-linearities. Non-linear equations are typical in many fields of human endeavour, especially mathematics and physics, to describe the motion of waves that are isolated and localised in a small space. Some of the applications of non-linear equations include hydrodynamic, non-linear optics, and plasma physics, among others. Alquran and Al-Khaled (2010) argue that there is a significant interest in solving non-linear equations by simplifying the process. This is mainly because the systems of equations are treated with emphasis on various real-world applications including electrical devices, engineering, and mechanics of solids and fluids, among others. The differential transform method (DTE) is emerging as a reliable technique for solving non-linear equations.

Non-linear Partial Differential Equations

Non-linear partial differential equations are common in fields of human endeavour including biology, chemistry, mathematics, and physics, among others. Closed-form solutions of differential equations are useful for understanding the qualitative features of phenomena and processes in natural science. The usefulness of non-linear equations is in graphically demonstrating and understanding the mechanisms of complex non-linear phenomena. Exact solutions are useful because they allow researchers to design and perform experiments. Equations can be general, elliptic, hyperbolic, parabolic, and mixed types. Equations could also be second-, third, fourth-, and higher-order non-linear equations. In academia and the industry, solutions of equations are spread across many different applications. Polyanin and Zaitsev (2017) argue that there are over 1,600 non-linear mathematical physics equations and non-linear partial differential equations along with solutions. Simple solutions are not only useful for teaching courses, but they also provide evidence for the critical tenets of a theory, which are based on mathematical formulation. The exact solutions of non-linear equations are useful for understanding phenomena such as spatial localisation of processes, and absence or multiplicity of steady states under various conditions, among others.

Exact analytical methods solve non-linear equations. In practice, practicality is given great preference as the approach allows the methods to construct the exact solutions effectively. A wide range of methods has been developed in the past decade. Some reliable techniques include the non-classical and direct methods for symmetry reductions, differential constraints method, and the method of generalised separation of variables, among others. It is important to note that simple problems could be solved by several approaches, while difficult problems require a sophisticated approach. Some problems could remain unsolved, especially if they are too complicated to characterise. The emphasis of this investigation is to demonstrate the usefulness of DTM for solving the non-linear partial differential equation.

Differential Transform Method

DTM is dedicated to finding the Taylor expansion of differential equations. Sarp, Evirgen, and Ikikardes (2018) argue that functions can be expressed as polynomials, and the method helps obtain the differential equations of polynomial solutions. Zhou (1986) was the first researcher to suggest the differential transformation for the design of electrical circuits. The original model has been enhanced to achieve a better structure to solve more problems. The enhanced approach is gaining popularity as it has many applications. Researchers and industry experts have shown that the technique is highly effective. The development is useful because new solutions could emerge in the future. Transformations could extend from one-variable differential transformation to three-variable transformations. Evidence from the body of knowledge shows that DTM could effectively be applied to solve ordinary and partial differential equations. In the following sections, the discussion shows how the solutions generated by DTM match the solutions obtained by other methods. It is of significant interest to reformulate the differential equation or inequality into a problem, which could be solved by intuitive techniques such as minimisation by compactness arguments. As more and more researchers and experts engage the technique for understanding phenomena, further evidence or solutions based on the method could emerge.

Application of the Differential Transform Method

The following discussion shows the application of the Differential Transform Method. Alquran (2012) argues that the derivations provide evidence for how the technique is applied, and how the solutions are useful for practical reasons.

Consider a function $u(x,t)$, which is expressed as follows:

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k\,\partial t^h}\right]_{(x_0,t_0)} \cdots (1)$$

In this expression, $u(x,y)$ is the original function and $U(k,h)$ is the transformed function.

$U(k,h)$ has the following differential inverse transform.

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x-x_0)^k(t-t_0)^h \quad \ldots (2)$$

The function $u(x,t)$ could be modified as a finite series by considering (x_0,t_0) as $(0,0)$.

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k\,\partial t^h}\right]x^k t^h \ \ldots (3)$$

The equation shows that the two-dimensional differential transform is achieved by deploying a two-dimensional Taylor series expansion. Equations 1 and 2 show that the transformed functions comply with the mathematical functions in the following expressions.

The m-dimensional transform of $u(x_1,x_2,\ldots x_m)$ is expressed as follows.

$$U(k_1,k_2,k_3,\ldots k_n) = \frac{1}{k_1!k_2!\ldots k_m!}\left[\frac{\partial^{k_1+k_2+\cdots+k_m}u(x_1,x_2,\ldots x_m)}{\partial x_1^{k_1}\,\partial x_2^{k_2}\ldots\partial x_m^{k_m}}\right]_{(0,0,\ldots 0)} \ldots (4)$$

The differential inverse transform of $u(x_1,x_2,\ldots x_m)$ is given by the following expression.

$$u(x_1,x_2,\ldots x_m) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_m=0}^{\infty} U(k_1,k_2,k_3,\ldots k_n)x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}$$

$\ldots(5)$

Equations 4 and 5 help conclude that.

$$u(x_1,x_2,\ldots x_m) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_m=0}^{\infty}\frac{1}{k_1!k_2!\ldots k_m!}\left[\frac{\partial^{k_1+k_2+\cdots+k_m}u(x_1,x_2,\ldots x_m)}{\partial x_1^{k_1}\,\partial x_2^{k_2}\ldots\partial x_m^{k_m}}\right]_{(0,0,\ldots 0)} x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m} \ldots (6)$$

These equations provide evidence that the transformed equations comply with the following mathematical operations.

For $u(x_1,x_2,\ldots x_n)$=$f_1(x_1,x_2,\ldots x_n) \mp f_2(x_1,x_2,\ldots x_n)$, the transformed equation is given by:

$U(k_1,k_2,\ldots k_n)$=$F_1(k_1,k_2,\ldots k_n) \mp F_2(k_1,k_2,\ldots k_n)$

For $u(x_1,x_2,\ldots x_n)$=$\lambda g(x_1,x_2,\ldots x_n)$, where λ is a constant, the transformed equation is expressed as:

$U(k_1,k_2,\ldots k_n)$=$\lambda G(k_1,k_2,\ldots k_n)$

For $u(x_1,x_2,\ldots x_n)$=$\frac{\delta g(x_1,x_2,\ldots x_n)}{\delta x_1}$, the transformed equation is written as follows:

$U(k,h) = (k_i+1)G(k_1,\ldots k_i+1,\ldots k_{1n}),\ \ 1 \le i \le n$

For $u(x,t) = \frac{\partial^{r+s}g(x_1,x_2,\ldots x_n)}{\partial x_i^r\,\partial x_j^s}$, where $1 \le i \neq j \le n$, the transformed equation is as follows:

$U(k_1,k_2,\ldots k_n)$=$(k_i+1)(k_i+2)+\cdots+$= $(k_i+r)(k_j+1)(k_j+2)+\cdots(k_j+s)G(k_1,\ldots k_i+r,\ldots,k_j+s,\ldots k_n)$

For $u(x_1,x_2,\ldots x_n) = x_1^{h_1}x_2^{h_2}\ldots x_n^{h_n}$, the transformed equation is given by the following equation:

$U(k_1,k_2,\ldots k_n) = \partial(k_1-h_1)\,\partial(k_2-$ h2…∂kn−hn, where ∂ki−hi={0 otherwise1,ki=hi

For $u(x_1,x_2) = f_1(x_1,x_2)f_2(x_1,x_2)$, the transformed equation is written as follows:

$$U(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} F_1\,(r,h-s)F_2(k-r,s)$$

For $u(x_1,x_{2,}x_{3,}) = f_1(x_1,x_2,x_3)f_2(x_1,x_2,x_3)$, the transformed equation is given by the expression.

For $U(k_1,k_{2,}k_{3,}) = \sum_{r=0}^{k_1}\sum_{s=0}^{k_2}\sum_{p=0}^{k_3} F_1(r,k_2-$ s,k3−p)F2(k1−r,s,p)

The methods provide evidence for the effectiveness of the differential transform method. The technique is useful for reducing computational tasks.

Examples

The following examples show the application of the differential transform method for solving nonlinear partial differential equations.

Example 1

Consider the following Klein Gordonequation, which as

$$u_{tt} - u_{xx} + u^2 = -xcost + x^2cos^2t \qquad (1.1)$$

The initial condition is expressed as follows:

$$u(x,0) = x\,, \qquad u_t(x,0) = 0 \qquad (1.2)$$

where $u = u(x,t)$ is a function of the variables x and t.

This problem has the exact solution $u(x,t) = xcost$.

The transformed equation of 1.1 is derived from the principles of reduced differential transformation.

$$(h+1)(h+2)U(k,h+2) - (k+1)(k+2)U(k+2,h) + \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)U(k-r,s) = -\delta k - 1 1\, h!\, cos2\pi h + \delta k - 4h!\, cos2\pi h \; (1.3)$$

From the initial condition in 1.2,

$$U(k,0) = \delta(k-1)\ , U(k,1) = 0 \qquad (1.4)$$

Substituting Equation 1.4 in 1.3,

$$(1,0) = 1, U(1,2) = -\frac{1}{2}, U(1,4) = \frac{1}{24}, U(1,6) = -\frac{1}{720}, \ldots.. \qquad (1.5)$$

Finally, the inverse transform of $U_k(x)$ is as follows:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h \; (1.6)$$

$$u(x,t) = x - \frac{xt^2}{2} + \frac{xt^4}{24} - \frac{xt^6}{720} + \cdots \qquad (1.7)$$

$$u(x,t) = x(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots) \qquad (1.8)$$

Therefore, the exact solution is as follows:

$$u(x,t) = xcost. \qquad (1.9)$$

Example 2

For the second example, consider a Gas dynamic equation.

$$u_t - \frac{1}{2}u_x^2 - u(1-u) = 0 \qquad (2.1)$$

The initial conditions are:

$$u(x,0) = e^{-x} \qquad (2.2)$$

The differential transform equation of 2.1 is given by the expression.

$$(h+1)U(k,h+1) = \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s) + U(k,h) - \sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)U(k-r,s) \qquad (2.3)$$

Based on the initial conditions in Equation 2.2,

$$u(k,0) = \frac{-1^k}{k!} \qquad k = 0,1,2, \ldots\ldots.. \qquad (2.4)$$

By substituting 2.4 in 2.3, the values of $U_k(x)$ are as follows.

$$U(0,1) = 1$$
$$U(1,1) = -1$$
$$U(0,2) = \frac{1}{2!}$$
$$U(1,2) = -\frac{1}{2!}$$

Finally, the inverse transform of $U_k(x)$ is expressed as follows.

$$u(x,t) = \left(1 - x + t - xt + \frac{t^2}{2!} + \frac{x^2}{2!} + \frac{x^2t}{2!} - \frac{xt^2}{2!} + \cdots\right) \qquad (2.5)$$

Therefore, the exact solution is as follows.

$$u(x,t) = e^{t-x} \qquad (2.6)$$

II. CONCLUSION

The differential transform method helps find the approximate solution for non-linear partial differential equations by transforming or changing the variables to a new wave variable. The examples in this study show the effectiveness of the technique as the results agree with the precise solution. In conclusion, a non-linear partial differential equation is converted to an ordinary differential equation by using a wave variable, and the differential transformation method is applied to the ordinary differential equation. These results are useful because they have engineering applications, which continue to suffer due to drawbacks in the previous-gen approaches. DTM could be applied to many non-linear partial differential equations, which makes the approach highly effective in solving the problems that are too cumbersome to solve with the standard approaches.

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