RESEARCH ARTICLE OPEN ACCESS

Fractional vision of the fluid-particle interaction and its models

José Roberto Mercado Escalante*, Pedro Antonio Guido-Aldana**
**Independent Researcher, Mexico.*
*** Mexican Institute of Water Technology.*
Corresponding Author: José Roberto Mercado Escalante

ABSTRACT

Our analysis is based on the interaction of small particles and fluid, under the fractional vision of Navier-Stokes equations. A proposal for a relationship between Feigenbaum constant and spatial occupation index, through the link with the diameter exponent of the particle considered is obtained. Power of particle size decreases from the value 2 for the linear viscous layer, move through inertial range, until reaching the inverse of Feigenbaum constant, in the developed turbulence regime. An arboreal fractal structure for laminar sublayer and inertial range is also proposed. Subsequently, a generalized Rubey model and a discrepancy number to evaluate models with respect to Stokes are formulated.

Keywords-Feigenbaum constant, fluid-particle interaction, Navier Stokes fractional equations, sedimentation, spatial occupation index.

Date of Submission: 20-07-2018 Date of acceptance: 04-08-2018

I. INTRODUCTION

Inverse problems are a very valuable technical tool to discover the rich wealth that is often hidden in the depths, below from the feet of a nation. But on the other hand, as in this paper, also allow us to solve problems about theoretical foundations that lead to formulate the body force that in a couple of action and reaction, a fluid exerts on a particle.

We want to see how analysis and results are modified when spherical droplets experiment deformations that lead us to consider a variety of forms as diverse as snow or ice particles. In our previous description of droplets in clouds we imagine them spherical as a result of surface efficiency for a given volume; and also we saw drops in the rain with the same spherical shape. The new context places us in a similar situation to the case of sediments, already considered by us in paper [1].

As we also announce in paper [1], analysis context will be that of particles in interaction with a fluid, under the Navier-Stokes equations fractional vision, but we enunciate it from an abstract general simplicity foundation, for later direct us towards the complexity of the most concrete; thus we come to a more general relationship between free velocity and size than that formulated by Stokes; a link between spatial occupation index and Feigenbaum constant; and also, an arboreal fractal structure for laminar sublayer and inertial range for the case of a flat and extensive boundary surface for the fluid.

Two variables of fluid field are velocity and pressure, as primitive variables. Movement action on field variables is represented by a linear differential operator, which contains the physical parameter of fluid viscosity and the spatial differential operators of divergence and gradient. In its fractional version, explicit reference is doing to fluid movement scales, while in the version called by us “classical” there is no reference to these movement scales. In addition, boundary conditions are associated with an operator that nullifies field variables on said contour. Then, we transform the description into its “variational” and “weak Q” modes with two bilinear forms, each one associated with one of the field variables. Now, Galerkin method can be applied to obtain the solution in a recurrently way.

An alternative form arises when considering a particle embedded in the fluid in motion, analogous to an irrigation channel. Equations that link velocity evolution and water level are Saint-Venant equations, in which the force that particle exerts on fluid must be added, as a couple action and reaction, which is a body force; and that in its dimensionless form resembles a friction slope. The above refers to the result that states: Lie derivative on hydraulic slope is proportional to the hydraulic slope itself; with which hydraulic slope is determined as a quotient between a velocity power, free, and another power of hydraulic radius, which for the case is proportional to the diameter, [2]. Then we take care of the different modes.

II. MODES

2.1 Abstract

Problem of fluid and a particle interaction can be formulated in a general abstract way as a linear differential operator and a boundary operator. For this the unknown quantity is presented in the column$u = \begin{pmatrix} \vec{u} \\ p \end{pmatrix}$of velocity and pressure, as primitive variables, and external force too, as a column$f = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix}$, differential operators are grouped into the matrix, $L_\beta = \begin{pmatrix} -\nu_\alpha \Delta^\alpha & \nabla \\ div & 0 \end{pmatrix}$, $\alpha = 1 + \beta$, with $(\beta, \nu_\alpha, \nu = \nu_2)$ as spatial occupation index,the α- kinematic viscosity, [3], and the boundary operator as$B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$; so interaction problem is stated as in (1), where the asterisk denotes the possible subset of the boundary, [4],

$L_\beta u = f, \text{in } \Omega; Bu = 0, \text{on } \delta\Omega^*$(1)

2.2 Variational

Remember that we are dealing with a velocities space, with two indexes: first, the order of its differentiability and the second, the integrability order. Whenever the second is 2 it is a Hilbert space. Besides, if the first is 1 and the velocity has its domain located within a compact, velocity can be measured by the diagonal of the parallelepiped of its three partial derivatives along coordinate axes and we are inside of the denoted space $\left(H_0^1(\Omega)\right)^n$, $n = 3$. Set of zero divergence distributions is considered $V = \{v \in D(\Omega) : divv = 0\}$, with domain in Ω and being its closurethe Hilbert space $\overline{V} = \left(H_0^1(\Omega)\right)^n$. External force is presumed of type $f \in \left(L^2(\Omega)\right)^n$, being $L^2(\Omega)$ the energy space of velocities that have integrable kinetic energies. Distributions are evaluated in any $v \in (D(\Omega))^n$, $(\nabla p/\rho, v) = \nu_\alpha \left(\nabla^{1+\beta} u, v\right) + (f, v) - a_0(u, v)$ and as, $divv = 0$

$\nu_\beta\left(\nabla^\beta u, \nabla v\right) = (f, v) - a_0(u, v), \quad \forall v \in V$ (2)

Following space is defined $V_{div} = \left\{v \in \overline{V} \in \left(H_0^1(\Omega)^n\right) : divv = 0\right\}$, which is a closed subspace of $\overline{V}$, but because of Poincaré inequality, [4],it is also a Hilbert space with the norm: $\|v\| = \|\nabla v\|_0$ It follows that bilinear form is coercive in $V_{div} \times V_{div}$,

$$a(w, v) = a_0(w, v) + \nu_\beta\left(\nabla^\beta u, \nabla v\right), \quad \forall w, v \in \overline{V} \quad (3)$$

but also $v \mapsto (f, v)$ is linear and continuous on V_{div}, therefore the following problem admits a unique solution, according to Lax-Milgram lemma, [4]:

$\exists u \in V_{div} : a(u, v) = (f, v), \quad \forall v \in V_{div}$ (4)

Within Galerkin method: let $h > 0$ be the mesh size or step, where $\{V_{div,h}\}$ denotes a family of dimensional finite subspaces of V_{div} that satisfies the hypothesis of consistency:

$$\forall v \in V_{div} : \inf_{v_h \in V_{div,h}} \|v - v_h\| \to 0, \text{if } h \to 0$$

His Galerkin approximation raises: find $u_h \in V_{div,h}$ such that

$a(u_h, v_h) = (f, v_h), \forall v_h \in V_{div,h}$ (5)

An application of the Lax-Milgram lemma, [4], to V_{div} and subspaces $V_{div,h}$, produces the existence and uniqueness of the solution of the problem (5). Solution is stable and converges to Stokes problem solution (5). That is, there is a constant C, which does not depend on h, such that

$\|u - u_h\| \le C \inf_{v_h \in V_{div,h}} \|u - v_h\|$ (6)

2.3 Weak Q

It is distinguished from variational form because pressures Hilbert space is considered, as $Q = L_0^2(\Omega) = \left\{q : \int_\Omega q = 0\right\}$ which correspond to $q \leftrightarrow p/\rho$ and the bilinear form:

$b(v, q) = -(q, divv), q \in Q, v \in V = \left(H_0^1(\Omega)\right)^n$ (7)

So, two bilinear forms are available.

It can be show that the problem: Find $u \in V$, $p \in Q$

$$\begin{aligned} a(u, v) + b(v, p) &= (f, v), \quad \forall v \in V \\ b(u, q) &= 0, \quad \forall q \in Q \end{aligned} \quad (8)$$

has unique solution, (although pressure is determined up to an additive constant).

In the Galerkin method two subspaces families are introduced: V_h of V and Q_h of Q. The goal is approximated with discrete problem: findu_h in V_h and p_h in Q_h,

$$\begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= (f, v)_h, \quad \forall v_h \in V_h \\ b(u_h, q_h) &= 0, \quad \forall q_h \in Q_h \end{aligned}$$

2.4 Classic

We consider a dynamic quartet formed by: particle weight, Archimedes floating force, hydrodynamic pressure force and viscous friction.

For weight and Archimedes flotation forceswe have the effective weight force: $F_e = \left(\frac{\rho - \rho_f}{\rho_f}\right)(\rho_f V)g$, with (ρ, ρ_f) the particle and fluid density, and Vthe volume in its relation with sphere area: $F_e = \frac{g\Delta}{\nu}\left(\rho_f \nu \frac{A^{2/3}}{(6\pi)^{1/2}}\right)g$, pair of forces contributes with the effective weight, with (ν, g) cinematic viscosity and gravity. If the other pair, of hydrodynamic pressure force and viscous friction that we will call drag, produce: $F_D = C\rho_f \nu Ud$, with (U, d) the free velocity of the fluid and the particle diameter; so the balance is reached under the condition: $\frac{U}{Bd} = g\Delta, \quad Bd = \frac{1}{C6\pi^{1/2}\nu}\left(A/d^2\right)^{3/2} d^2$ so that $C = 3\pi, \quad Bd = \frac{1}{18\nu}d^2$ and equilibrium condition results$\frac{\mathrm{U}}{\mathrm{Bd}} = \mathrm{g}\Delta$. Instead of the stock triplet, $\nu_r = \sqrt{\tau_0/\rho_f}, \quad l_r = \nu/\nu_r, \quad \nu_r^2/l_r$ shear velocity, the fundamental length and the reference acceleration [5], we give it a dimensionless representation with $Ud/\nu, \quad d_* = \left(\frac{g\Delta}{\nu^2}\right)^{1/3} d$ the particle Reynolds number and its dimensionless diameter [6]. Equilibrium condition: $\left(\frac{\mathrm{Ud}}{\nu}\right) = \frac{1}{18}\mathrm{d}_*^3$. We call $F(d_*)$ the form factor and for Stokes model it's:

$$Ud/\nu = d_*^3 F(d_*), \quad F(d_*) = \frac{1}{18} \quad (9)$$

Hydrodynamic pressure force is found: $F_p \propto \rho_f \nu Ud$, Stokes approximation expresses the balance between viscous force and hydrodynamic pressure gradient:

$$\frac{\partial}{\partial x^i} p = \nu_\alpha \nabla^\alpha u_i, \quad \frac{\partial}{\partial x^i} u^i = 0 \quad (10)$$

In the laminar limit $\beta \to 1$ it is obtained:

$$\frac{\partial}{\partial x^i} p = \nu_2 \nabla^2 u_i, \quad \frac{\partial}{\partial x^i} u^i = 0 \quad (11)$$

With divergence acting on momentum equation and mass conservation results a Poisson equation for the pressure

$$\frac{\partial^2}{\partial x^{i2}} p = 0 \quad (12)$$

Due small particle symmetry, spherical coordinates are used and accompanies Green function method. A solution in the serial form is sought: $p = \sum_{n=0}^{\infty} p_n, \quad p_n = A_n \frac{\partial^n}{\partial z^n}\frac{1}{r}$ with $z = r\cos\theta$. Due to boundary conditions on the sphere of radius $d/2$ and at distances sufficiently far away from it, series is reduced to a monomial because the following term remains only n=1: $p_1 = A_1 \frac{\partial}{\partial z}\frac{1}{r}, \quad A_1 = \frac{3}{4}\rho_f \nu Ud$ and it results $p_1 = -\frac{3}{4}\rho_f \nu Ud \frac{\cos\theta}{r^2}$. By integrating hydrodynamic pressure on the sphere, it provides the force: $F_p = \pi(\rho_f \nu Ud)$, [7]. In addition viscous friction force is: $F_f = 2\pi(\rho_f \nu Ud)$ what can be seen by the well-known Stokes formula or by solving differential equations.

Internal viscous friction is responsible for variations in deformation. If $F_e = (\rho - \rho_f)\left(\frac{1}{6}\pi d^3\right)g = F_f + \pi(\rho_f \nu Ud)$ and with $U = \frac{1}{18}\frac{g\Delta}{\nu}d^2$ it is obtained the viscous internal friction force: $F_f = 2\pi(\rho_f \nu Ud)$; in this case, it is observed that viscous force is twice pressure force. So total drag force or drag, hydrodynamic pressure plus internal friction, is: $F_D = F_p + F_f = 3\pi(\rho_f \nu Ud)$.

From the point of view of the drag coefficient or drag: $C_D = \frac{F_D}{\frac{\pi}{8}\rho_f U_0^2 d_s^2}$, with $U_0 \leftrightarrow U, \quad d_s \leftrightarrow d$ as free velocity and particle size, in positions $r >> d/2$. It is convenient to decompose into two: $C_f = \frac{F_f}{\frac{\pi}{8}\rho_f U^2 d^2} = \frac{16}{Ud/\nu}$ and $C_p = \frac{F_p}{\frac{\pi}{8}\rho_f U^2 d^2} = \frac{8}{Ud/\nu}$, and the sum is $C_D = \frac{24}{R_{ep}}, \quad R_{ep} = Ud/\nu$. However, when Reynolds number grows, deformation variations expressed by friction weaken as long as hydrodynamic pressure remains, until the first becomes a constant and that of pressure approaches the value $C_f = \frac{A}{R_{ep}}$, so $C_D = \frac{A}{R_{ep}} + B$. This description is compatible with the Ossen force: $F_D = 3\pi\left(\rho_f \nu^2\right)\left(R_{ep} + \frac{3}{16}R_{ep}^2\right), \quad r \gg d/2$ the radial coordinate, [4], [14]. In effect, $C_D = \frac{F_D}{\frac{\pi}{8}\rho_f \nu^2 R_{ep}^2} = \frac{24}{R_{ep}} + \frac{9}{2}, \quad r \gg d/2$, which is of the type: $C_D = \frac{A}{R_{ep}} + B$. But also, in equilibrium, the pair of effective weight is balanced with the pair of drag, thus $\frac{A}{R_{ep}} + B = C\frac{d_*^3}{\left(R_{ep}\right)^2}$, then $R_{ep} = d_*^{3/2}F\left(d_*\right)$,

$$F\left(d_*\right) = \sqrt{\left(\frac{A}{2Bd_*^{3/2}}\right)^2 + \frac{C}{B}} - \sqrt{\left(\frac{A}{2Bd_*^{3/2}}\right)^2}$$

where C is a number linked to the volume that depends on the particle shape.Moreover, the so-called "drag crisis" is included in Bvalue, considering it constant until before the critical Reynolds number for which it occurs, when it descends from B to B_1, with$B_1 \approx 0.6B$, for smooth surface; however for rough surface, B_1grows approximately linearly with R_{ep}so that the value of R_{ep}remains determined by the model implicitly, [8].

III. GENERALIZATION

There are several aspects that we must put in perspective. Formulas cited in paper [9] are of experimental origin and show that particles size is manifested through a variety of diameter exponents, exponents that assume different values and lower than the value 2 of Stokes formula; and that some can even be seen as interpolations of two of these different exponents. In Stokes formula, of theoretical origin, participation of particles size can be seen as a fractional exponent of the area of its boundary, so it is possible to imagine a fractal process of Cantor type, which progressively modifies the area of the boundary and produce new exponents.

Previous results [9], we had written them as: $\left(C_1\nu_\alpha\phi'\left(d/2r\right)\right)U^b = g\Delta d^c, \quad \frac{U^b}{Bd} = g\Delta$;we imagine it as aB operator that transforms diameter power and producesBd; but then we seeit as a fractional derivative and we ended up being dimensionless with the addition of the particle Reynolds numberand its dimensionless diameter $Ud/\nu, \quad d_* = \left(\frac{g\Delta}{\nu^2}\right)^{1/3} d$; in the form:$\mathrm{Bd} = \mathrm{CD}^{\mathrm{s}}_{\mathrm{d}_*}(\mathrm{d}_*)^3$. We already formulated Stokes model as: $Ud/\nu = d_*^3 F\left(d_*\right), \quad F\left(d_*\right) = \frac{1}{18}$. Now generalizing, we formulate a model like $\left(\frac{Ud/\nu}{CD^s_{d_*}\left(d_*\right)^3}\right)^b = F\left(d_*\right)$, with the form factor $F\left(d_*\right)$ to be specified. In addition, we see the original factor $\left(\phi'(d/2r\right)$ as dependent on the dimensionless diameter d_* and inversely proportional to the form factor: $C_1\nu_\alpha\phi'\left(d/2r\right) \propto 1/F\left(d_*\right)$.

First, in the vision of the operator that transforms diameter power and produces $Bd \approx d^c$ we had imagined c as an interpolation between $1/2$ and 2, as conceived by Rubey [10], but for us depending on the parameterized flow regime by β, so$c = (1-\beta)\frac{1}{2} + \beta 2$, what produces the following scenarios:$\beta \to 1, c \to 2$ if it'slaminate, and$\beta \to 0, c \to 1/2$if it's turbulent.

But then, when considering sediments case, we know that velocities formulas are also proposed that arise from doing an interpolation between two diameter powers such as the formula of Scotti-Foglieni, [11], [9]. So, we return to that idea and generalize it by representing the exponent of the diameter as an interpolation between the laminar exponent and another that we seek to specify, but using the spatial occupation index so we intend to link two exponents: one relative to laminar movement with another relative to turbulent movement; so we set out to go a step further and interpolate:$c = (1-\beta)\frac{1}{\sigma} + \beta 2$, but the value $1/\sigma$remains to be specified later.

For this and within the second aspect, the fractal, we consider the Cantor process described by the Feigenbaum tree. For what we propose: Feigenbaum relation $\frac{\beta_L - \beta_m}{\beta_L - \beta_1} = \left(\frac{1}{\delta}\right)^{m-1}, \quad \frac{1}{\delta} \approx \frac{1}{5}$, if we assume $\beta_L \to 0, \quad \beta_1 = 1$, it is reduced, $\beta_m = \left(\frac{1}{\delta}\right)^{m-1}$; so the other extreme that we propose is$\frac{1}{\sigma} \to \frac{1}{\delta}$ and it results $\frac{c_m - 1/\delta}{2 - 1/\delta} = \beta_m$.

Therefore, particle size participates as diameter exponent d, seeing this exponent as an interpolation between the inverse of Feigenbaum constant and Stokes exponent, and being the spatial occupation index β the interpolator: $c = (1-\beta)\frac{1}{\delta} + \beta 2$.

From the irregular surface we will highlight its roughness. We see irregular surface as sections of irregular curves that could be characterized as

Brownian curves of Hurst persistence index H, $0 < H < 1$, which have fractal dimension of $2 - H$ value. Roughness is dimensionlessby the diameter; we also perform its potential representation by $(1/q)^{2-H}$, $0 < q < 1$; therefore, a rougher surface than another corresponds to a larger dimension and, therefore, to a lower Hurst index, [12].

In paper [3] we saw that for a given fluid there is an inverse correlation between roughness height and kinetic energy transfer rate, so that on the rough surface the laminar layer cannot be formed, thus roughness extreme is linked to a β_m for some $m > 1$, which in turn corresponds to a valuec_m of the diameter exponent:$c_m = (1 - \beta_m)\frac{1}{\delta} + \beta_m 2$.

Furthermore, if we assume $s_m = (1 - \beta_m)(2 - 1/\delta)$, $2 - s_m = c_m$, the expression Bd transform into$Bd \to D_d^{s_m} d^2$, $D_d^{s_m} d^2 = cte(s^{c_m})$.

On the other hand, in the dimensionless case we have: $3 - s_m = 1 + c_m$, $D_{d_*}^{s_m} d_*^3 = cte\ d_*^{3-s_m}$. Therefore, there is an interrelation between occupation index, derivative order, size exponent, Feigenbaum constant and m branch order described by:

$$\beta = \frac{c - 1/\delta}{2 - 1/\delta}, \quad s = (1 - \beta)(2 - 1/\delta), \quad \beta_m = \left(\frac{1}{\delta}\right)^{m-1} \tag{13}$$

We can enunciate two types of formulas: one for the dimensional case of hydraulic slope; the other for the dimensionless case:

$$\frac{U^b}{C_\beta D_d^s d^2} = g\Delta, \quad \left(\frac{Ud/\nu}{CD_{d_*}^s (d_*)^3}\right)^b = F(d_*) \tag{14}$$

Where in the dimensional case, left side, we have the manifestation of the drag force pair and in the right, that of the pair of effective weight or weight and flotation; while in the dimensionless, right side, ratio of drag force or drag is expressed with respect to the pair of weight and flotation as a form factor, being the form factor a generalization of the Rubey model with different formulations that we address in the following subsection.

3.1 Formulas

Below we present a sample of formulas that describe movement of small particles in fluids. Both those considered in [13], where experimentally the fall velocity in air of ice and snow particles are studying as a function of their size measured by a diameter, as well as some of the useful formulas for calculating velocity in the case of sediments [11].

We begin by exposing the generalization of one of the formulas coming from sediments issue, because this one opens the path of generalization. Rubey formula (1933), [10], which was proposed to obtain the falling velocity of roughness natural particleswith size between silt and gravel; for example silts, which are somewhat small particles that have diameters between 0.002 and 0.06 mm.

In the impact velocity model of Rubey, Uvelocity is factorized as: $U = (g\Delta d)^{1/2} F$, with $F(d_*) = \sqrt{\frac{36}{d_*^3} + \frac{2}{3}} - \sqrt{\frac{36}{d_*^3}}$; or in the dimensionless formulation: $Ud/\nu = d_*^{3/2} F(d_*)$. A generalized model of Rubey has the form:

$$\left(\frac{Ud}{\nu}\right)^b = (d_*)^c F(d_*) \tag{15}$$

with different formulations for the form factor as it can be $F(d_*) = \sqrt[n]{\frac{cte}{d_*^\sigma} + h} - \sqrt[n]{\frac{cte}{d_*^\sigma}}$,or the form factor found from Ossen drag coefficient: $F(d_*) = \sqrt{\left(\frac{A}{2Bd_*^{3/2}}\right)^2 + \frac{C}{B}} - \sqrt{\left(\frac{A}{2Bd_*^{3/2}}\right)^2}$;these then we will also represent them as $F(d_*) \approx \frac{1}{n+1}(x^{-1})^{\frac{n-1}{n}}$, $x(d_*)$, with x dependent on d_*.

First we analyze dimensional formulations and then the dimensionless ones. We begin by considering models that arise from clouds phenomenon and their empirical formulas; and then those that have originated in sediments. Next, a data box is presented that includes: index, derivative order, branches order and branches number.

- Magono (1953), [13]: he determined fall velocities of crystals and big and small snow particles by the stroboscopic method. Hypothesis is a drag force external to the crystal structure which determines that the square of the velocity is proportional to the square of its size, complemented by an internal force to the structure with the square of the velocity proportional to the cube of its size; so for the velocity, it is also an interpolation of powers 1 and 3/2 of the crystal size. For small particles the velocity is assumed proportional to the square root of its size. Their experimental results can be adjusted by concave curves of square root type. His model is: $U = \sqrt{K}\left(\frac{d}{a + bd}\right)^{1/2}$,and we are going to consider:

$\frac{U^b}{C_\beta D_d^s d^2} = g\Delta$ under the form: $\frac{U^2}{C_\beta D_d^s d^2} = K$ and $D^s d = \frac{1}{a}\frac{(a/d)^2}{(b+a/d)^2}$,with $C_\beta D_d^s d^2 = D^{s-1}d$, then $D^{s-1}x = \frac{1}{a}\int\frac{(a/x)^2}{(b+a/x)^2}dx$; it becomes: $D^{s-1}d = \frac{d}{(a+bd)}$. Therefore: $\frac{U^2}{d/(a+bd)} = K$ with $(a, b, \sqrt{K}) = (0.8, 0.63, 132)$ we get $U = 132\,(d/(0.8+0.63d))^{1/2}$ and Magono model is obtained. Data box is (see equation 13):

	b	c	β	s	m
Magono	1	1/2	0.16005	1.3800	2

- Langleben (1954), [13]: he also calculated velocities of falling snowflakes by means photographs on a dark background. By means of the dimensional formula (14), $\frac{U^b}{C_\beta D_d^s d^2} = g\Delta$, $U^b = (Cg\Delta)d^c$ it is produced Langleen's model: $U^b = (Cg\Delta)d^c$, with $(b, c) = (2, 0.62)$, or: $U = kd^{(0.31)}$. Data box is:

	b	c	β	s	m
Langleben	2	0.62	0.22725	1.3800	2

- Leslie model of "competent" velocity,"law of competent velocities" or "sixth power law", [14], where the weight or volume of the largest pebble which can be moved along the bottom of a stream varies with as the sixth power of the stream velocity. At particle scale, the square of this "competent" velocity would be proportional to the power $1/3$ of the diameter. Data box is:

	b	c	β	s	m
Leslie	2	1/3	6.6727x10^{-2}	1.6667	3

Authors quoted by [10] are: Dubuat, Robinson, Blackwell, Login and Forbes (1857), Suchier (1924), Owens (1908), [15].

- Litvinov (1956), [13], with the dimensionless formula: $\frac{U^b}{C_\beta Bd} = g\Delta$, $(g\Delta C_\beta)Bd = kd^c$, with $U^2 = kd^{0.32}$, $(b, c) = (2, 0.32)$, $k < 100$ which is the Litvinov model for snowflakes. Data box is:

	b	c	β	s	m
Litvinov	2	0.32	5.9256x10^{-2}	1.68	3

A model with an exponent of the value $c = 8/35$, would produce a branch order of $m \approx 4$.

- Allen [11]: with $U = 0.2\frac{(g\Delta)^{2/3}}{\nu^{1/3}}d$, $(b, c) = (3/2, 3/2)$, we have the model that is attributed to Allen. We calculate the data box:

	b	c	β	s	m
Allen	3/2	3/2	0.72002	1/2	1

It can also be seen as part of generalized Rubey type (15), if it is described by: $\frac{Ud}{\nu} = 0.2d_*^2$, $F = 0.2$.

- Newton [11]: this is the result that Newton proposed for a turbulent regime: $U = 1.82\sqrt{g\Delta d}$, $(b, c) = (2, 1)$, it can be presented as: $\frac{U^2}{(1.82)^2 d} = g\Delta$. Data box is:

	b	c	β	s	m
Newton	2	1	0.44003	1	2

As part of the generalized Rubey type it is described by: $\frac{Ud}{\nu} = 1.82d_*^{3/2}$, $F = 1.82$, (15).

- Owens [11]: it was proposed in 1979 to obtain the fall velocity with a proportional constant that varies with sediment form and nature. But data box and Rubey model generalized are the same, difference lies in the empirical constant ($U = k\sqrt{g\Delta d}$). Constant k is dimensionless and varies with increasing round shape; but it also changes with grains nature being larger for sand than for quartz; it also seems to increase with grains size. Their values are of the type: 9.35,8.25,6.12,1.28. It is observed that Newton's model is within these values.

- Maza and García [11]: in 1996 Maza and García result was available to estimate the critical average velocity of particles of diameter d, or as a function of the critical Froude number, which are considered applicable in the interval $0.0001 < d < 0.4$ [m]. It can be enunciated as: $\frac{U^2}{\left(\frac{4.7}{\sqrt{g}R_h^{0.2}}\right)^2 d^{7/10}} = g\Delta, \quad (b,c) = (2, 7/10)$.

Data box is:

	b	c	β	s	m
Maza and García	2	7/10	0.27204	1.3	2

- Scotti - Foglieni [11]: on the other hand, sedimentation velocity can be estimate through an interpolation of a formula like Owens with another proportional to the first power of the grain size, so the Scotti - Foglieni formula arise $U = 3.8\sqrt{d} + 8.3d$.

- Gold number: it is a model of theoretical order where the power is the gold number$c = \frac{1+\sqrt{5}}{2}$. Data box is:

	b	c	β	s	m
Gold number	1	$\frac{1+\sqrt{5}}{2}$	0.7861	0.38199	1

Know we can order data by the decreasing cvalue, see Table 1.

Table 1. Results ordered according to the decreasing c value.

	b	c	β	s	m
Gold number	1	$\frac{1+\sqrt{5}}{2}$	0.7861	0.38199	1
Allen	1	3/2	0.72002	1/2	1
Newton	2	1	0.44003	1	2
Maza and García	2	7/10	0.27204	1.3	2
Langleben	2	0.62	0.22725	1.3800	2
Magono	1	1/2	0.16005	1.3800	2
Leslie	2	1/3	$6.6727x10^{-2}$	1.6667	3
Litvinov	2	0.32	$5.9256x10^{-2}$	1.68	3

Regarding dimensionless formulations we had already considered Rubey, where velocity U is factored as: $U = (g\Delta d)^{1/2} F$. $F(d_*) = \sqrt{\frac{36}{d_*^3} + \frac{2}{3}} - \sqrt{\frac{36}{d_*^3}}$; being its dimensionless formulation: $(Ud/\nu = d_*^{3/2} F(d_*))$, and a generalized Rubey model that acquires the form $\left(\frac{Ud}{\nu}\right)^b = (d_*)^c F(d_*)$ with different formulations for form factor (15).

We already mentioned (15) that for several models we can represent form factor as $F(d_*) = \sqrt[n]{\frac{A}{d_*^\sigma} + h} - \sqrt[n]{\frac{A}{d_*^\sigma}}$, with A and h positive constants, σ a positive exponent, the idea is that the particle be so small that satisfy $\frac{A}{d_*^\sigma} \gg h$; then the following approach is valid $\left(x + \frac{n}{n+1}\right)^{1/n} - x^{1/n} \approx \frac{n}{n+1}\frac{d}{dx}(x^{1/n}) = \frac{1}{n+1}(x^{-1})^{\frac{n-1}{n}}$; and we obtain therefore the approximation: $\left(\frac{Ud}{\nu}\right)^b \approx \frac{1}{n+1}\left(\frac{1}{A}\right)^{\frac{n-1}{n}} d_*^{b\varepsilon}, \qquad b\varepsilon = c + \sigma\frac{(n-1)}{n}$ where the exponent of the diameterd_*can be observe in the dimensionless expression. To formulate a model that provides something new with respect to Stokes, discrepancy number is required: $\varepsilon = \frac{c}{b} + \frac{\sigma}{b}\left(\frac{n-1}{n}\right) < 3$. Also, we could simplify a bit if we assume that $\sigma = cn$, $\frac{cn}{b} < 3$.

- In [6] the model is described by: $\frac{Ud}{\nu} = \left(\sqrt{25 + 1.2d_*^2} - 5\right)^{3/2}$, $\left(\frac{Ud}{\nu}\right)^{2/3} = d_* F(d_*)$ with the form factor: $F(d_*) = \sqrt{\frac{25}{d_*^2} + 1.2} - \sqrt{\frac{25}{d_*^2}}$; and $\varepsilon = 3$,then the discrepancy criterion is not satisfied for Cheng.
- Also in [6] Zhang (1989) is cited as: $U = \sqrt{\left(13.95\frac{\nu}{d}\right)^2 + 1.09g\Delta d} - \sqrt{\left(13.95\frac{\nu}{d}\right)^2}$, or

$\frac{Ud}{\nu} = d_*^{3/2} F(d_*)$ being $F(d_*) = \sqrt{\left(\frac{13.95}{d_*^{3/2}}\right)^2 + 1.09} - \sqrt{\left(\frac{13.95}{d_*^{3/2}}\right)^2}$. We evaluate for Rubey and Zhang: $\varepsilon = 3$ and neither is satisfied.

- In [6] Zanke model (1977) is cited: $U = 10\frac{\nu}{d}\left(\sqrt{1 + 0.01 d_*^2} - 1\right)$, which is: $\frac{Ud}{\nu} = d_* F(d_*)$, with $F(d_*) = \sqrt{\frac{100}{d_*^2} + 1} - \sqrt{\frac{100}{d_*^2}}$.

We evaluate Zanke: $\varepsilon = 2$ and this one satisfies it.

- In [16] Camenen model is considered: $\left(\frac{Ud}{\nu}\right)^{1/k} = (d_*)^{3/2k} F(d_*)$, with $F(d_*) = \sqrt{\left(\frac{1}{4}\left(\frac{A}{B}\frac{1}{d_*^{3/2}}\right)\right)^{2/k} + \left(\frac{4}{3B}\right)^{1/k}} - \sqrt{\left(\frac{1}{4}\left(\frac{A}{B}\frac{1}{d_*^{3/2}}\right)\right)^{2/k}}$

When evaluating discrepancy criterion: $\varepsilon = 3$, we see that it does not satisfy either; therefore the criterion is not satisfied neither Camenen, nor Rubey, nor Cheng, nor Julien, [1]. It is observed in (15) that this model is contained in the generalized Rubey with $k = 1$, $C = \frac{4}{3}$, value linked to the volume that arises in specific cases of rigid sphere and ellipsoid, while for the cylinder it is $C = 2$. It also contains Rubey case: with $k = 1$, $\frac{1}{4}\left(\frac{A}{B}\right)^2 = 36$, $\frac{4}{3B} = \frac{2}{3}$, and doing $B = 2$, $A = 24$ it recovers $F(d_*) = \sqrt{\frac{36}{d_*^3} + \frac{2}{3}} - \sqrt{\frac{36}{d_*^3}}$.

Therefore, after evaluating the aforementioned models with respect to discrepancy number, we observe that only Zanke brings something new compared with the classic Stokes model.

IV. GRAPHS

We now graph the proposed relationship between the index β and the diameter power (see Figure 1), if we use the dimensionless diameter, the straight line would move to the right and descend from the value 3 for Stokes model.

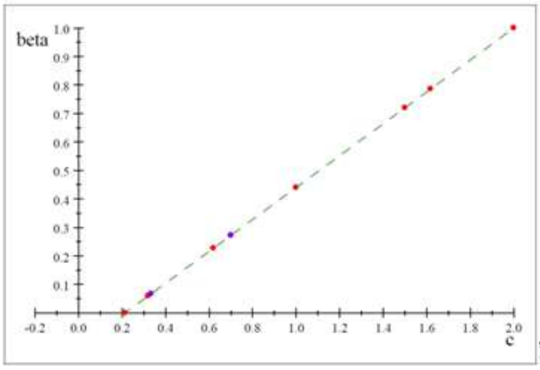

Figure 1: $1/\delta = 0.214$ **, Index** β **as function of power.**

Graph in Figure 1 highlights values obtained experimentally, except the first two, by different researchers and in various disciplines. Data correspond to Stokes, Rubey, Allen, Owen, Maza, and others.

Now, in Figure 2, we graph the proposed relationship between branch order $(m)$and derivative order(s).

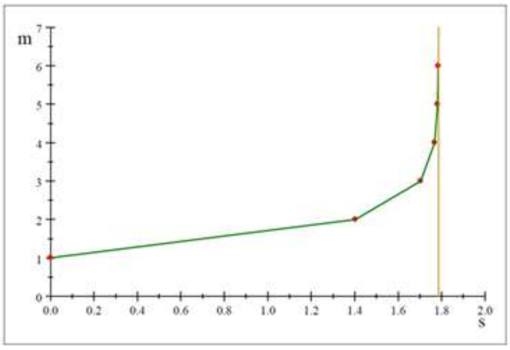

Figure 2: Relationship between branch order (m) **and derivative order**(s)**, both variables dimensionless.**

It is observed that in only 6 steps it reaches an almost saturation stage which corresponds to developed turbulence, derivative order is of $s = 1.7858$, and branches number of sixth order of $2^6 = 64$, in Feigenbaum tree.

V. CONCLUSIONS

A relation proposal between Feigenbaum constant and spatial occupation index is obtained, through the link with the diameter exponent of the considered particle. Particle size power decreases from the value 2 for the linear viscous layer, it goes through the inertial range, until to reach the inverse of the Feigenbaum constant, in the developed turbulence regime. Concomitantly, the spatial occupation index drops from the value 1 in the laminar layer, going through the inertial range until approaching to 0 in the fully developed regime.

The basis of our analysis is the fractional version of the Navier-Stokes equation. We take a shortcut by means the Sain-Venant fractional equation as an alternative argument, and reconsider the result published in [2] to obtain the presentation (14).

We reinterpret Rubey's formula to generalize it, which offers us an alternative way to enunciate various models of formulas considered. We developed a criterion to differentiate models of the classic Stokes.

It is observed that almost all formulas are located in the inertial range of the fluid movement but scarcely describe the sublaminar layer, which we would consider to be described by branch order or bifurcation of order 2 or 3. In particular, Litvinov stands out, which produces the highest branch order.

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Authors biographies

Dr. José Roberto Mercado-Escalante

Independent Researcher. Doctor of Sciences (Mathematics) by the National Autonomous University of Mexico (UNAM), Master Degree in Mathematics by the Meritorious Autonomous University of Puebla (BUAP), Mexico; graduates in Physics by the National University of Colombia. Research areas: inverse problems, fractals and fractional derivatives, mathematical aspects of hydraulics.

Dr. Pedro Antonio Guido-Aldana

Researcher at the Mexican Institute of Water Technology- IMTA, Mexico. Affiliate to the Professional and Institutional Development Coordination. Doctorof Engineering and Master Degree in Hydraulics by the National Autonomous University of Mexico-UNAM. Civil Engineer. Associate Professor of the Engineering Faculty of UNAM. Research interest: hydraulics, potamology, energy and water planning, implementation of PIV and LDA measurements systems in open channel flows.

José Roberto Mercado Escalante " Fractional vision of the fluid-particle interaction and its models"International Journal of Engineering Research and Applications (IJERA) , vol. 8, no.7, 2018, pp.24-32