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The Weight Enumerator for a Class of Linear Codes

Qi Zhong[1], Yuchao Zhang[1], Xiangli Kong[1,*]
School of Mathematical Sciences, Qufu Normal University, Shandong 273165, P R China

ABSTRACT: Recently, linear codes constructed from defining sets have been studied widely since they have many applications in cryptography and communication systems. In this paper, we consider a defining set
$D=\{(x_1,x_2)\in(\mathbb{F}_q^*)^2:\mathrm{Tr}(x_1+x_2)=1)\}$,
where $q=p^m$ for a positive integer m and an odd prime p, and Tr is the absolute trace function from $\mathbb{F}_q$ onto $\mathbb{F}_p$. Define a class of p-ary linear codes by
$C_D=\{\mathbf{c}(a_1,a_2):(a_1,a_2)\in\mathbb{F}_q^2\}$,
where
$\mathbf{c}(a_1,a_2)=(\mathrm{Tr}(a_1x_1^2+a_2x_2^2))_{(x_1,x_2)\in D}$.
We compute the weight enumerators of the punctured codes C_D.

Keywords: Linear codes; Gaussian periods; Gauss sums; Cyclotomic numbers; Weight distribution.

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I. INTRODUCTION

Throughout this paper, let $q=p^m$ for an odd prime p and a positive integer m. Denote by $\mathbb{F}_q$ a finite field with q elements. An $[n,k,d]$ linear code C over $\mathbb{F}_p$ is a k-dimensional subspace of $\mathbb{F}_p^n$ with minimum distance d . Let A_i denote the number of codewords in C with Hamming weight i . Then the weight enumerator of C is defined by $1+A_1z+A_2z^2+\cdots+A_nz^n$. The sequences $(1,A_1,A_2,\cdots,A_n)$ is called the weight distribution of C. For more information in coding theory, we refer the reader to [20].

In modern communication society, linear codes have found many applications in cryptography, error correction, data storage systems and network coding due to their efficient encoding and decoding algorithms. However, there are still many unsolved problems in coding theory, such as, the determination of the weights and forms of codewords. They have been an interesting topic of study for a long time. The weight distributions of linear codes have been studied by evaluating the corresponding exponential sums over finite fields, see [2, 3, 7, 11, 16, 17, 21, 29, 30, 32-35]. The authors in [4, 5] dealt with codes constructed from finite geometries. Recently there are also many papers considered traces codes over rings and they also constructed many interesting codes over finite fields, which are distance optimal and minimal [22-28]. Note that the motivation of such research is that the weight distribution of a code allows the computation of the error probability of error detection and correction with respect to some algorithms.

In the work [6, 8, 9], the authors introduced a generic construction of linear codes. Set $D=\{d_1d_2,\cdots,d_n\}\in\mathbb{F}_q$, where $q=p^m$. Denote by Tr the absolute trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. A linear code of length $n=\#D$ is defined by
$\mathcal{C}_D=\{(\mathrm{Tr}(bd_1),\mathrm{Tr}(bd_2),\cdots,\mathrm{Tr}(bd_n)):b\in\mathbb{F}_q\}$
.

The set D is called the defining set of $\mathcal{C}_D$. This construction technique is general in the sense that many classes of known codes could be produced by properly selecting the defining set $D\subseteq\mathbb{F}_q$. Many classes of linear codes over finite fields were constructed using this method, see [1, 2, 12-15, 19, 31, 34, 36-38]. Particularly, the authors in [13, 31, 38] constructed some linear codes and presented their complete weight enumerators, by choosing $D_c'=\{x\in\mathbb{F}_q^*:\mathrm{Tr}(ax^{p^k+1})=c\}$, where $\mathbb{F}_q^*=\mathbb{F}_q\backslash\{0\}$, $a\in\mathbb{F}_q^*$, $c\in\mathbb{F}_p$. Ahn and Ka [1] investigated the weight enumerators of a class of p-ary linear codes C_{D_0} defined by

$$C_{D_0}=\{\mathbf{c}(a_1,a_2):(a_1,a_2)\in\mathbb{F}_q^2\}, \quad (1)$$

where

$$\mathbf{c}(a_1,a_2)=(\mathrm{Tr}(a_1x_1^2+a_2x_2^2))_{(x_1,x_2)\in D_0},$$

and the defining set is chosen to be

$$D_0 = \{(x_1, x_2) \in (\mathbb{F}_q^*)^2 : \mathrm{Tr}(x_1 + x_2) = 0\}$$
.

In this paper, strongly inspired by [1] and the above construction, we define a set $D \subset \mathbb{F}_q^2$ such that

$$D = \left\{(x_1, x_2) \in (\mathbb{F}_q^*)^2 : \mathrm{Tr}(x_1 + x_2) = 1\right\} \qquad (2)$$
.

We contribute on the weight enumerator of the linear codes C_D of (1). The main results are shown in Theorems 1 and 2 in Section 3. For $b \in \mathbb{F}_p^*$, define

$$D_b = \left\{(x_1, x_2) \in (\mathbb{F}_q^*)^2 : \mathrm{Tr}(x_1 + x_2) = b\right\}$$
.

One checks that $D = D_1$ and the codes C_D are actually the codes C_{D_b} for $b \neq 0$. Thus the weight enumerators of C_{D_b} are also presented. Moreover, compared with the codes C_{D_0} in [1], our codes C_D have better parameters if we take the same p and m (see Remark 1). Besides, the codes presented in this paper are minimal in the sense of Yuan and Ding [39]. So they are suitable to be applied in secret sharing schemes.

The remainder of this paper is organized as follows. In Section 2, we briefly recall some definitions and results on exponential sums. In Section 3, we present the main results and additionally we give some examples. Finally, in Section 4, we make a conclusion.

II. PRELIMINARIES

We begin with the concept of additive characters and Gauss sums over finite fields. Let p be an odd prime and $q = p^m$ for a positive integer m. Let $\mathrm{Tr} : \mathbb{F}_q \longrightarrow \mathbb{F}_p$ be the absolute trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$. Then the function ψ_a defined by

$$\psi_a(x) = \zeta_p^{\mathrm{Tr}(ax)} \text{ for all } a \in \mathbb{F}_q$$

is an additive character of $\mathbb{F}_q$, where $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ is a p-th primitive root of unity. It is clear that $\psi_0(x) = 1$ for all $x \in \mathbb{F}_q$ and it is called the trivial additive character. The character ψ_1 is called the canonical additive character of $\mathbb{F}_q$. All additive characters ψ_a of $\mathbb{F}_q$ can be expressed in terms of ψ_1, i.e., $\psi_a(x) = \psi_1(ax)$for all $x \in \mathbb{F}_q$. The orthogonal property of additive characters is given by

$$\sum_{x \in \mathbb{F}_q} \psi_a(x) = \begin{cases} q & \text{if } a = 0, \\ 0 & \text{if } a \in \mathbb{F}_q^*. \end{cases}$$

Suppose that η is the quadratic character of $\mathbb{F}_q^*$ and η_p is the quadratic character of $\mathbb{F}_p^*$. For all $z \in \mathbb{F}_p^*$, it is easily checked that

$$\eta(z) = \begin{cases} 1 & \text{if } m \text{ is even}, \\ \eta_p(z) & \text{if } m \text{ is odd}. \end{cases}$$

Now we define the quadratic Gauss sums over $\mathbb{F}_q$,

$$G(\eta) = \sum_{x \in \mathbb{F}_q^*} \eta(x)\psi_1(x).$$

According to Theorem 5.15 in [18], we have

$$G(\eta) = (-1)^{m-1}\sqrt{(p^*)^m},$$

and $G(\eta_p) = \sqrt{p^*}$, where $p^* = \eta_p(-1)p = (-1)^{\frac{p-1}{2}}p$. Let $q - 1 = sN$ for two positive integers $s > 1$ and $N > 1$. The Gaussian periods of order N are defined by

$$\eta_i^{(N,q)} = \sum_{x \in C_i^{(N,q)}} \psi_1(x),$$

for $i = 0, 1, \cdots, N-1$, where $C_i^{(N,q)}$ is the i-th cyclotomic class of order N in $\mathbb{F}_q$.

The following lemmas will be needed when we calculate the weight distributions of our codes in the next section.

Lemma 1 (Propositions 1 and 19, [10]). *When $N = 2$, the Gaussian periods are given by*

$$\eta_0^{(2,q)} = \begin{cases} \frac{-1+(-1)^{m-1}\sqrt{q}}{2} & \text{if } p \equiv 1 \pmod 4, \\ \frac{-1+(-1)^{m-1}(\sqrt{-1})^m\sqrt{q}}{2} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and $\eta_1^{(2,q)} = -1 - \eta_0^{(2,q)}$.

Lemma 2 (Theorem 5.33, [18]). *If q is odd and $f(x)=a_2x^2+a_1x+a_0\in\mathbb{F}_q[x]$ with $a_2\neq 0$, then*

$$\sum_{x\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}(f(x))}=\zeta_p^{\mathrm{Tr}(a_0-a_1^2(4a_2)^{-1})}\eta(a_2)G(\eta).$$

III. WEIGHT ENUMERATORS OF THE LINEAR CODES *CD*

In this section, we present our main results of the weight distribution of the linear codes C_D defined by (1), where the defining set D is given by

$$D=\left\{(x_1,x_2)\in(\mathbb{F}_q^*)^2:\mathrm{Tr}(x_1+x_2)=1\right\}.$$

We should firstly compute the length of C_D. Denote

$$n=\#\{(x_1,x_2)\in(\mathbb{F}_q^*)^2:\mathrm{Tr}(x_1+x_2)=1\}.$$

It is easily obtained that the length of the linear codes C_D is $n=\frac{(q-1)^2-1}{p}$.

To get the weight distribution of C_D, we need to compute the weight of every codeword and count the frequency of each weight occurring in all codewords. For a codeword $\mathbf{c}(a_1,a_2)$ of C_D, let $N_0:=N(a_1,a_2)$ be the number of components $\mathrm{Tr}(a_1x_1^2+a_2x_2^2)$ of $\mathbf{c}(a_1,a_2)$ which are equal to 0. Then the weight of $\mathbf{c}(a_1,a_2)$ is given by $wt(\mathbf{c}(a_1,a_2))=n-N_0$. By definition,

$$N_0=\sum_{x_1,x_2\in\mathbb{F}_q^*}\left(\frac{1}{p}\sum_{y\in\mathbb{F}_p}\zeta_p^{y\mathrm{Tr}(x_1+x_2)-y}\right)\left(\frac{1}{p}\sum_{z\in\mathbb{F}_p}\zeta_p^{z\mathrm{Tr}(a_1x_1^2+a_2x_2^2)}\right)$$

$$=\frac{(q-1)^2}{p^2}+\frac{1}{p^2}(\Omega_1+\Omega_2+\Omega_3), \qquad (3)$$

where

$$\Omega_1=\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\sum_{x_1\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}(yx_1)}\sum_{x_2\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}(yx_2)}=-1,$$

$$\Omega_2=\sum_{z\in\mathbb{F}_p^*}\sum_{x_1\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}(za_1x_1^2)}\sum_{x_2\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}(za_2x_2^2)},$$

$$\Omega_3=\sum_{y,z\in\mathbb{F}_p^*}\zeta_p^{-y}\sum_{x_1\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}(za_1x_1^2+yx_1)}\sum_{x_2\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}(za_2x_2^2+yx_2)}.$$

Now let us determine the values of Ω_2 and Ω_3 in the next two lemmas. For simplicity, we denote $G=G(\eta)G(\eta_p)$ and $G_i=G(\eta)\eta(a_i)$ for $i\in\{1,2\}$.

Lemma 3 (Lemma 5, [1]). *The values of Ω_2 are given as follows.*

(1) *If m is even, then*

$$\Omega_2=\begin{cases}(q-1)^2(p-1) & \textit{if } a_1=0,a_2=0,\\ (q-1)(p-1)(G_1-1) & \textit{if } a_1\neq 0,a_2=0,\\ (q-1)(p-1)(G_2-1) & \textit{if } a_1=0,a_2\neq 0,\\ (p-1)(G_1-1)(G_2-1) & \textit{if } a_1\neq 0,a_2\neq 0.\end{cases}$$

(2) *If m is odd, then*

$$\Omega_2=\begin{cases}(q-1)^2(p-1) & \textit{if } a_1=0,\ a_2=0,\\ -(q-1)(p-1) & \textit{if } a_1\neq 0,\ a_2=0,\\ -(q-1)(p-1) & \textit{if } a_1=0,\ a_2\neq 0,\\ (p-1)(G_1G_2+1) & \textit{if } a_1\neq 0,\ a_2\neq 0.\end{cases}$$

Lemma 4. *Denote $\lambda_i=\mathrm{Tr}(a_i^{-1})$ if $a\neq 0$. The values of Ω_3 are given as follows.*

(1) *When m is even, we have*

(1.1) *if* $a_1 = 0$ *and* $a_2 = 0$, *then* $\Omega_3 = -(p-1)$;

(1.2) *if* $a_1 \neq 0$ *and* $a_2 = 0$, *then*

$$\Omega_3 = \begin{cases} (p-1)(G_1 - 1) & \text{if } \lambda_1 = 0, \\ -G_1 - (p-1) & \text{if } \lambda_1 \neq 0; \end{cases}$$

(1.3) *if* $a_1 = 0$ *and* $a_2 \neq 0$, *then*

$$\Omega_3 = \begin{cases} (p-1)(G_2 - 1) & \text{if } \lambda_2 = 0, \\ -G_2 - (p-1) & \text{if } \lambda_2 \neq 0; \end{cases}$$

(1.4) *if* $a_1 \neq 0$ *and* $a_2 \neq 0$, *then*

$$\Omega_3 = \begin{cases} -(p-1)(G_1-1)(G_2-1) & \text{if } \lambda_1 = \lambda_2 = 0, \\ (G_2-1)(G_1+p-1) & \text{if } \lambda_1 \neq 0, \lambda_2 = 0, \\ (G_1-1)(G_2+p-1) & \text{if } \lambda_1 = 0, \lambda_2 \neq 0, \\ -(p-1)(G_1G_2+1) - G_1 - G_2 & \text{if } \lambda_1\lambda_2 \neq 0, \lambda_1 + \lambda_2 = 0, \\ G_1G_2 - G_1 - G_2 - (p-1) & \text{if } \lambda_1\lambda_2 \neq 0, \lambda_1 + \lambda_2 \neq 0. \end{cases}$$

(2) *When* m *is odd, by denoting* $A_i = \eta(a_i)\eta_p(-\lambda_i)$ *for* $a_i \neq 0$, *we have*

(2.1) *if* $a_1 = 0$ *and* $a_2 = 0$, *then* $\Omega_3 = -(p-1)$;

(2.2) *if* $a_1 \neq 0$ *and* $a_2 = 0$, *then*

$$\Omega_3 = \begin{cases} -p-1 & \text{if } \lambda_1 = 0, \\ GA_1 - (p-1) & \text{if } \lambda_1 \neq 0; \end{cases}$$

(2.3) *if* $a_1 = 0$ *and* $a_2 \neq 0$, *then*

$$\Omega_3 = \begin{cases} -(p-1) & \text{if } \lambda_2 = 0, \\ GA_2 - (p-1) & \text{if } \lambda_2 \neq 0; \end{cases}$$

(2.4) *if* $a_1 \neq 0$ *and* $a_2 \neq 0$, *then*

$$\Omega_3 = \begin{cases} -(p-1)(G_1G_2+1) & \text{if } \lambda_1 = \lambda_2 = 0, \\ G_1G_2 + GA_1 - (p-1) & \text{if } \lambda_1 \neq 0, \lambda_2 = 0, \\ G_1G_2 + GA_2 - (p-1) & \text{if } \lambda_1 = 0, \lambda_2 \neq 0, \\ G_1A_1 + G_2A_2 - (p-1)(G_1G_2+1) & \text{if } \lambda_1\lambda_2 \neq 0, \lambda_1 + \lambda_2 = 0, \\ G_1G_2 + G_1A_1 + G_2A_2 - (p-1) & \text{if } \lambda_1\lambda_2 \neq 0, \lambda_1 + \lambda_2 \neq 0; \end{cases}$$

Proof. There are four cases to consider separately: $\begin{cases} (1)\ a_1 = 0, a_2 = 0\ , \\ (2)\ a_1 \neq 0, a_2 = 0\ , \\ (3)\ a_1 = 0, a_2 \neq 0\ , \\ (4)\ a_1 = 0, a_2 \neq 0\ . \end{cases}$

Case 1 If $a_1 = 0$ and $a_2 = 0$, then

$$\begin{aligned} \Omega_3 &= \sum_{y,z\in\mathbb{F}_q^*} \zeta_p^{-y} \sum_{x_1\in\mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}(yx_1)} \sum_{x_2\in\mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}(yx_2)} \\ &= \sum_{z\in\mathbb{F}_q^*} \sum_{y\in\mathbb{F}_q^*} \zeta_p^{-y} = -(p-1). \end{aligned}$$

Case 2 If $a_1 \neq 0$ and $a_2 = 0$, then

$$\begin{aligned}\Omega_3 &= \sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\Big(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}(za_1x_1^2+yx_1)}-1\Big)\Big(\sum_{x_2\in\mathbb{F}_q^*}\zeta_p^{\mathrm{Tr}(yx_2)}\Big)\\ &= -\sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\Big(\zeta_p^{-\frac{y^2}{4z}\lambda_1}\eta(za_1)G(\eta)-1\Big)\\ &= -G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-\frac{\lambda_1}{4z}y^2-y}-(p-1)\\ &= \begin{cases}-G_1\sum\limits_{z\in\mathbb{F}_p^*}\eta(z)\sum\limits_{y\in\mathbb{F}_p^*}\zeta_p^{-y}-(p-1) & \text{if } \lambda_1=0,\\ -G_1\sum\limits_{z\in\mathbb{F}_p^*}\eta(z)\zeta_p^{\frac{z}{\lambda_1}}\eta_p(-\frac{\lambda_1}{4z})G(\eta_p)+G_1\sum\limits_{z\in\mathbb{F}_p^*}\eta(z)-(p-1) & \text{if } \lambda_1\neq 0,\end{cases}\end{aligned}$$

where we denote $\lambda_1=\mathrm{Tr}(a_1^{-1})$.

Suppose that m is even, then we have $\eta(z)=1$ for $z\in\mathbb{F}_p^*$. So

$$\begin{aligned}\Omega_3 &= \begin{cases}-G_1\sum\limits_{z\in\mathbb{F}_p^*}\sum\limits_{y\in\mathbb{F}_p^*}\zeta_p^{-y}-(p-1) & \text{if } \lambda_1=0\\ -G_1\sum\limits_{z\in\mathbb{F}_p^*}\zeta_P^{\frac{z}{\lambda_1}}\eta_p(-\frac{\lambda_1}{4z})G(\eta_p)+G_1\sum\limits_{z\in\mathbb{F}_p^*}1-(p-1) & \text{if } \lambda_1\neq 0\end{cases}\\ &= \begin{cases}(p-1)(G_1-1) & \text{if } \lambda_1=0,\\ -G_1-(p-1) & \text{if } \lambda_1\neq 0.\end{cases}\end{aligned}$$

Suppose that m is odd, then we have $\eta(z)=\eta_p(z)$ for $z\in\mathbb{F}_p^*$. Hence

$$\begin{aligned}\Omega_3 &= \begin{cases}-G_1\sum\limits_{z\in\mathbb{F}_p^*}\eta_p(z)\sum\limits_{y\in\mathbb{F}_p^*}\zeta_p^{-y}-(p-1) & \text{if } \lambda_1=0,\\ -G_1\sum\limits_{z\in\mathbb{F}_p^*}\zeta_p^{\frac{z}{\lambda_1}}\eta_p(-\lambda_1)G(\eta_p)-(p-1) & \text{if } \lambda_1\neq 0.\end{cases}\\ &= \begin{cases}-(p-1) & \text{if } \lambda_1=0\\ G_1\eta_p(-\lambda_1)G(\eta_p)-(p-1) & \text{if } \lambda_1\neq 0\end{cases}\\ &= \begin{cases}-(p-1) & \text{if } \lambda_1=0,\\ GA_1-(p-1) & \text{if } \lambda_1\neq 0,\end{cases}\end{aligned}$$

where $G=G(\eta)G(\eta_p)$ and $A_1=\eta(a_1)\eta_p(-\lambda_1)$.

Case 3 If $a_1=0$ and $a_2\neq 0$, then the results are obtained similarly.

Case 4 If $a_1\neq 0$ and $a_2\neq 0$, then

$$\begin{aligned}\Omega_3 &= \sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y}\Big(\sum_{x_1\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}(za_1x_1^2+yx_1)}-1\Big)\Big(\sum_{x_2\in\mathbb{F}_q}\zeta_p^{\mathrm{Tr}(za_2x_2^2+yx_2)}-1\Big)\\ &= \sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_P^*}\zeta_p^{-y}(\zeta_p^{-\frac{y^2}{4z}\lambda_1}\eta(za_1)G(\eta)-1)(\zeta_p^{-\frac{y^2}{4z}\lambda_2}\eta(za_2)G(\eta)-1)\\ &= G_1G_2\sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-\frac{y^2}{4z}(\lambda_1+\lambda_2)-y}-G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-\frac{y^2}{4z}\lambda_1-y}\\ &\quad -G_2\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-\frac{y^2}{4z}\lambda_2-y}-(p-1),\end{aligned}$$

where we denote $\lambda_1 = \mathrm{Tr}(a_1^{-1})$ and $\lambda_2 = \mathrm{Tr}(a_2^{-1})$.
Let us divide the rest of the proof into five subcases according to the values of λ_1 and λ_2.
Subcase 4.a If $\lambda_1 = \lambda_2 = 0$, then

$$\Omega_3 = G_1G_2 \sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*} \zeta_p^{-y} - (G_1+G_2)\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y} - (p-1)$$

$$= \begin{cases} -(p-1)(G_1-1)(G_2-1) & \text{if } m \text{ is even,} \\ -(p-1)(G_1G_2+1) & \text{if } m \text{ is odd.} \end{cases}$$

Subcase 4.b If $\lambda_1 \neq 0$ and $\lambda = 0$, then

$$\Omega_3 = G_1G_2 \sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*} \zeta_p^{-\frac{y^2}{4z}\lambda_1 - y} - G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-\frac{y^2}{4z}\lambda_1 - y}$$
$$-G_2\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-y} - (p-1).$$

When m is even, then $\eta(z) = 1$ for all $z \in \mathbb{F}_p^*$. So

$$\Omega_3 = G_1(G_2-1)\sum_{z\in\mathbb{F}_p^*}(\zeta_p^{\frac{z}{\lambda_1}}\eta_p(-\frac{\lambda_1}{4z})G(\eta_p) - 1) + G_2(p-1) - (p-1)$$
$$= G_1(G_2-1) + (p-1)(G_2-1)$$
$$= (G_2-1)(G_1+p-1).$$

When m is odd, then $\eta(z) = \eta_p(z)$ for all $z \in \mathbb{F}_p^*$. So

$$\Omega_3 = G_1G_2\sum_{z\in\mathbb{F}_p^*}(\zeta_p^{\frac{z}{\lambda_1}}\eta_p(-\frac{\lambda_1}{4z})G(\eta_p) - 1)$$
$$-G_1\sum_{z\in\mathbb{F}_p^*}\eta_p(z)(\zeta_p^{\frac{z}{\lambda_1}}\eta_p(-\frac{\lambda_1}{4z})G(\eta_p) - 1) + G_2\sum_{z\in\mathbb{F}_p^*}\zeta_p(z) - (p-1)$$
$$= G_1G_2 + GA_1 - (p-1)$$

where $A_1 = \eta(a_1)\eta_p(-\lambda_1)$.
Subcase 4.c If $\lambda_1 = 0$, $\lambda_2 \neq 0$, we perform a similar calculation and obtain

$$\Omega_3 = \begin{cases} (G_1-1)(G_2+p-1) & \text{if } m \text{ is even,} \\ G_1G_2 + GA_2 - (p-1) & \text{if } m \text{ is odd,} \end{cases}$$

where $A_2 = \eta(a_2)\eta_p(-\lambda_2)$.
Subcase 4.d If $\lambda_1\lambda_2 \neq 0$ and moreover $\lambda_1 + \lambda_2 = 0$, then

$$\Omega_3 = G_1G_2 \sum_{z\in\mathbb{F}_p^*}\sum_{y\in\mathbb{F}_p^*} \zeta_p^{-y} - G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-\frac{y^2}{4z}\lambda_1 - y}$$
$$-G_2\sum_{z\in\mathbb{F}_p^*}\eta(z)\sum_{y\in\mathbb{F}_p^*}\zeta_p^{-\frac{y^2}{4z}\lambda_2 - y} - (p-1).$$

When m is even,

$$\Omega_3 = -(p-1)G_1G_2 - G_1\sum_{z\in\mathbb{F}_p^*}(\zeta_p^{\frac{z}{\lambda_1}}\eta_p(-\frac{\lambda_1}{4z})G(\eta_p) - 1)$$
$$-G_2\sum_{z\in\mathbb{F}_p^*}(\zeta_p^{\frac{z}{\lambda_2}}\eta_p(-\frac{\lambda_2}{4z})G(\eta_p) - 1) - (p-1)$$
$$= -(p-1)G_1G_2 - G_1 - G_2 - (p-1).$$

When m is odd,

$$\Omega_3 = -(p-1)G_1G_2 - G_1\sum_{z\in\mathbb{F}_p^*}\eta_p(z)(\zeta_p^{\frac{z}{\lambda_1}}\eta_p(-\frac{\lambda_1}{4z})G(\eta_p) - 1)$$

$$-G_2\sum_{z\in\mathbb{F}_p^*}\eta_p(z)(\zeta_p^{\frac{z}{\lambda_2}}\eta_p(-\frac{\lambda_2}{4z})G(\eta_p)-1)-(p-1)$$

$$=-(p-1)G_1G_2+G_1A_1+G_2A_2-(p-1).$$

Subcase 4.e If $\lambda_1\lambda_2\neq 0$ and moreover $\lambda_1+\lambda_2\neq 0$, then

$$\Omega_3=G_1G_2\sum_{z\in\mathbb{F}_p^*}(\zeta_p^{\frac{z}{\lambda_1+\lambda_2}}\eta_p(-\frac{\lambda_1+\lambda_2}{4z})G(\eta_p)-1)$$

$$-G_1\sum_{z\in\mathbb{F}_p^*}\eta(z)(\zeta_p^{\frac{z}{\lambda_1}}\eta_p(-\frac{\lambda_1}{4z})G(\eta_p)-1)$$

$$-G_2\sum_{z\in\mathbb{F}_p^*}\eta(z)(\zeta_p^{\frac{z}{\lambda_2}}\eta_p(-\frac{\lambda_2}{4z})G(\eta_p)-1)-(p-1).$$

This leads to the conclusion that

$$\Omega_3=\begin{cases}G_1G_2-G_1-G_2-(p-1) & \text{if } m \text{ is even,}\\ G_1G_2+G_1A_1+G_2A_2-(p-1) & \text{if } m \text{ is odd.}\end{cases}$$

The whole proof of this lemma is completed.

Lemma 5 (Lemma 3.4[2]). *For any* $c\in\mathbb{F}_p$, *define* $m_c=\#\{a\in\mathbb{F}_p^*:\mathrm{Tr}(a^{-1})=c\}$.
Then we have

$$m_c=\begin{cases}p^{m-1}-1 & \text{if } c=0,\\ p^{m-1} & \text{if } c\neq 0.\end{cases}$$

Lemma 6 (Lemma 3.5, [2]). *For any* $c\in\mathbb{F}_p$, *let*
$n_{i,c}=\#\{a\in\mathbb{F}_p^*:\eta(a)=i,\mathrm{Tr}(a^{-1})=c\}$ for $i\in\{-1,1\}$.

(1) *If* m *is even, then*

$$n_{i,c}=\begin{cases}\frac{1}{2p}(q-p+i(p-1)G(\eta)) & \text{if } c=0,\\ \frac{1}{2p}(q-iG(\eta)) & \text{if } c\neq 0.\end{cases}$$

(2) *If* m *is odd, then*

$$n_{i,c}=\begin{cases}\frac{1}{2p}(q-p) & \text{if } c=0,\\ \frac{1}{2p}(q+i\eta_p(-c)G) & \text{if } c\neq 0.\end{cases}$$

Lemma 7 (Lemma 3.7, [2]). *Suppose that* m *is odd, let*

$$n'_{i,j}=\#\{a\in\mathbb{F}_q^*:\eta(a)=i,\eta_p(\text{-}\mathrm{Tr}(a^{-1}))=j\}\text{ for } i,j\in\{-1,1\}.$$

Then we have

$$n'_{i,j}=\frac{p-1}{4p}(q+ijG).$$

Theorem 1. *Let* C_D *be a linear code defined by (1) where*

$$D=\{(x_1,x_2)\in(\mathbb{F}_q^*)^2:\mathrm{Tr}(x_1+x_2)=1\}.$$

Suppose that m is even.
(1) If $m=2$ *, then the weight distribution of* C_D *is given by Table 1 and the code* C_D *has parameters* $[p^3-2p,4,d]$, *where*

$$d=\begin{cases}9 & \text{if } p=3,\\ (p-1)(p^2-p-1) & \text{if } p>3.\end{cases}$$

(2) If $m\geqslant 4$, *then the weight distribution of* C_D *is given by Table 2 and the code* C_D *has parameters* $[p^{2m-1}-2p^{m-1},2m,(p-1)p^{m-2}(p^m-p^{\frac{m}{2}}-1)]$.

Table 1. Weight distribution of C_D for $m=2$

Weight w	Frequency A_w
0	1
$(p-1)(p^2-p-1)$	$2(p-1)$
$(p-1)(p^2-p-1)+1$	$(p-1)^2$
$(p-1)(p^2+p-1)-1$	p^2-1
$(p-1)(p^2-2)$	$\frac{p+3}{2}(p-1)^2$
$(p-1)(p^2-1)$	$(p-1)(p^2-1)$
$(p-1)(p^2-3)$	$(p-1)^3$
$(p-1)(p^2-2)\pm 2$	$\frac{p-1}{4}(p\mp 1)^2$
$(p-1)(p^2-2)-p\pm 2$	$\frac{(p-1)(p-2)}{4}(p\mp 1)^2$
$(p-1)(p^2-2)+p$	$\frac{(p-1)(p-2)}{2}(p^2-1)$

Table 2. Weight distribution of C_D for even $m>4$

Weight w	Frequency A_w
0	1
$(p-1)p^{m-2}(p^m\pm p^{\frac{m}{2}}-1)$	$p^{m-1}-1\mp(p-1)p^{\frac{m}{2}-1}$
$(p-1)p^{m-2}(p^m+p^{\frac{m}{2}}-1)-p^{\frac{m}{2}-1}$	$(p-1)(p^{m-1}+p^{\frac{m}{2}-1})$
$(p-1)p^{m-2}(p^m-p^{\frac{m}{2}}-1)+p^{\frac{m}{2}-1}$	$(p-1)(p^{m-1}-p^{\frac{m}{2}-1})$
$(p-1)p^{m-2}(p^m-2)$	$(p^{m-1}-1)^2+\frac{p-1}{2}p^{m-2}(p^m-1)$
$(p-1)p^{m-2}(p^m-2)\pm p^{m-1}-p^{\frac{m}{2}-1}$	$\frac{p-1}{2}(p^{m-1}+p^{\frac{m}{2}-1})(p^{m-1}-1\pm(p-1)p^{\frac{m}{2}-1})$
$(p-1)p^{m-2}(p^m-2)\pm p^{m-1}+p^{\frac{m}{2}-1}$	$\frac{p-1}{2}(p^{m-1}-p^{\frac{m}{2}-1})(p^{m-1}-1\mp(p-1)p^{\frac{m}{2}-1})$
$(p-1)p^{m-2}(p^m-2)\pm 2p^{\frac{m}{2}-1}$	$\frac{p-1}{4}(p^{m-1}\mp p^{\frac{m}{2}-1})^2$
$(p-1)p^{m-2}(p^m-2)-p^{m-1}\pm 2p^{\frac{m}{2}-1}$	$\frac{p-1}{4}(p-2)(p^{m-1}\mp p^{\frac{m}{2}-1})^2$
$(p-1)p^{m-2}(p^m-2)+p^{m-1}$	$\frac{p-1}{2}(p-2)p^{m-2}(p^m-1)$

Proof. Recall that the weight of $\mathbf{c}(a_1,a_2)$ in C_D is given by

$$wt(c(a_1,a_2))=n-N_0,$$

where $n=\frac{(q-1)^2-1}{p}=p^{2m-1}-2p^{m-1}$ and $N_0=\frac{(q-1)^2-1}{p^2}+\frac{1}{p^2}(\Omega_2+\Omega_3)$. We employ Lemmas 3 and 4 to compute the values of N_0. For convenience, we denote $\lambda_i=\mathrm{Tr}(a_i^{-1})$ if $a_i\neq 0$.

Case 1 Assume $a_1\neq 0$ and $a_2=0$.

If $\lambda_1=0$, then we obtain

$$\begin{aligned}N_0&=\frac{(q-1)^2-1}{p^2}+\frac{1}{p^2}(\Omega_2+\Omega_3)\\&=\frac{(q-1)^2-1}{p^2}+\frac{1}{p^2}\Big((q-1)(p-1)(G_1-1)+(p-1)(G_1-1)\Big)\\&=\frac{(q-1)^2-1}{p^2}+\frac{q}{p^2}(p-1)(G(\eta)\eta(a_1)-1)\\&=\begin{cases}p^{m-2}(p^{m-2})+p^{m-2}(p-1)(G(\eta)-1) & \text{if } \eta(a_1)=1,\\ p^{m-2}(p^{m-2})-p^{m-2}(p-1)(G(\eta)+1) & \text{if } \eta(a_1)=-1.\end{cases}\end{aligned}$$

Now the frequencies are

$$n_{1,0} = \frac{1}{2}(p^{m-1} - 1) + \frac{p-1}{2p}G(\eta),$$
$$n_{-1,0} = \frac{1}{2}(p^{m-1} - 1) - \frac{p-1}{2p}G(\eta),$$

by Lemma 6, respectively.
If $\lambda_1 \neq 0$, then we obtain

$$\begin{aligned} N_0 &= \frac{(q-1)^2 - 1}{p^2} + \frac{1}{p^2}(\Omega_2 + \Omega_3) \\ &= \frac{(q-1)^2 - 1}{p^2} + \frac{1}{p^2}\Big((q-1)(p-1)(G_1 - 1) - G_1 - (p-1)\Big) \\ &= \begin{cases} p^{m-2}(p^m - 2) + p^{-2}\left((p-1)q(G(\eta) - 1) - pG(\eta)\right) & \text{if } \eta(a_1) = 1, \\ p^{m-2}(p^m - 2) - p^{-2}\left((p-1)q(G(\eta) + 1) - pG(\eta)\right) & \text{if } \eta(a_1) = -1. \end{cases} \end{aligned}$$

Again from Lemma 6, their frequencies are

$$\sum_{c \neq 0} n_{1,c} = \frac{p-1}{2p}(q - G(\eta)),$$
$$\sum_{c \neq 0} n_{-1,c} = \frac{p-1}{2p}(q + G(\eta)),$$

respectively.
Case 2 If $a_1 = 0$ and $a_2 \neq 0$, then we also have the same weights and the same frequencies as those in Case 1.
Case 3 Now let us assume that $a_1 \neq 0$ and $a_2 \neq 0$. There are five subcases to consider.
Subcase 3.a If $\lambda_1 = \lambda_2 = 0$, then we obtain

$$\begin{aligned} N_0 &= \frac{(q-1)^2 - 1}{p^2} + \frac{1}{p^2}(\Omega_2 + \Omega_3) \\ &= p^{m-2}(p^m - 2), \end{aligned}$$

since $\Omega_2 = -\Omega_3 = (p-1)(G_1 - 1)(G_2 - 1)$. The frequency is $(p^{m-1} - 1)^2$ by Lemma 5.
Subcase 3.b If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, then we have

$$\begin{aligned} N_0 &= \frac{(q-1)^2 - 1}{p^2} + \frac{1}{p^2}(\Omega_2 + \Omega_3) \\ &= \frac{(q-1)^2 - 1}{p^2} \\ &\quad + \frac{1}{p^2}\Big((p-1)(G_1 - 1)(G_2 - 1) + G_1G_2 - G_1 + (p-1)(G_2 - 1)\Big) \\ &= \frac{(q-1)^2 - 1}{p^2} + \frac{1}{p}G_1(G_2 - 1) \\ &= \begin{cases} p^{m-2}(p^m - 2) + p^{-1}G(\eta)(G(\eta) - 1) & \text{if } \eta(a_1) = \eta(a_2) = 1, \\ p^{m-2}(p^m - 2) - p^{-1}G(\eta)(G(\eta) + 1) & \text{if } \eta(a_1) = 1,\ \eta(a_2) = -1, \\ p^{m-2}(p^m - 2) - p^{-1}G(\eta)(G(\eta) - 1) & \text{if } \eta(a_1) = -1,\ \eta(a_2) = 1, \\ p^{m-2}(p^m - 2) + p^{-1}G(\eta)(G(\eta) + 1) & \text{if } \eta(a_1) = -1,\ \eta(a_2) = -1. \end{cases} \end{aligned}$$

According to Lemma 6, their frequencies are

$$\Big(\sum_{c\neq 0} n_{1,c}\Big)n_{1,0}=\frac{p-1}{4p^2}(q-G(\eta))(q-p+(p-1)G(\eta)),$$
$$\Big(\sum_{c\neq 0} n_{1,c}\Big)n_{-1,0}=\frac{p-1}{4p^2}(q-G(\eta))(q-p-(p-1)G(\eta)),$$
$$\Big(\sum_{c\neq 0} n_{-1,c}\Big)n_{1,0}=\frac{p-1}{4p^2}(q+G(\eta))(q-p+(p-1)G(\eta)),$$
$$\Big(\sum_{c\neq 0} n_{-1,c}\Big)n_{-1,0}=\frac{p-1}{4p^2}(q+G(\eta))(q-p-(p-1)G(\eta)),$$

respectively.

Subcase 3.c If $\lambda_1=0$ and $\lambda_2\neq 0$, then we have the same weights and the same frequencies as those in Subcase 3.b.

Subcase 3.d If $\lambda_1\neq 0, \lambda_2\neq 0$ and $\lambda_1+\lambda_2=0$, then we have

$$\begin{aligned}
N_0&=\frac{(q-1)^2-1}{p^2}+\frac{1}{p^2}(\Omega_2+\Omega_3)\\
&=\frac{(q-1)^2-1}{p^2}\\
&\quad+\frac{1}{p^2}\Big((p-1)(G_1-1)(G_2-1)-(p-1)(G_1G_2+1)-G_1-G_2\Big)\\
&=\frac{(q-1)^2-1}{p^2}-\frac{1}{p}(G_1+G_2)\\
&=\frac{(q-1)^2-1}{p^2}-\frac{1}{p}G(\eta)\Big(\eta(a_1)+\eta(a_2)\Big)\\
&=\begin{cases} p^{m-2}(p^m-2)-2p^{-1}G(\eta) & \text{if } \eta(a_1)=\eta(a_2)=1,\\ p^{m-2}(p^m-2) & \text{if } \eta(a_1a_2)=-1,\\ p^{m-2}(p^m-2)+2p^{-1}G(\eta) & \text{if } \eta(a_1)=\eta(a_2)=-1.\end{cases}
\end{aligned}$$

Their frequencies are given below

$$\sum_{c\neq 0} n_{1,c}^2=\frac{p-1}{4p^2}(q-G(\eta))^2,$$
$$2\sum_{c\neq 0} n_{1,c}n_{-1,0}=\frac{p-1}{2p^2}(q^2-G(\eta)^2),$$
$$\sum_{c\neq 0} n_{-1,c}^2=\frac{p-1}{4p^2}(q+G(\eta))^2,$$

respectively.

Subcase 3.e If $\lambda_1\neq 0, \lambda_2\neq 0$ and moreover $\lambda_1+\lambda_2\neq 0$, then we have

$$\begin{aligned}
N_0&=\frac{(q-1)^2-1}{p^2}+\frac{1}{p^2}\Big((p-1)(G_1-1)(G_2-1)+(G_1-1)(G_2-1)-p\Big)\\
&=\frac{(q-1)^2-1}{p^2}+\frac{1}{p}(G_1G_2-G_1-G_2)\\
&=\frac{(q-1)^2-1}{p^2}+\frac{1}{p}G(\eta)\big(G(\eta)\eta(a_1a_2)-\eta(a_1)-\eta(a_2)\big)
\end{aligned}$$

$$= \begin{cases} p^{m-2}(p^m-2)+p^{-1}G(\eta)(G(\eta)-2) & \text{if } \eta(a_1)=\eta(a_2)=1, \\ p^{m-2}(p^m-2)-p^{-1}G(\eta)^2 & \text{if } \eta(a_1a_2)=-1, \\ p^{m-2}(p^m-2)+p^{-1}G(\eta)(G(\eta)+2) & \text{if } \eta(a_1)=\eta(a_2)=-1. \end{cases}$$

Let us compute their frequencies. From Lemma 6 they are given by

$$\sum_{c\neq 0} n_{1,c} \sum_{\substack{k\neq 0\\ k\neq -c}} n_{1,k} = \frac{(p-1)(p-2)}{4p^2}(q-G((\eta))^2,$$

$$2\sum_{c\neq 0} n_{1,c} \sum_{\substack{k\neq 0\\ k\neq -c}} n_{-1,k} = \frac{(p-1)(p-2)}{2p^2}(q^2-G(\eta)^2),$$

$$\sum_{c\neq 0} n_{-1,c} \sum_{\substack{k\neq 0\\ k\neq -c}} n_{-1,k} = \frac{(p-1)(p-2)}{4p^2}(q+G(\eta))^2,$$

respectively. Thus we get the desired conclusions.

Example 1. (1) Let $(p,m)=(5,2)$. By Theorem 1, the code C_D is a [115,4, 76] linear code over $\mathbb{F}_5$. Its weight enumerator is

$$1+8z^{76}+16z^{77}+108z^{85}+64z^{88}+48z^{89}+36z^{90}$$
$$+64z^{92}+16z^{94}+96z^{96}+144z^{97}+24z^{115},$$

which is checked by Magma programs.

(2) Let $(p,m)=(3,4)$. By Theorem 1, the code C_D is a [2133, 8,1278] linear code over $\mathbb{F}_3$. Its weight enumerator is

$$1+32z^{1278}+48z^{1281}+450z^{1389}+600z^{1392}+768z^{1398}$$
$$+288z^{1401}+450z^{1416}+1396z^{1422}+288z^{1428}+960z^{1446}$$
$$+720z^{1449}+480z^{1452}+60z^{1599}+20z^{1602},$$

which is checked by Magma programs.

Theorem 2. *Let C_D be a linear code defined by* (1) *where*

$$D=\{(x_1,x_2)\in \left(\mathbb{F}_q^*\right)^2 : \mathrm{Tr}(x_1+x_2)=1\}.$$

Suppose that m is odd and $m>1$. Then the weight distribution of C_D is given by Table 3 and the code C_D has parameters

$$[p^{2m-1}-p^{m-1}, 2m, d],$$

where

$$d=\begin{cases} 2\cdot 3^{m-2}(3^m-2)-3^{m-1}-3^{\frac{m-3}{2}} & \text{if } p=3, \\ (p-1)p^{m-2}(p^m-2)-p^{m-1}-2p^{\frac{m-3}{2}} & \text{if } p>3. \end{cases}$$

Table 3. Weight distribution of C_D for odd $m>1$

Weight w	Frequency A_w
0	1
$(p-1)(p^{2m-2}-p^{m-2})$	$2(p^{m-1}-1)$
$(p-1)(p^{2m-2}-p^{m-2})\pm p^{\frac{m-3}{2}}$	$(p-1)(p^{m-1}\mp p^{\frac{m-1}{2}})$
$(p-1)p^{m-2}(p^m-2)$	$(p^{m-1}-1)^2+\frac{p-1}{2}(p^{2m-2}-p^{m-1})$
$(p-1)p^{m-2}(p^m-2)-p^{m-1}\pm p^{\frac{m-3}{2}}$	$\frac{p-1}{2}(p^{m-1}\mp p^{\frac{m-1}{2}})(p^{m-1}-1)$
$(p-1)p^{m-2}(p^m-2)+p^{m-1}\pm p^{\frac{m-3}{2}}$	$\frac{p-1}{2}(p^{m-1}\mp p^{\frac{m-1}{2}})(p^{m-1}-1)$
$(p-1)p^{m-2}(p^m-2)\pm 2p^{\frac{m-3}{2}}$	$\frac{p-1}{4}(p^{m-1}\mp p^{\frac{m-1}{2}})^2$
$(p-1)p^{m-2}(p^m-2)-p^{m-1}\pm 2p^{\frac{m-3}{2}}$	$\frac{p-1}{4}\frac{p-3}{2}(p^{m-1}\mp p^{\frac{m-1}{2}})^2$
$(p-1)p^{m-2}(p^m-2)+p^{m-1}$	$\frac{p-1}{2}\frac{p-3}{2}(p^{2m-2}-p^{m-1})$

$(p-1)p^{m-2}(p^m-2)-p^{m-1}$	$\frac{p-1}{2}\frac{p-1}{2}(p^{2m-2}-p^{m-1})$
$(p-1)p^{m-2}(p^m-2)+p^{m-1}-2p^{\frac{m-3}{2}}$	$\frac{p-1}{4}\frac{p-1}{2}(p^{m-1}+p^{\frac{m-1}{2}})^2$
$(p-1)p^{m-2}(p^m-2)+p^{m-1}+2p^{\frac{m-3}{2}}$	$\frac{p-1}{4}\frac{p-1}{2}(p^{m-1}-p^{\frac{m-1}{2}})^2$

Proof. Recall that $N_0 = \frac{(q-1)^2-1}{p^2} + \frac{1}{p^2}(\Omega_2+\Omega_3)$. We employ Lemmas 3 and 4 to compute the values of N_0. Note that $\lambda_i = \mathrm{Tr}(a_i^{-1})$ if $a_i \neq 0$.

Case 1 Suppose that $a_1 \neq 0$ and $a_2 = 0$.

If $\lambda_1 = 0$, then we obtain

$$\begin{aligned} N_0 &= \frac{(q-1)^2-1}{p^2} + \frac{1}{p^2}(\Omega_2+\Omega_3) \\ &= \frac{(q-1)^2-1}{p^2} + \frac{1}{p^2}\Big(-(q-1)(p-1)-(p-1)\Big) \\ &= p^{m-2}(p^m-2)-(p-1)p^{m-2}. \end{aligned}$$

From Lemma 5, its frequency is $m_0 = p^{m-1}-1$.

If $\lambda_1 \neq 0$, then we obtain

$$\begin{aligned} N_0 &= \frac{(q-1)^2-1}{p^2} + \frac{1}{p^2}(\Omega_2+\Omega_3) \\ &= \frac{(q-1)^2-1}{p^2} + \frac{1}{p^2}\Big(-(q-1)(p-1)+GA_1-(p-1)\Big) \\ &= p^{m-2}(p^m-2)-(p-1)p^{m-2}+\frac{1}{p^2}GA_1 \\ &= \begin{cases} p^{2m-2}-p^{m-1}-p^{m-2}+p^{-2}G & \text{if } A_1 = 1, \\ p^{2m-2}-p^{m-1}-p^{m-2}-p^{-2}G & \text{if } A_1 = -1, \end{cases} \end{aligned}$$

where $A_1 = \eta(a_1)\eta_p(-\lambda_1)$. Their frequencies are computed from Lemma 7

$$\begin{aligned} n'_{1,1}+n'_{-1,-1} &= \frac{p-1}{2p}(q+G), \\ n'_{-1,1}+n'_{1,-1} &= \frac{p-1}{2p}(q-G), \end{aligned}$$

respectively.

Case 2 If $a_1 = 0$ and $a_2 \neq 0$, then we also have the same weights and the same frequencies as those in Case 1.

Case 3 Suppose that $a_1 \neq 0$ and $a_2 \neq 0$.

Subcase 3.a If $\lambda_1 = \lambda_2 = 0$, then we obtain

$$\begin{aligned} N_0 &= \frac{(q-1)^2-1}{p^2} + \frac{1}{p^2}(\Omega_2+\Omega_3) \\ &= p^{m-2}(p^m-2), \end{aligned}$$

Since $\Omega_2 = -\Omega_3 = (p-1)(G_1G_2+1)$. The frequency is $m_0^2 = (p^{m-1}-1)^2$.

Subcase 3.b If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, then we have

$$\begin{aligned} N_0 &= \frac{(q-1)^2}{p^2} + \frac{1}{p^2}(\Omega_2+\Omega_3) \\ &= \frac{(q-1)^2-1}{p^2} + \frac{1}{p^2}\Big((p-1)(G_1G_2+1)+G_1G_2+GA_1-(p-1)\Big) \\ &= p^{m-2}(p^m-2)+p^{-1}G_1G_2+p^{-2}GA_1 \end{aligned}$$

$$= \begin{cases} p^{m-2}(p^m-2)+p^{-1}G(\eta)^2+p^{-2}G & \text{if } \eta(a_1a_2)=1, A_1=1, \\ p^{m-2}(p^m-2)+p^{-1}G(\eta)^2-p^{-2}G & \text{if } \eta(a_1a_2)=1, A_1=-1, \\ p^{m-2}(p^m-2)-p^{-1}G(\eta)^2+p^{-2}G & \text{if } \eta(a_1a_2)=-1, A_1=1, \\ p^{m-2}(p^m-2)-p^{-1}G(\eta)^2-p^{-2}G & \text{if } \eta(a_1a_2)=-1, A_1=-1. \end{cases}$$

According to Lemmas 6 and 7, their frequencies are

$$n'_{1,1}n_{1,0}+n'_{-1,-1}n_{-1,0}=\frac{p-1}{4p}(p^{m-1}-1)(q+G),$$
$$n'_{1,-1}n_{1,0}+n'_{-1,1}n_{-1,0}=\frac{p-1}{4p}(p^{m-1}-1)(q-G),$$
$$n'_{1,1}n_{-1,0}+n'_{-1,-1}n_{1,0}=\frac{p-1}{4p}(p^{m-1}-1)(q+G),$$
$$n'_{1,-1}n_{-1,0}+n'_{-1,1}n_{1,0}=\frac{p-1}{4p}(p^{m-1}-1)(q-G),$$

respectively.
Subcase 3.c If $\lambda_1=0$ and $\lambda_2\neq 0$, then we have the same weights and the same frequencies as those in Subcase 3.b.
Subcase 3.d If $\lambda_1\neq 0, \lambda_2\neq 0$ and $\lambda_1+\lambda_2=0$, then we have

$$\begin{aligned} N_0 &= \frac{(q-1)^2-1}{p^2}+\frac{1}{p^2}(\Omega_2+\Omega_3) \\ &= \frac{(q-1)^2-1}{p^2}+\frac{1}{p^2}G(A_1+A_2) \\ &= \begin{cases} p^{m-2}(p^m-2)+2p^{-2}G & \text{if } A_1=A_2=1, \\ p^{m-2}(p^m-2)-2p^{-2}G & \text{if } A_1=A_2=-1, \\ p^{m-2}(p^m-2) & \text{if } A_1A_2=-1. \end{cases} \end{aligned}$$

If $p\equiv 1(mod\ 4)$, then $\eta_p(-1)=1$. So their frequencies are given from Lemma 6

$$\sum_{c\in C_0^{(2,p)}} n_{1,c}n_{1,-c}+\sum_{c\in C_1^{(2,p)}} n_{-1,c}n_{-1,-c}=\frac{p-1}{4p^2}(q+G)^2,$$
$$\sum_{c\in C_0^{(2,p)}} n_{-1,c}n_{-1,-c}+\sum_{c\in C_1^{(2,p)}} n_{1,c}n_{1,-c}=\frac{p-1}{4p^2}(q-G)^2,$$

and

$$\sum_{c\in C_0^{(2,p)}} (n_{1,c}n_{-1,-c}+n_{-1,c}n_{1,-c})+\sum_{c\in C_1^{(2,p)}} (n_{-1,c}n_{1,-c}+n_{1,c}n_{-1,-c})$$
$$=\frac{p-1}{2p^2}(q^2-G^2),$$

respectively. In the case of $p\equiv 3(mod\ 4)$, we compute similarly by noting that $\eta_p(-1)=-1$. Their frequencies are given below

$$\sum_{c\in C_0^{(2,p)}} n_{-1,c}n_{1,-c}+\sum_{c\in C_1^{(2,p)}} n_{1,c}n_{-1,-c}=\frac{p-1}{4p^2}(q+G)^2,$$
$$\sum_{c\in C_0^{(2,p)}} n_{1,c}n_{-1,-c}+\sum_{c\in C_1^{(2,p)}} n_{-1,c}n_{1,-c}=\frac{p-1}{4p^2}(q-G)^2,$$

and

$$\sum_{c\in C_0^{(2,p)}}(n_{-1,c}n_{-1,-c}+n_{1,c}n_{1,-c})+\sum_{c\in C_1^{(2,p)}}(n_{-1,c}n_{-1,-c}+n_{1,c}n_{1,-c})$$
$$=\frac{p-1}{2p^2}(q^2-G^2),$$

respectively.

Subcase 3.e If $\lambda_1\neq 0,\lambda_2\neq 0$ and moreover $\lambda_1+\lambda_2\neq 0$, then we have

$$\begin{aligned}N_0&=\frac{(q-1)^2-1}{p^2}+\frac{1}{p^2}\Big((p-1)(G_1G_2+1)+G_1G_2+G(A_1+A_2)-(p-1)\Big)\\&=\frac{(q-1)^2-1}{p^2}+\frac{1}{p}G_1G_2+\frac{1}{p^2}G(A_1+A_2)\\&=\begin{cases}p^{m-2}(p^m-2)+p^{-1}G(\eta)^2+2p^{-2}G & \text{if } \eta(a_1a_2)=A_1=A_2=1,\\ p^{m-2}(p^m-2)+p^{-1}G(\eta)^2 & \text{if } \eta(a_1a_2)=1, A_1A_2=-1,\\ p^{m-2}(p^m-2)+p^{-1}G(\eta)^2-2p^{-2}G & \text{if } \eta(a_1a_2)=1, A_1=A_2=-1,\\ p^{m-2}(p^m-2)-p^{-1}G(\eta)^2+2p^{-2}G & \text{if } \eta(a_1a_2)=-1, A_1=A_2=1,\\ p^{m-2}(p^m-2)-p^{-1}G(\eta)^2 & \text{if } \eta(a_1a_2)=-1, A_1A_2=-1,\\ p^{m-2}(p^m-2)-p^{-1}G(\eta)^2-2p^{-2}G & \text{if } \eta(a_1a_2)=A_1=A_2=-1.\end{cases}\end{aligned}$$

Note that $G(\eta)^2=\eta_p(-1)p^m$ for odd m. Now let us compute the frequency for the case of $\eta(a_1a_2)=A_1=A_2=1$. The other cases will be computed similarly, which are omitted here. If $p\equiv 1(mod\ 4)$, then $\eta_p(-1)=1$. By Lemma 6, the frequency is

$$\begin{aligned}&\sum_{c\in C_0^{(2,p)}}n_{1,c}\sum_{\substack{d\in C_0^{(2,p)}\\ d\neq -c}}n_{1,d}+\sum_{c\in C_1^{(2,p)}}n_{-1,c}\sum_{\substack{d\in C_1^{(2,p)}\\ d\neq -c}}n_{-1,d}\\&=\sum_{c\in C_0^{(2,p)}}\frac{1}{2p}(q+\eta_p(-c)G)\sum_{\substack{d\in C_0^{(2,p)}\\ d\neq -c}}\frac{1}{2p}(q+\eta_p(-d)G)\\&\quad+\sum_{c\in C_1^{(2,p)}}\frac{1}{2p}(q-\eta_p(-c)G)\sum_{\substack{d\in C_1^{(2,p)}\\ d\neq -c}}\frac{1}{2p}(q-\eta_p(-d)G)\\&=\frac{p-1}{4p^2}\cdot\frac{p-3}{2}(q+G)^2.\end{aligned}$$

In the case of $p\equiv 3(mod\ 4)$, we have $\eta_p(-1)=-1$. Again from Lemma 6, the frequency is

$$\begin{aligned}&\sum_{c\in C_1^{(2,p)}}n_{1,c}\sum_{\substack{d\in C_1^{(2,p)}\\ d\neq -c}}n_{1,d}+\sum_{c\in C_0^{(2,p)}}n_{-1,c}\sum_{\substack{d\in C_0^{(2,p)}\\ d\neq -c}}n_{-1,d}\\&=\sum_{c\in C_1^{(2,p)}}\frac{1}{2p}(q+\eta_p(-c)G)\sum_{d\in C_1^{(2,p)}}\frac{1}{2p}(q+\eta_p(-d)G)\\&\quad+\sum_{c\in C_0^{(2,p)}}\frac{1}{2p}(q-\eta_p(-c)G)\sum_{d\in C_0^{(2,p)}}\frac{1}{2p}(q-\eta_p(-d)G)\\&=\frac{p-1}{4p^2}\cdot\frac{p-1}{2}(q+G)^2.\end{aligned}$$

Thus we get the desired conclusions.

Example 2. (1) Let $(p,m)=(3,3)$. By Theorem 2, the code C_D is a [225, 6,140] linear code over $\mathbb{F}_3$. Its weight enumerator is

$$1+96z^{140}+72z^{141}+48z^{142}+72z^{148}+136z^{150}+18z^{152}$$
$$+24z^{155}+16z^{156}+84z^{157}+96z^{158}+48z^{160}+18z^{161},$$

which is checked by Magma programs.

(2)Let $(p,m)=(5,3)$. By Theorem 2, the code C_D is a [3075, 6, 2433] linear code over $\mathbb{F}_5$. Its weight enumerator is

$$1+900z^{2433}+1440z^{2434}+2400z^{2435}+960z^{2436}+400z^{2437}+900z^{2458}$$
$$+1776z^{2460}+400z^{2462}+120z^{2479}+48z^{2480}+80z^{2481}+1800z^{2483}$$
$$+1440z^{2484}+1200z^{2485}+960z^{2486}+800z^{2487},$$

which is checked by Magma programs.

Remark 1. We remark that our codes in Theorems 1 and 2 have better parameters than those in [1]. The details are shown in Table 4 by comparing the parameters of our codes in Examples 1 and 2 with those in [1].

Table 4. Comparison of codes

(p,m)	(5, 2)	(3,4)	(3, 3)	(5, 3)
Parameters of our codes	[115, 4, 76]	[2133, 8, 1278]	[225, 6, 140]	[3075, 6, 2433]
Parameters of codes in [1]	[116, 4, 72]	[2134, 8, 1278]	[226, 6, 128]	[3076, 6, 2352]

IV. CONCLUDING REMARKS

In this paper, we employed exponential sums to demonstrate the weight enumerators of linear codes C_D with defining set D of (2). As introduced in [39], any linear code over $\mathbb{F}_p$ can be employed to construct secret sharing schemes with interesting access structures provided that

$$\frac{w_{min}}{w_{max}}>\frac{p-1}{p},$$

where w_{min} and w_{max} denote the minimum and maximum nonzero weights in C_D, respectively. By Theorem 1, we easily check

$$\frac{w_{min}}{w_{max}}=\frac{(p-1)p^{m-2}(p^m-p^{\frac{m}{2}}-1)}{(p-1)p^{m-2}(p^m+p^{\frac{m}{2}}-1)}>\frac{p-1}{p},$$

where m is even and $m\geqslant 4$. Moreover, by Theorems 2, we easily check

$$\frac{w_{min}}{w_{max}}=\frac{(p-1)p^{m-2}(p^m-2)-p^{m-1}-2p^{\frac{m-3}{2}}}{(p-1)p^{m-2}(p^m-2)+p^{m-1}+2p^{\frac{m-3}{2}}}>\frac{p-1}{p},$$

where m is odd and $m\geqslant 3$. Hence the linear codes constructed in this paper are suitable for applications in secret sharing schemes with interesting access structures.

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