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A Modified Observer Method for the Joint Estimation of States and Parameters for the Class of Linear Uncertain Discrete-Time Systems

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ABSTRACT
Proposed is the simultaneous state and parameter estimations for a class of stable linear uncertain discrete-time systems where at least one state is measured. The model uncertainties are represented as additive matrices to the state and input matrices in the state space system representation. The proposed methodology is a recursive process whereby the elements of the uncertain state matrices are estimated using the states predicted by a Luenberger observer. These new estimates are used to update the Luenberger observer which then forecasts the subsequent states. Adaptations to the uncertain state matrices are then made using the newly estimated states. The utility of this formulation is that the model uncertainties are confined to the structure of the state matrices thus identifying the location(s) within the state matrices requiring parameter adaptation. Based on the location(s) and size of these adaptations, real-time health monitoring and health degradation isolation for a system can be realized thereby enabling prognostics, remaining useful life estimation, and forecasting. To benchmark the effectiveness of the proposed methodology, a comparison of the state and parameter estimations produced using the proposed method, a Kalman filter and the augmented extended Kalman filter is provided.
Keywords - state observer, least squares estimation, model uncertainties

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I. INTRODUCTION

Simultaneous state and parameter estimations are pivotal concepts applicable to a broad range of disciplines requiring forecasting, monitoring, and control. Historically, state estimation has relied on Luenberger observers [1, 2, 3, 4] and Kalman filters [5, 6, 7, 8]. These methods are traditionally based on time-invariant models with known parameters. In the case when the physical system parameters do not match the observer model, as is usually the case, there is the potential for a large bias in the estimation of the unmeasured states. As accurate state predictions are necessary for forecasting, monitoring and control, the need for adaptive observer parameter estimation arises.

Historically the need for simultaneous state and parameter estimation is addressed with a Kalman filter (KF) augmented with additional states [9, 10]. Specifically, the additional states, representing the uncertain model parameters, allows for the asymptotic convergence of the unknown parameter estimates. Collectively the new state vector enables the simultaneous state and parameter estimation.

The augmented Kalman filter has evolved to an extended version for nonlinear systems where the augmented extended Kalman filter (AEKF) has proven effective [11, 12]. Its successful application has broadly appeared in finance [13], biology and medicine [14], transportation [12], robotics and other areas of engineering [15, 16, 17]. In this EKF formulation, the uncertain system parameters are incorporated as additional states and are therefore estimated simultaneously alongside with the inherent states of the system. In [11] AEKF was successfully employed to estimate the state of charge of Lithium-Ion batteries and in [18] AEKF was used for the joint estimation of the dynamic states and parameters of a moving vehicle.

Alternative methods in the literature jointly employ the Kalman filter for estimating system states and the Least Squares Estimation algorithm

(LSE) for estimating model parameters. In these approaches, the Kalman filter updates the state estimates sequentially, while a batch Least Squares step is periodically performed to estimate the model parameters. The integration of both methods enhances the ability of the Kalman filter to handle systems with time-varying parameters. However, the computational complicity increases, and the probability of numerical instabilities arise due the batch processing nature of LSE. In [19] KF-LSE has been used for the structural health monitoring and the early detection of structural damage via the on-line identification of the structural parameters. In [20], the KF-LSE effectiveness in estimating the system states and parameters was demonstrated on an observer in canonical state space form. In [21] a generalized extended recursive least squares estimation algorithm (RLSE) has been used along with KF for the joint parameters and states estimation for observer state space systems with colored noise.

Another approach that has been employed for uncertain system state and parameters estimation is based on observer theory and incorporates adaptive control techniques, known as an adaptive observer. The methodology is designed to not only estimate unmeasured states but also adapt its internal parameters to handle uncertainties and variations in system parameters. Although the method requires a fine-tuning of various parameters, its real-time adaptation makes it well-suited for systems with time-varying parameters and operating conditions. In [22] an adaptive observer approach to analyze a Rössler hyper-chaotic system featuring two unstable poles has been studied. The method aimed to estimate both model coefficients and states by assessing the error between measured and estimated states. In [23] an adaptive observer has been employed to identify the parameters of a chemical reaction. In that investigation the unknown parameters were assumed to be constant, thus not able to address time-varying model parameters. In [24], an iterative learning observer was designed by augmenting the Luenberger observer with an additional term, aiming to simultaneously identify states and time-varying faults. Additional observer-based system identification techniques have been implemented for model-based fault detection and diagnosis [25, 26]. An augmented Luenberger observer is provided in [27] that estimates both internal and external uncertainty of the system. This estimation cancels the effect of uncertainty on the state feedback control design thereby offering improved robustness and disturbance rejection.

In [28] an observer-based method was employed in the problem of parameter estimation within a parametric uncertain system where the states and output matrices are expressed as the sum of a nominal and a perturbation matrix. The methodology performs on the cases where all the states within the system are measurable, and its effectiveness was demonstrated on an example of non-stimulated system and without output uncertainties. Therefore, the developed method does not address the class of systems with unmeasurable states. Inspired by the later work, the present paper presents a solution for the simultaneous estimation of states and parameters in the problem of parametric uncertain system where the states and input matrices are expressed as the sum of a nominal and a perturbation matrix and partial state measurement. The proposed method employs a sliding window Recursive Least Squares estimation algorithm [29] with covariance reset (RLSE-CR) to capture the parameter variability and a Luenberger observer to handle internal states estimation. The algorithm runs in parallel with the system and uses the available input/output data to generate an optimal estimate of the unmeasured states and system parameters. A stability analysis of the observer is carried out ensuring the convergence of the state estimation in the presence of model uncertainty. The present study addresses the case where the model structure is known, and the nominal model parameter are known as well. The goal is to estimate both the unmeasured states of the system and the uncertainties in the model parameters that are incorporated in the uncertain state space matrices.

The paper is organized as follows. Presented in section II is the problem statement. Developed in section III are the basic design concepts of the Luenberger state observer. A simulation example is presented in section IV. Illustrated in the same section a comparison between the proposed method results and those of KF-RLSE-CR and AEKF methods. A brief conclusion is presented in section V.

II. PROBLEM STATEMENT

Consider a stable observable linear discrete-time model with time varying model uncertainties defined as

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k) \qquad (1)$$

$$y(k) = C\,x(k)$$

where $\boldsymbol{x}(k) \in \mathcal{R}^n$ is the system state, $\boldsymbol{u}(k) \in \mathcal{R}^m$ is the system input and $\boldsymbol{y}(k) \in \mathcal{R}^p$ is the measured system output. The system matrices are defined as $\boldsymbol{A}(k) = \boldsymbol{A_0} + \boldsymbol{\Delta A}(k) \in \mathcal{R}^{n\times n}$ and $\boldsymbol{B}(k) = \boldsymbol{B_0} + \boldsymbol{\Delta B}(k) \in \mathcal{R}^{n\times m}$ where $\boldsymbol{A_0}$ and $\boldsymbol{B_0}$ are the nominal system parameters, and $\boldsymbol{\Delta A}(k)$ and $\boldsymbol{\Delta B}(k)$ are the unknown time-varying parameters. The system output matrix, $\boldsymbol{C} \in \mathcal{R}^{p\times n}$, defines the individual state(s) that are measured. The objective is to estimate the uncertain matrices $\boldsymbol{\Delta A}(k)$ and $\boldsymbol{\Delta B}(k)$ and the states $\boldsymbol{x}(k)$ of the system using the given input $\boldsymbol{u}(k)$ and the available state measurement(s) denoted as $\boldsymbol{y}(k)$.

III. SIMULTANEOUS STATE AND PARAMETER ESTIMATION METHODS

Detailed in this section are the proposed state/parameter estimation methodology. The class of uncertain parametric systems considered is defined in (1). Shown in **Figure 1** is the parallel passive execution of the observer**.**

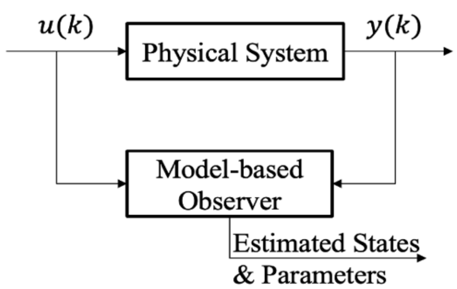

Figure 1. Block Diagram of the Physical System and the Observer.

1. Observer-RLSE-CR Estimator

Developed is an observer-based method for the joint estimation of the internal states and parameters for the class of systems defined in (1). The proposed method integrates a Luenberger observer with a covariance-reset RLSE algorithm to estimate the unknown system parameter matrices $\boldsymbol{\Delta A}$ and $\boldsymbol{\Delta B}$. Here, covariance reset is employed to identify when the parameter estimate(s) have converged.

Thus, following a Luenberger observer formulation, the state estimation is realized as

$$\begin{aligned}\hat{x}(k+1) &= \left(A_0 + \widehat{\Delta A}(k)\right)\hat{x}(k) \\ &\quad + L\left(y(k) - C_0\,\hat{x}(k)\right) \\ &\quad + \left(B_0 + \widehat{\Delta \mathrm{B}}(k)\right)u(k)\end{aligned} \qquad (2)$$

where $\boldsymbol{L}$ represents the observer gain matrix.

Stability of the Observer States

Ensuring observer asymptotic stability is necessary for state estimate(s) convergence. Let the estimation uncertainty be bounded as

$$\left\|\boldsymbol{\Delta A} - \widehat{\boldsymbol{\Delta A}}\right\|_F < \delta_1$$

$$\left\|\boldsymbol{\Delta \mathrm{B}} - \widehat{\boldsymbol{\Delta \mathrm{B}}}\right\|_F < \delta_2 \qquad (3)$$

where $\|.\|_F$ represents the Frobenius norm defined for a matrix $\boldsymbol{M} \in \mathcal{R}^{n\times n}$ as

$$\|\boldsymbol{M}\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left|M_{ij}\right|^2} \qquad (4)$$

Given the boundedness of the estimated uncertainties in (3), the asymptotic stability of the observer is guaranteed provided the eigenvalues lie within the unit disc. Thus, the observer gain $\boldsymbol{L}$ can be designed using the Ackermann method [30] such that the observer poles are within the unit disc $\forall\, \boldsymbol{\Delta A}(k)$, that is

$$|\lambda_i(\boldsymbol{A_0} + \boldsymbol{\Delta A})| < 1$$

$$and\ |\lambda_i(\boldsymbol{A_0} + \boldsymbol{\Delta A} - \boldsymbol{L}\,\boldsymbol{C_0})| < 1\ for \qquad (5)$$

$$i = 1, \dots, n$$

Parametric Uncertainties Estimation

The estimation of the matrices $\boldsymbol{\Delta A}$ and $\boldsymbol{\Delta B}$ can be performed using the recursive least squares estimation [31]. From equation (1),

$$\hat{x}_i(k+1) = \widehat{\boldsymbol{A}}_{\boldsymbol{i}}(k)\,\hat{\boldsymbol{x}}(k) + \widehat{\boldsymbol{B}}_{\boldsymbol{i}}(k)\boldsymbol{u}(k) \quad (6)$$

where the subscript "i" corresponds to the i^{th} row in a matrix or a vector. Thus,

$$\hat{x}_i(k+1) = \left(\boldsymbol{A}_{\boldsymbol{0}_i} + \widehat{\boldsymbol{\Delta A}}_{\boldsymbol{i}}(k)\right)\hat{\boldsymbol{x}}(k) + \left(\boldsymbol{B}_{\boldsymbol{0}_i} + \widehat{\boldsymbol{\Delta B}}_{\boldsymbol{i}}(k)\right)\boldsymbol{u}(k) \quad (7)$$

Isolating the system uncertainty gives

$$\begin{aligned} \hat{x}_i(k+1) - \boldsymbol{A}_{\boldsymbol{0}_i}\,\hat{x}(k) - \boldsymbol{B}_{\boldsymbol{0}_i}(k)\,\boldsymbol{u}(k) &\quad (8) \\ = \widehat{\boldsymbol{\Delta A}}_{\boldsymbol{i}}(k)\,\hat{x}(k) + \widehat{\boldsymbol{\Delta B}}_{\boldsymbol{i}}(k)\,\boldsymbol{u}(k) & \\ = \boldsymbol{\phi}^{\boldsymbol{T}}(k)\,\widehat{\boldsymbol{\theta}}_{\mathbf{i}}(k) & \end{aligned}$$

where $\boldsymbol{\phi}^{\boldsymbol{T}}$ denotes

$$\boldsymbol{\phi}^{\boldsymbol{T}}(k) = [\hat{\boldsymbol{x}}^{\boldsymbol{T}}(k) \quad \boldsymbol{u}^{\boldsymbol{T}}(k)] \quad (9)$$

and $\widehat{\boldsymbol{\theta}}_{\mathbf{i}}$ denotes the parameters vector having the i^{th} rows of the uncertainty matrices $\widehat{\boldsymbol{\Delta A}}(k)$ and $\widehat{\boldsymbol{\Delta B}}(k)$ as elements, namely

$$\widehat{\boldsymbol{\theta}}_{\mathbf{i}}^{\mathbf{T}}(k) = [\widehat{\boldsymbol{\Delta A}}_{\boldsymbol{i}}(k) \quad \widehat{\boldsymbol{\Delta B}}_{i}(k)] \quad (10)$$

Identifying the exact location(s) within the state matrices requiring parameter adaptation is crucial to reduce the size of $\widehat{\boldsymbol{\theta}}_{\mathbf{i}}$ by setting the appropriate elements in $\boldsymbol{\Delta A}$ and $\boldsymbol{\Delta B}$ to zero.

Let the lefthand side of (8) be defined as

$$\begin{aligned} z(k+1) = \hat{\boldsymbol{x}}_{\boldsymbol{i}}(k+1) - \boldsymbol{A}_{\boldsymbol{0}_i}\,\hat{x}(k) & \\ - \boldsymbol{B}_{\boldsymbol{0}_i}(k)\,\boldsymbol{u}(k) &\quad (11) \end{aligned}$$

The RLSE-CR algorithm is initialized as

$$\widehat{\boldsymbol{\theta}}_{\boldsymbol{i}}(0) = \boldsymbol{0} \quad (12)$$

$$\boldsymbol{P}(0) = \delta^{-1}\boldsymbol{I} \quad (13)$$

where δ is a small, positive constant. The estimation procedure executing RLSE is

$$\boldsymbol{K}_k = \frac{\boldsymbol{P}_{k-1}\,\boldsymbol{\phi}_k}{\lambda + \boldsymbol{\phi}_k^T\,\boldsymbol{P}_{k-1}\,\boldsymbol{\phi}_k} \quad (14)$$

$$\widehat{\boldsymbol{\theta}}_{\boldsymbol{i}}(k) = \widehat{\boldsymbol{\theta}}_{\boldsymbol{i}}(k-1) + \boldsymbol{K}_k\left[z(k) - \boldsymbol{\phi}(k) * \widehat{\boldsymbol{\theta}}_{\boldsymbol{i}}(k-1)\right] \quad (15)$$

$$\boldsymbol{P}_k = \frac{\boldsymbol{P}_{k-1} - \boldsymbol{K}_k\,\boldsymbol{\phi}_k^T\,\boldsymbol{P}_{k-1}}{\lambda} \quad (16)$$

where λ is the forgetting factor, typically chosen to be slightly less than or equal to one. Owing to the use of a sliding window in the RLSE-CR process, fine-tuning of λ is not as critical as it is in the standard RLSE algorithm. The covariance matrix of the RLSE algorithm is reset as in (13) with the first sample of every p^{th} frame. The window size p is often chosen through an iterative manual selection. Alternatively, a direct approach can be based on the locations of system poles and principles from linear system theory. The system uncertainties can then be calculated using the estimated state and input matrices $\widehat{\boldsymbol{\Delta A}}$ and $\widehat{\boldsymbol{\Delta B}}$.

Observer Based RLSE-CR Estimation

The objective is to concurrently calculate estimates of the uncertain state matrices elements and the states for the system described in (1) via the available input/output data. The estimation flowchart shown in **Figure 2** is described with two steps: (1) Use the state estimates via the observer at the k^{th} step for uncertain parameters estimation, and (2) Use the updated parameters estimation for the $(k+1)$ state estimations.

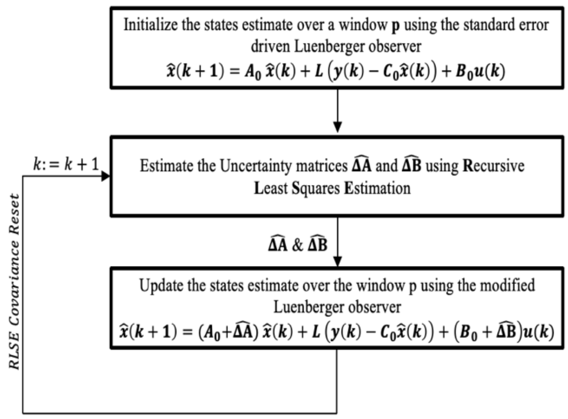

Figure 2. Flowchart of Computing the Parameter and State Estimate using Observer-RLSE-CR Method.

A window of p data samples is initialized using the standard error driven Luenberger observer equation. The RLSE runs within the window and estimates the $\widehat{\boldsymbol{\Delta A}}$ and $\widehat{\boldsymbol{\Delta B}}$. The estimated uncertainties are then used to re-estimate the states employing the modified Luenberger observer equation. As a new data measurement becomes available, the oldest data point is dropped followed by the RLSE-CR to update the parameter estimates within that new window of data. This process iteratively improves the state and parameters estimates.

IV. APPLICATION EXAMPLE

To demonstrate the proposed estimation methodology, a simulation study on a double mass-spring-damper system (**Figure *3***) is provided. The input/output data were generated via simulation. Model uncertainty in the system parameters stiffness/damping and on the input signal were introduced in the simulations. The goal is to accurately estimate the uncertain system parameters and to verify the efficacy of the method by observing how the estimated system parameters adapt throughout the simulation.

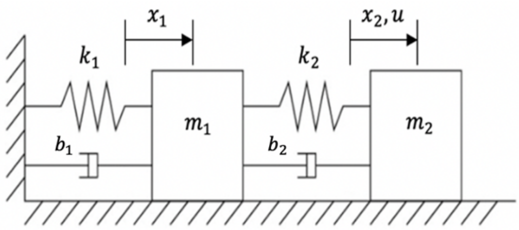

Figure 3. Double-Mass-Spring-Damper System.

Consider the system in **Figure *3***, the equation of motion governing the movements of the two-degree-of-freedom double-mass-spring-damper can be derived using Newton's second law

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + b_2(\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_2(x_1 - x_2) = 0 \tag{17}$$

$$m_2\ddot{x}_2 + b_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = u \tag{18}$$

where x_1 and x_2 are the displacements of the masses m_1 and m_2 respectively from their equilibrium positions, u is the input force applied to the mass m_2, k_1, k_2 are the springs stiffnesses and b_1, b_2 are the damping values.

The continuous form of the state space representation for the system described above can be expressed as

$$\dot{\boldsymbol{x}} = \boldsymbol{A_c}\,\boldsymbol{x} + \boldsymbol{B_c}\,\boldsymbol{u} \tag{19}$$

$$\boldsymbol{y} = \boldsymbol{C_c}\,\boldsymbol{x}$$

where $x = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \end{bmatrix}$ represents the state vector, comprised of the displacements and the velocities of the two masses, $y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ represents the output vector comprised of the measured displacements.

The state matrix $\boldsymbol{A_c}$, the input matrix $\boldsymbol{B_c}$ and the output matrix $\boldsymbol{C_c}$ are obtained as follows

$$\boldsymbol{A_c} = \begin{bmatrix} -\frac{b_1+b_2}{m_1} & \frac{b_2}{m_1} & -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{b_2}{m_2} & -\frac{b_2}{m_2} & \frac{k_2}{m_2} & -\frac{k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{20}$$

$$\boldsymbol{B_c} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

$$\boldsymbol{C_c} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

Consider the discretization of the continuous-time double-mass-spring-damper system using Euler method. For a sampling time T_s, the Euler discretization scheme of the system is

$$\boldsymbol{x}(k+1) = \boldsymbol{A_d}\,\boldsymbol{x}(k) + \boldsymbol{B_d}\,\boldsymbol{u}(\boldsymbol{k}) \tag{23}$$

$$\boldsymbol{y}(k) = \boldsymbol{C_d}\,\boldsymbol{x}(k)$$

where $\boldsymbol{A_d}$, $\boldsymbol{B_d}$ and $\boldsymbol{C_d}$ are the discrete counterparts of the continuous-time matrices $\boldsymbol{A_c}$, $\boldsymbol{B_c}$ and $\boldsymbol{C_c}$, that can be calculated using the equations $\boldsymbol{A_d} = e^{\boldsymbol{A_c}\,T_s} = \boldsymbol{I} + \boldsymbol{A_c}\,T_s$, $\boldsymbol{B_d} = \int_0^{T_s} e^{\boldsymbol{A_c}\,\lambda}\boldsymbol{B_c}\,d\lambda$ and $\boldsymbol{C_d} = \boldsymbol{C}$, Thus

$$\boldsymbol{A_d} = \begin{bmatrix} 1-\frac{b_1+b_2}{m_1}T_s & \frac{b_2}{m_1}T_s & -\frac{k_1+k_2}{m_1}T_s & \frac{k_2}{m_1}T_s \\ \frac{b_2}{m_2}T_s & 1-\frac{b_2}{m_2}T_s & \frac{k_2}{m_2}T_s & -\frac{k_2}{m_2}T_s \\ T_s & 0 & 1 & 0 \\ 0 & T_s & 0 & 1 \end{bmatrix} \tag{24}$$

$$\boldsymbol{B_d} = \begin{bmatrix} 0 \\ T_s \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

and

$$\boldsymbol{C_d} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

For the representation of the physical system, time-varying uncertainties on the springs stiffnesses Δk_1 and Δk_2 and on the damping Δb_1 and Δb_2 are introduced, as these parameters may be unknown or change over time due to structural damage or parameter aging. Thus, the resulting system model can be written as function of the uncertain parameters as

$$\begin{aligned} \boldsymbol{x}(k+1) = (\boldsymbol{A_d} + \boldsymbol{\Delta A(k)})\, \boldsymbol{x}(k) \\ + (\boldsymbol{B_d} + \boldsymbol{\Delta B(k)})\, \boldsymbol{u}(k) \end{aligned} \tag{27}$$

$$\boldsymbol{y}(k) = \boldsymbol{C_d}\, \boldsymbol{x}(k)$$

where $\boldsymbol{\Delta A}$ and $\boldsymbol{\Delta B}$ are

$$\boldsymbol{\Delta A} = \begin{bmatrix} -\frac{\Delta b_1 + \Delta b_2}{m_1} T_s & \frac{\Delta b_2}{m_1} T_s & -\frac{\Delta k_1 + \Delta k_2}{m_1} T_s & \frac{\Delta k_2}{m_1} T_s \\ \frac{\Delta b_2}{m_2} T_s & -\frac{\Delta b_2}{m_2} T_s & \frac{\Delta k_2}{m_2} T_s & -\frac{\Delta k_2}{m_2} T_s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

$$\boldsymbol{\Delta B} = \begin{bmatrix} 0 \\ B_s \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

The objective is to estimate the unmeasured velocities and the elements of the uncertain state matrices $\boldsymbol{\Delta A}$ and $\boldsymbol{\Delta B}$. To that end, the proposed observer-based method has been employed for validation and assessment purposes. For the simulation, the mass-spring-damper system parameters nominal values are set to $m_1 = 5\,kg, m_2 = 3\,kg, b_1 = 5\,Nm^{-1}, b_2 = 5\,Nm^{-1}$, $k_1 = 25\,Nm^{-1}, k_2 = 100\,Nm^{-1}$. The sampling time T_s is related to the sampling frequency by $f_s = \frac{1}{T_s} = 0.1\,kHz$. Step variations on Δk_1 and Δk_2, were introduced on this simulation. An input uncertainty B_s was incorporated as well. Since there are no uncertainties in the damping coefficients, the ΔA matrix is written as

$$\boldsymbol{\Delta A} = \begin{bmatrix} 0 & 0 & \delta a_{13} & \delta a_{14} \\ 0 & 0 & \delta a_{23} & \delta a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

For the implementation of the proposed Observer-RLSE-CR approach, the initial step is the design of the observer gain matrix through pole placement method. The observer poles positions are selected to the left of the continuous-time nominal system poles in the s-plane. This placement ensures that the observer tracks the plant dynamics at the desired rate yet places the observer bandwidth such that sensor noise is attenuated. The continuous-time nominal plant poles are $p_{1,2} = -1.5401 \pm 7.2664\,i$ and $p_{3,4} = -0.2933 \pm 1.7131\,i$. Based on those values, the observer poles are placed at $p_1 = -45; p_2 = -40; p_3 = -35\ and\ p_4 = -30.$ Thus, using Ackerman method, the calculated observer gain matrix is

$$K_e = \begin{bmatrix} 1.24028 & -0.192081 \\ -0.192081 & 1.24028 \\ 0.0731667 & -0.00744765 \\ -0.00744765 & 0.0731667 \end{bmatrix} \tag{31}$$

For an assessment purposes, the Observer-RLSE-CR results were compared with those of an augmented extended Kalman filter (AEKF) and a KF-RLSE-CR method. The later one has the same structure as the proposed Observer-RLSE-CR approach except that the Kalman filter fulfills the role of the observer for states estimation. A detailed description of the mathematical formulations of the KF and AEKF algorithms can be found in [32] and [11] respectively. The KF state vector X_1, the AEKF state vector X_2 and its observation matrices F_1 and F_2, respectively, can be written as

$$X_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{32}$$

$$X_2 = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \\ \delta a_{13} \\ \delta a_{14} \\ \delta a_{23} \\ \delta a_{24} \\ B_s \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \delta a_{13} \\ \delta a_{14} \\ \delta a_{23} \\ \delta a_{24} \\ B_s \end{bmatrix} \tag{33}$$

As shown in equation (33), the AEKF state vector is augmented by the elements of the matrices $\boldsymbol{\Delta A}$ and $\boldsymbol{\Delta B}$.

Presented in **Figure *4*** is the input force u acting on the mass m_2 and in **Figure 5** the resulting displacements y_1 and y_2 of the masses m_1 and m_2 respectively.

$$F_1 = \begin{bmatrix} 1-\frac{b_1+b_2}{m_1}T_s & \frac{b_2}{m_1}T_s & -\frac{k_1+k_2}{m_1}T_s+\delta a_{13} & \frac{k_2}{m_1}T_s+\delta a_{14} \\ \frac{b_2}{m_2}T_s & 1-\frac{b_2}{m_2}T_s & \frac{k_2}{m_2}T_s+\delta a_{23} & -\frac{k_2}{m_2}T_s+\delta a_{24} \\ T_s & 0 & 1 & 0 \\ 0 & T_s & 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1-\frac{b_1+b_2}{m_1}T_s & \frac{b_2}{m_1}T_s & -\frac{k_1+k_2}{m_1}T_s+\delta a_{13} & \frac{k_2}{m_1}T_s+\delta a_{14} & x_3 & x_4 & 0 & 0 & 0 \\ \frac{b_2}{m_2}T_s & 1-\frac{b_2}{m_2}T_s & \frac{k_2}{m_2}T_s+\delta a_{23} & -\frac{k_2}{m_2}T_s+\delta a_{24} & 0 & 0 & x_3 & x_4 & u \\ T_s & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_s & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

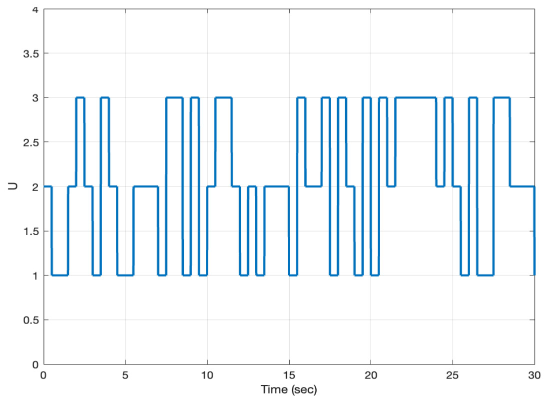

Figure 4. Input Force.

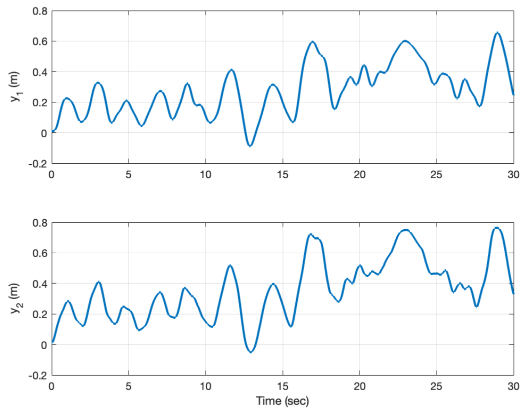

Figure 5. Measured Displacements.

Consider the time-varying uncertainties $\Delta k_1, \Delta k_2$ and B_s listed in the **Table *1***.

	First 15 sec	Last 15 sec
Δk_1	25	10
Δk_2	50	25
B_s	$\frac{T_s}{2}$	T_s

Table 1. Time-Varying Uncertainties Imposed to the System.

Considering the locations of the system's poles, the settling time associated with the slowest pole is $t_{settle} = 17.04\ seconds$. Thus, the window size associated with the estimation process can be calculated as $p = \frac{t_{settle}}{h} = 1705\ samples$.

Applying the Observer-RLSE-CR method described in Section 3, the superposition of the actual system states and the estimated states are presented in **Figure *6*** and **Figure *7***.

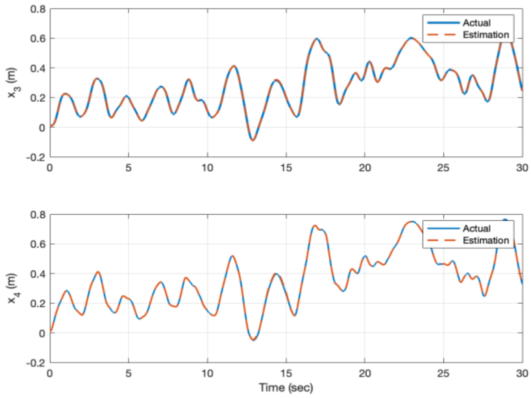

Figure 6. Superposition of the Actual and Estimated Displacements.

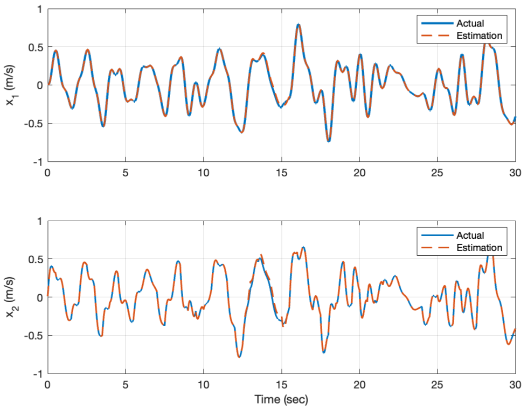

Figure 7. Superposition of the Actual and Estimated Velocities.

Provided in **Figure *8*** and **Figure 9** are the comparison of the Observer-RLSE-CR, KF-RLSE-CR and AEKF results in estimating the elements of the uncertain state matrices $\boldsymbol{\Delta A}$ and $\boldsymbol{\Delta B}$. Presented in **Table 2**, **Table 3** and **Table 4** are the estimation error percentage. As shown in the plots all methods efficiently estimated the uncertainties δa_{13}, δa_{14}, δa_{23} and δa_{24} with variable estimation errors. As listed in the **Table 2**, **Table 3** and **Table 4**, the Observer-RLSE-CR approach outperform the AEKF and the KF-RLSE-CR in the estimation of the uncertainty B_s that was deliberately introduced in the input.

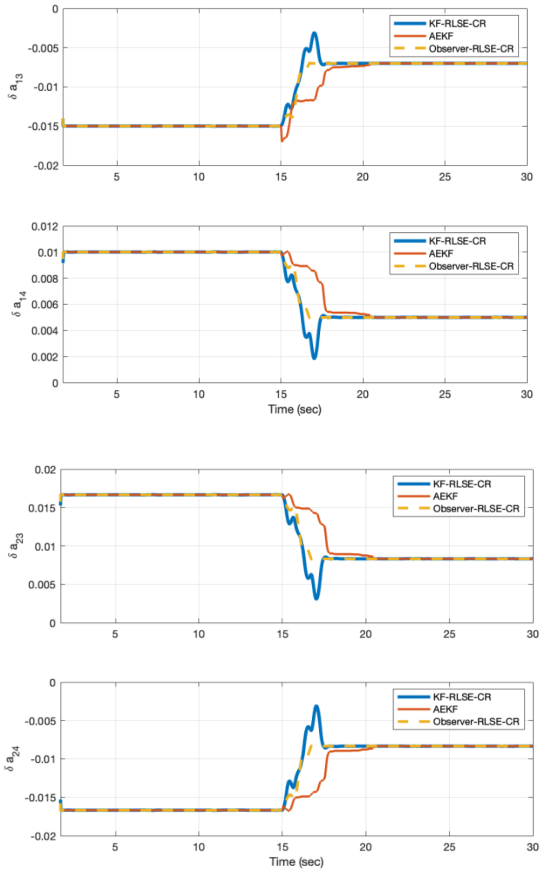

Figure 8. Estimated Uncertainties of the $\boldsymbol{\Delta A}$ State Matrix.

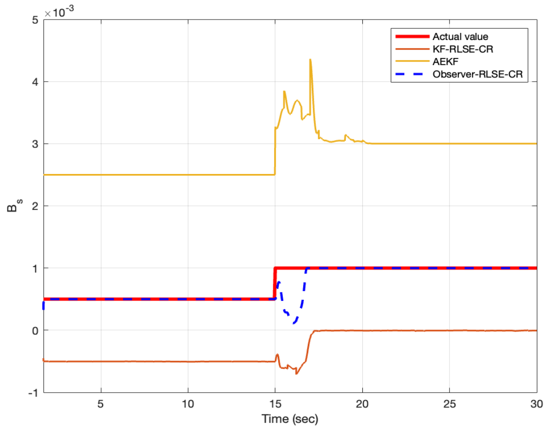

Figure 9. Estimated Uncertain Input.

		δa_{13}	δa_{14}	δa_{23}	δa_{24}	B_s
1st Combination	Actual Value	−0.015	0.01	0.0167	−0.0167	0.0005
	Estimated Value	−0.015	0.01	0.0167	−0.0167	0.0005
	Percent Error	0 %	0 %	0 %	0 %	0 %
2nd Combination	Actual Value	−0.007	0.005	0.0083	−0.0083	0.001
	Estimated Value	−0.007	0.005	0.0083	−0.0083	0.001
	Percent Error	0 %	0 %	0 %	0 %	0 %

Table 2. Estimation Error of the Observer-RLSE-CR Method.

		δa_{13}	δa_{14}	δa_{23}	δa_{24}	B_s
1st Combination	Actual Value	−0.015	0.01	0.0167	−0.0167	0.0005
	Estimated Value	−0.015	0.01	0.0166	−0.0166	0.0025
	Percent Error	0 %	0 %	0.6 %	0 .6%	−
2nd Combination	Actual Value	−0.007	0.005	0.0083	−0.0083	0.001
	Estimated Value	−0.0071	0.005	0.00836	−0.00836	0.003
	Percent Error	1.4 %	0 %	0 .72%	0.72 %	−

Table 3. Estimation Error of the AEKF Method.

		δa_{13}	δa_{14}	δa_{23}	δa_{24}	B_s
1st Combination	Actual Value	−0.015	0.01	0.0167	−0.0167	0.0005
	Estimated Value	−0.0149	0.0099	0.0166	−0.0166	−0.0005
	Percent Error	0.7 %	1 %	0.6 %	0 .6%	−
2nd Combination	Actual Value	−0.007	0.005	0.0083	−0.0083	0.001
	Estimated Value	−0.0069	0.0049	0.00828	−0.00828	$-1.455\ 10^{-5}$
	Percent Error	1.42%	2%	0 .24%	0.24%	−

Table 4. Estimation Error of the KF-RLSE-CR Method.

To assess the effectiveness of the Observer-RLSE-CR approach in the presence of noisy data, a zero mean white noise (**Figure *10***) is introduced into the sensor measurements. The results of the Observer-RLSE-CR approach are presented in **Figure *11*** and **Figure 12**. Despite the challenges associated with the sensor noise, the proposed approach effectively mitigates the impact of noise on the estimation process and provides accurate and reliable uncertainties estimations.

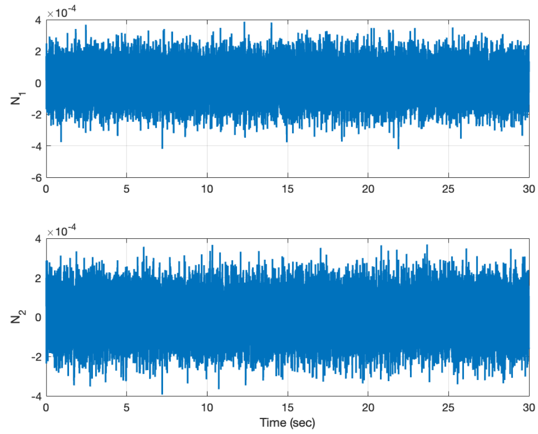

Figure 10. Measurement Noise.

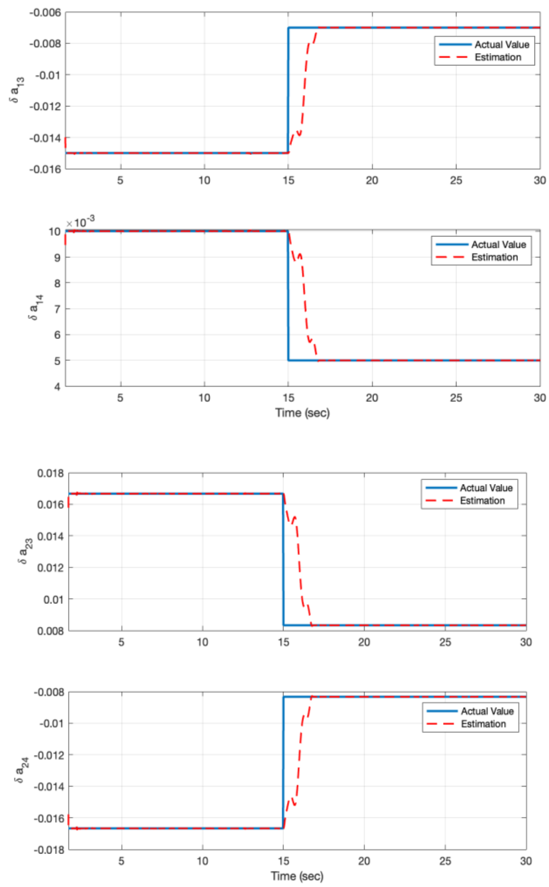

Figure 11. Estimated Elements of the ***ΔA*** State Matrix using Observer-RLSE-CR Method.

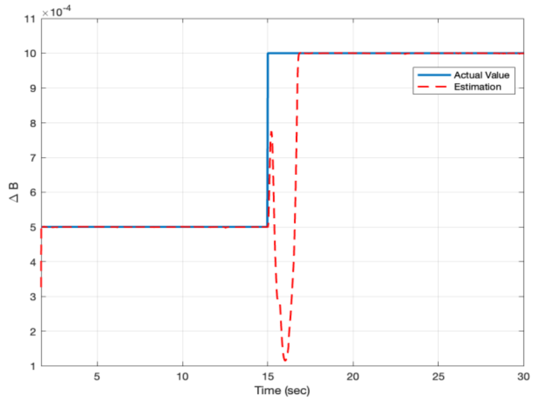

Figure 12. Estimated Uncertain Input using Observer-RLSE-CR Method.

The simulations demonstrate that the observer-RLSE-CR method constructs a robust and effective approach for simultaneous states and uncertainty estimation, even in the presence of sensor noise. The results indicate that the three methods accurately estimate model uncertainties, with the observer-RLSE-CR outperforming the Kalman-based methods in the estimation of the input uncertainties. Unlike Kalman filters, that requires manual tuning of the process and measurement covariance matrices, the observer-based method involves tuning of the observer gain L that is employed to stabilize the error dynamics and optimize the observer performance.

V. CONCLUSION

This paper introduces an approach addressing the challenge of concurrent estimation of states and parameters for an ensemble of linear uncertain discrete state space systems. Through the integration of a modified Luenberger observer with sliding window recursive least squares estimation, a robust framework is established capable of tracking the system's evolving states while simultaneously identifying its time-varying parameters. Through simulations, the developed method demonstrates improved performance compared to Kalman-based estimators like AEKF and KF-RLSE-CR. By incorporating sensor measurement noise into the data, the developed approach mitigates the impact of noise and provides reliable state and parameter estimation.

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