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On A New Technique For Studying The Resolving Kernel of Volterra Integral Equation

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ABSTRACT: Here, we use resolving kernel method as a successive approximation method to solve the solution of Volterra integral equation. Some numerical examples are considered and the error of the method is computed

Key Wards: Volterra integral equation, resolving kernel, approximate method, the error of the method.

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I. INTRODUCTION:

The theory of integral equations has close contact with many different areas of different sciences. These different problems have led researches to establish different methods for solving integral equations of different kinds with continuous kernel. There are many well–written texts on the theory and applications of integral equation in different sciences. Among such, we noteGreen,1969);(Hochstadt,1971); (Golberg.ed,1979), (Tricomi, 1985); (Burten1983); (Kanwal, 1996); (Schiavone at.al.,2002) and (Muskhelishvili, 1953). The reader must know that the importance of the singular integral equations came from the work of(Muskhelishvili1953); , who has established the theory of singular integral equation (Cauchy method), that gives the solution of the singular integral equation, analytically.

At the same time, approximately from 1960, many new numerical methods have been developed for the solution of many types of integral equation. We note especially (Linz, 1985); (Atkinson, 1976. 1997);(Baker , 1082), (Delves and Mohamed, 1985) and (Golberg ,ed. 1990).

Consider the linear Volterra integral equation of the second kind,

$$\varphi(t) = f(t) + \lambda \int_0^t k(t,s)\, \varphi(s)\, ds \quad (1)$$

Here, k(t, s) and f(t) are known continuous functions called the kernel and free term, respectively, while φ(s) is the unknown function.

Theorem 1.(without proof): If k(t, s) and f(t) are continuous in $0 \le t \le T$, then the integral equation (1) possesses a unique continuous solution in $0 \le t \le T < 1$.

Here, in this paper the existence and uniqueness solution of Volterra integral equation of the second kind is considered. In addition, the solution of the linear Volterra equation with continuous kernel is obtained using a new technique for studying the resolvent kernel. Some examples are considered and the estimate error, with respect to the kernel,is computed.

II. THE RESOLVENT KERNEL METHOD

We pick up continuous function $\varphi_0(x) = f(t)$ then, from (1) we define the sequences

$$\varphi_n(t) = f(t) + \lambda \int_0^t k(t,s)\, \varphi_{n-1}(s)\, ds, \quad n = 1, 2, \ldots (2)$$

and

$$\varphi_{n-1}(t) = f(t) + \lambda \int_0^t k(t,s)\, \varphi_{n-2}(s)\, ds \quad (3)$$

By subtracting, we have

$$\varphi_n(t) - \varphi_{n-1}(t) = \lambda \int_0^t k(t,s)\, [\varphi_{n-1}(s) - \varphi_{n-2}(s)]\, ds$$

(4)

For easy of manipulation it is convenient to introduce

$$\lambda^n\, \psi_n(t) = \varphi_n(t) - \varphi_{n-1}(t); \; \psi_0(t) = f(t), \; n = 1, 2, \ldots (5)$$

By using equation (5), the formula (4) becomes

$$\psi_n(t) = \int_0^t k(t,s)\, \psi_{n-1}(s)\, ds, \quad n = 1, 2, \ldots (6).$$

In addition, from equation (5), we get

$$\varphi_n(t) = \sum_{i=0}^{n} \lambda^i\, \psi_i(t) \quad (7)$$

Using the recurrence relations and mathematical and the factthat: If the kernel $k(t,s)$ and the function $f(t)$ are continuous, then the order of integration can be interchanged, we get

$$\psi_n(t)=\int_0^t k_n(t,s)f(s)ds;\ k_n(t,s)=\int_s^t k(t,\tau)k_{n-1}(\tau,s)d\tau \quad (8).$$

with $k_1(t,s)=k(t,s)$.

The kernels $k_n(t,s), n=1,2,...$ are called the iterated kernels.

From equation (7), we follow

$$\varphi_n(t)=\sum_{i=0}^{n}\lambda^i\ \psi_i(t)=f(t)+\lambda\sum_{i=1}^{n}\lambda^{i-1}\ \psi_i(t)$$

Therefore, we get

$$\varphi_n(t)=f(t)+\lambda\int_0^t\Gamma_n(t,s;\lambda)f(s)ds, \quad (9).$$

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Where:

$$\Gamma_n(t,s;\lambda)=\sum_{i=1}^{n}\lambda^{i-1}k_i(t,s) \quad (10)$$

If the kernel $k(t,s)$ is continuous and $\left|k(t,s)\right|\le M$, $0\le s\le t\le T$, then we can prove by induction that

$$\left|k_n(t,s)\right|\le\frac{M^n\ (T-s)^{n-1}}{(n-1)!},\quad T=\max t, \quad (11)$$

Hence, the sequence in equation (10) converges and we can write

$$\Gamma(t,s;\lambda)=\sum_{i=1}^{\infty}\lambda^{i-1}k_i(t,s) \quad (12)$$

The function $\Gamma(t,s;\lambda)$ is the resolving kernel for k(t, s).

Theorem(2) (without proof): If k(t,s) and f(t) are continuous then the unique continuous solution of equation (1) is given by

$$\varphi(t)=f(t)+\lambda\int_0^t\Gamma(t,s;\lambda)f(s)ds \quad (13)$$

Theorem (3) Under the assumptions of theorem (2), the resolving kernel $\Gamma(t,s;\lambda)$ satisfies the equation

$$\Gamma(t,s;\lambda)=k(t,s)+\lambda\int_s^t k(t,\tau)\ \Gamma(t,\tau;\lambda)\ d\tau,$$

$$0\le s\le t\le T. \quad (14)$$

proof : Using equation (12), we see that

$$\lambda\int_s^t k(t,\tau)\ \Gamma(t,\tau;\lambda)d\tau=\lambda\int_s^t k(t,\tau)\sum_{i=1}^{\infty}\lambda^{i-1}k_i(\tau,s)d\tau \quad (15)$$

$$=\lambda\sum_{i=1}^{\infty}\lambda^{i-1}\int_s^t k(t,\tau)\,k_i(\tau,s)\,d\tau\ .$$

In addition, from equation (10), we have

$$\lambda\int_s^t k(t,\tau)\ \Gamma(t,\tau;\lambda)\ d\tau=\lambda\sum_{i=1}^{\infty}\lambda^{i-1}k_{i+1}(t,s)$$

$$=k_1(t,s)+\lambda\,k_2(t,s)+\lambda^2\,k_3(t,s)+\ ...\ -k(t,s)$$

Since $k_1(t,s)=k(t,s)$ then

$$\lambda\int_s^t k(t,\tau)\ \Gamma(t,\tau;\lambda)\ d\tau=\sum_{i=1}^{\infty}\lambda^{i-1}k_i(t,s)-k(t,s) \quad (16)$$

Using equation (12)in (16), we obtain

$$\lambda\int_s^t k(t,\tau)\ \Gamma(t,\tau;\lambda)\ d\tau=\Gamma(t,s;\lambda)-k(t,s)$$

Therefore, the following formula is satisfied

$$\Gamma(t,s;\lambda)=k(t,s)+\lambda\int_s^t k(t,\tau)\ \Gamma(\tau,s;\lambda)\ d\tau$$

III. APPLICATIONS:

Example (1): Find the resolving kernel of Volterra equation for k(t, s)=1

Solution: Assume

$$k_1(x,y)=k(x,y)=1$$

Hence, we have

$$k_2(x,y)=\int_y^x k(x,z)k_1(z,y)dz=\frac{(x-y)}{1!},$$

$$k_3(x,y)=\int_y^x k(x,z)k_2(z,y)dz=\int_y^x 1\ .(z\text{-}y)dz=\left.\frac{z^2}{2}\right|_y^x-yz\big|_y^x=\frac{(x-y)^2}{2!}$$

For n times, we get

$$k_n(x,y)=\frac{(x-y)^{n-1}}{(n-1)!}$$

Hence, the resolving takes the form

$$R(x,y;\lambda)=\sum_{n=0}^{\infty}\lambda^n k_{n+1}(x,y)=\sum_{n=0}^{\infty}\frac{\lambda^n(x-y)^n}{n!}=e^{\lambda(x-y)}$$

Example 3.2: Solve the integral equation

$$\varphi(t)=g(t)+\frac{1}{2}\int_0^t e^{(t-s)}\ \varphi(s)\ ds,\quad (0\le s\le t\le T)$$

Hence, find the solution when g(t)=1 and $\sinh(3t/2)$

Solution: Here, $k(t,s)=e^{t-s}$

Let, $k_1(t,s)=k(t,s)=e^{t-s}$

In addition, we have

$$k_2(t,s)=\int_s^t k(t,\tau)\,k_1(\tau,s)\,d\tau=\int_s^t e^{t-s}\ d\tau=(t-s)\ e^{t-s}$$

$$k_3(t,s)=\int_s^t k(t,\tau)\,k_2(\tau,s)\,d\tau=e^{t-s}\int_s^t(\tau-s)\,d\tau=\frac{1}{2}(t-s)^2\ e^{t-s}$$

So, in general, we obtain

$$k_n(t,s) = \frac{1}{(n-1)\,!}\,(t-s)^{n-1}\,e^{t-s}$$

Using equation (12), we have

$$\Gamma(t,s;\lambda) = e^{t-s}\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{i-1}\frac{(t-s)^{i-1}}{(i-1)\,!}$$

$$= e^{(t-s)}\;e^{\frac{1}{2}(t-s)} = e^{\frac{3}{2}(t-s)}$$

Finally, we can obtain

$$\phi(t) = g(t) + \frac{1}{2}\int_0^t e^{\frac{3}{2}(t-s)}\,ds = g(t) - \frac{1}{3}(1-e^{\frac{3}{2}t})$$

At

$i)$ At $g(t)=1 \rightarrow \phi(t) = \frac{1}{3}(2+e^{\frac{3}{2}t})$

$ii)$ At $g(t) = \sinh(3t/2) \rightarrow \phi(t) = \frac{1}{6}(5\,e^{\frac{3}{2}t} - 3e^{\frac{-3}{2}t} - 2).$

Now, we calculate some difference values of $k_n(t,s)$ and the corresponding error.

t	{s=0.1}	{s=0.2}	{s=0.7}	{s=0.9}
0	0.818731	0.67032	0.246597	0.165299
0.1	1	0.818731	0.301194	0.201897
0.2	1.221403	1	0.367879	0.246597
0.3	1.491825	1.221403	0.449329	0.301194
0.4	1.822119	1.491825	0.548812	0.367879
0.5	2.225541	1.822119	0.67032	0.449329
0.6	2.718282	2.225541	0.818731	0.548812
0.7	3.320117	2.718282	1	0.67032
0.8	4.0552	3.320117	1.221403	0.818731
0.9	4.953032	4.0552	1.491825	1

Table (1) contain the value of the kernel $K_n(t,s) = e^{2*(t-s)}$
For the values $s = 0.1$, $s = 0.2$, $s = 0.7$, $s = 0.9$
and the corresponding values t s.t $0 \leq t \leq 9$.

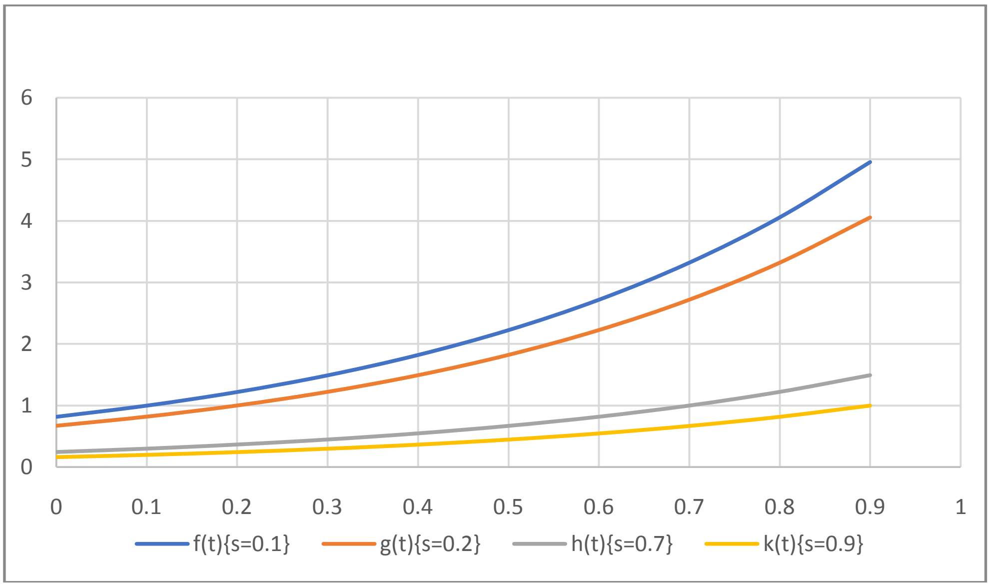

Figure (1)
The relation between t and kernel$K_n(t,s) = e^{2*(t-s)}$ for some values of s

t	{s=0.1}	{s=0.2}	{s=0.7}	{s=0.9}
0	0.860708	0.740818	0.349938	0.25924
0.1	1	0.860708	0.40657	0.301194
0.2	1.161834	1	0.472367	0.349938
0.3	1.349859	1.161834	0.548812	0.40657
0.4	1.568312	1.349859	0.637628	0.472367

0.5	1.822119	1.568312	0.740818	0.548812
0.6	2.117	1.822119	0.860708	0.637628
0.7	2.459603	2.117	1	0.740818
0.8	2.857651	2.459603	1.161834	0.860708
0.9	3.320117	2.857651	1.349859	1

Table (2) contain the value of the resolving $\Gamma_n\left(t, s; \frac{1}{2}\right) = e^{\frac{3}{2}(t-s)}$

For the values $s = 0.1, \quad s = 0.2, \quad s = 0.7, \quad s = 0.9$

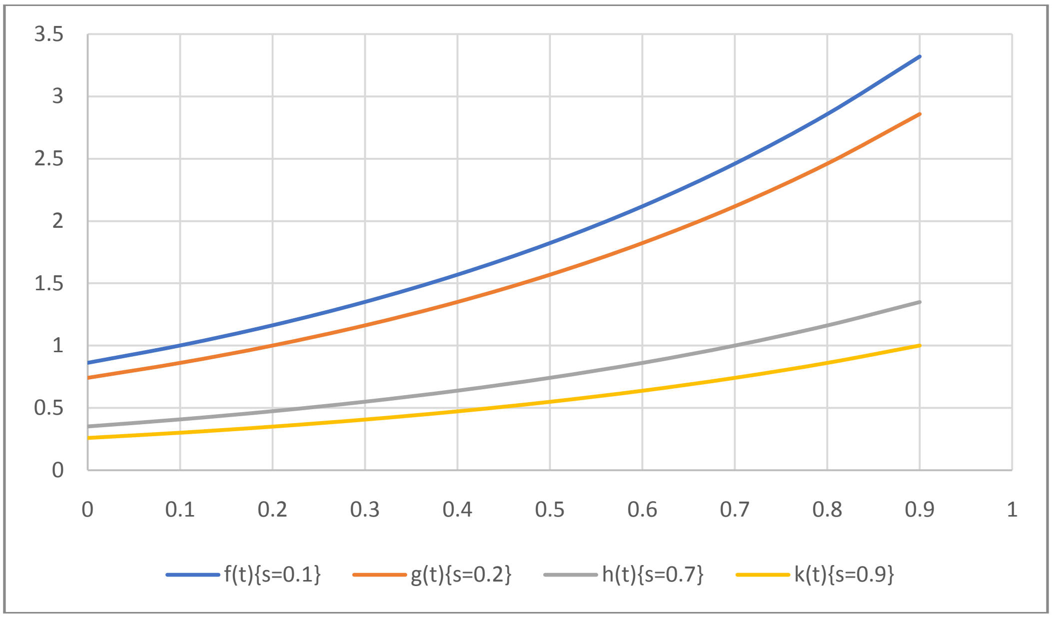

Figure (1)

The relation between t and resolving $\Gamma_n\left(t, s; \frac{1}{2}\right) = e^{\frac{3}{2}(t-s)}$ for some values of s

or the convolution kernel $k_n(t,s) = \frac{(t-s)^{n-1}}{(n-1)\,!} e^{t-s}$

(1) If n=1 then k_n (t, s) > 0 and the value increased for all points of interval t such that t>s and k_n (t, s) =1 when t = s

(2) If n>1, n= 3, 5, 7… the value of k_n(t, s) > 0 decreased for t<s,

and increased when t>s.

(3) If n > 1, n = 2, 4, 6… the value of k_n (t, s) < 0 increased when t>s,

Some difference Figures for the relation between the exact resolvent kernel Γ(t, s; λ) and the numerical resolvent Γ_n(t, s; λ) at $s = 0.2, \lambda = \frac{1}{2}$

or the resolvent kernel $\Gamma(t,s;\lambda) = e^{\frac{3}{2}(t-s)}$ we get $\Gamma(t,s;\lambda) > 0$ for all values of t, and for

$$\Gamma_n(t,s;\lambda) = \sum_{i=1}^{n} \lambda^{i-1} k_i(t,s), \lambda = \frac{1}{2}$$

and

$$k_i(t,s) = \frac{(t-s)^{i-1}}{(i-1)!} e^{t-s}$$

we get

$\Gamma_n(t,s;\lambda) > \Gamma(t,s;\lambda) > 0$ when t < s,

$\Gamma_n(t,s;\lambda) = \Gamma(t,s;\lambda) = 1$ when t = s,

and

$0 < \Gamma_n(t,s;\lambda) < \Gamma(t,s;\lambda)$ when t>s.

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