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Pyramidal Sum Labeling In Graphs

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ABSTRACT
Let G = (V, E) be a graph with p vertices and q edges. A graph G is said to admit Pyramidal Sum labeling if its vertices can be labeled by nonnegative integers $\{0,1,2 \ldots p_q\}$ such that the induced edge labels obtained by the Sum of the labels of the end vertices are the first q pyramidal numbers where p_q is the q[th] pyramidal number. A graph G which admits a Pyramidal Sum labeling is called a Pyramidal Sum graph. In this Paper we prove that the one vertex union of t copies of any Path, the graphs got by attaching the roots of different Stars to one vertex, Comb graph $P_n \odot K$ and all Lobsters are Pyramidal Sum Graphs and show that some graphs are Unpyramidal. By a graph we mean a finite, undirected graph without multiple edges or loops. For graph theoretic terminology, we refer to Harary [4] and Bondy and Murty [2]. For number theoretic terminology, we refer to M. Apostal [1] and Niven and Herbert S. Zuckerman [5].

***Key Words**: Comb, Helm, Lobsters, Pyramidal Number, Unpyramidal.*

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I. INTRODUCTION

All graphs in this Paper are finite, simple, undirected and connected. Graph labeling is an assignment of labels to the vertices or edges or to both the vertices and edges subject to certain conditions. In 1990 Harary introduced the notion of Sum Graphs. The concept of Triangular Sum labeling was introduced by S. Hedge and P. Sankaran. Based on these ideas here a new labeling called as Pyramidal Sum labeling is introduced. In this Paper we give some necessary conditions for a graph to admit Pyramidal Sum labeling and show that some families of graphs are Pyramidal Sum graphs while some others are termed as Unpyramidal as they do not satisfy the condition.

II. PYRAMIDAL SUM LABELING

Definition 2.1: A Triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n. If n[th] Triangular number is denoted by T_n then $T_n = \frac{n(n+1)}{2}$. Triangular numbers are found in the third diagonal of Pascal's Triangle starting at row 3. They are 1, 3, 6, 10, 15, 21...

Definition 2.2: The sum of consecutive Triangular numbers is known as Tetrahedral Numbers. They are found in the fourth diagonal of Pascal's Triangle. These numbers are 1, 1 + 3, 1 + 3 + 6, 1 + 3 + 6 + 10...
They are 1, 4, 10, 20, 35...

Definition 2.3: The Pyramidal numbers or Square Pyramidal numbers are the sums of consecutive pairs of Tetrahedral numbers. The following are some Pyramidal Numbers:
1. 1 + 4, 4 + 10, 10 + 20, 20 + 35...
They are 1, 5, 14, 30, 55...

Remark 2.4: The Pyramidal numbers are also calculated by the formula: $P_n = \frac{n(n+1)(2n+1)}{6}$

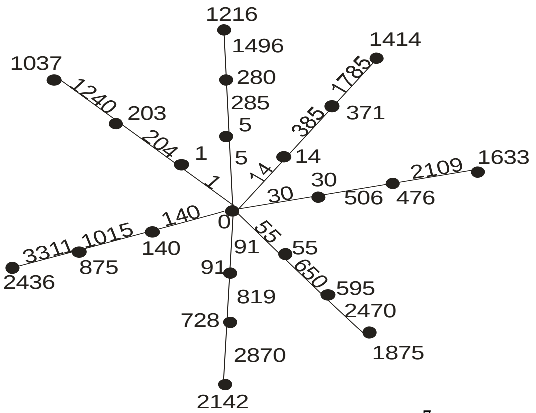
Definition 2.5: A Pyramidal Sum labeling of a graph G is a one-one function defined as $f:V(G) \to \{0,1,2 \ldots p_q\}$ that induces a bijection $f^+ : E(G) \to \{P_1, P_2, \ldots, P_q\}$ of the edges of G defined by $f^+(uv) = f(u) + f(v)$ for all $e = uv \in E(G)$ where p_i is the i[th] pyramidal number, i = 1,2,3,... The graph which admits such a labeling is called as Pyramidal Sum Graph.

Notation: The Notation p_i is used for each Pyramidal number where i = 1,2,3...

Theorem 2.6: The One Vertex Union of t isomorphic copies of any Path is a Pyramidal Sum graph.

Proof: Let P_n be a Path with n vertices. Let there be t copies of P_n. Let the central vertex be denoted by $v_{0,0}$. Let the vertices around the central vertex in the first layer be denoted by $v_{i,j}$, The vertices in the second layer by $v_{2,j}$ and so on. Let $e_{i,j}$ be the corresponding edges, where $1 \le j \le t$, $1 \le i \le n - 1$.
Define $f : V(G) \to \{0,1,2, \ldots, p_q\}$ as follows:
$f(v_{0,0}) = 0$
$f(v_{1,j}) = p_j$ Where $1 \le j \le t$
$f(v_{2,j}) = p_{t+j} - p_j$ Where $1 \le j \le t$

$f(v_{3,j}) = p_{2t+j} - p_{t+j}$ Where $1 \le j \le t$. In general, $f(v_{i,j}) = p_{(i-1)t+j} - p_{(i-2)t+j} + p_{(i-3)t+j} + \ldots + (-1)^{k-1} p_{(i-k)t+j}$, where $1 \le i \le n-1$, $1 \le k \le i$ and for each fixed i, j varies from 1 to t.



Pyramidal Sum Labeling Of P_4^7

The Graph P_4^7 is the union of 7 copies of the Path P_4. Here n = 4, t = 7, j = 1,2,3…7. Since there are 7 copies of P_4, the number of edges in P_4^7 = 3x7 = 21 and the edge labels defined by the sum of the labels of the end vertices are the first 21 Pyramidal numbers.

Theorem: 2.7: Every Comb Graph $P_n \odot K_1$ is a Pyramidal Sum Graph.

Proof: Let $G = P_n \odot K_1$ be a Comb Graph with 2n vertices and (2n –1) edges.

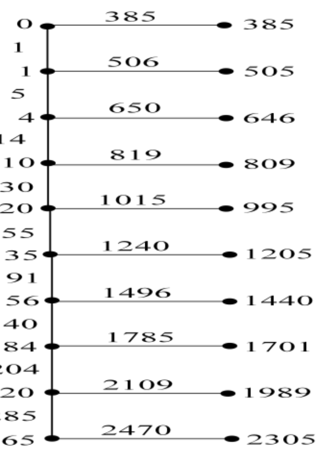
Let u_i, i = 1 to n be the vertices of the Path P_n and v_j, j= 1 to n be the end vertices. Let e_i, i= 1 to 2n-1 be the edges.

Define $f : V(G) \to \{0,1,2,\ldots,p_q\}$ as follows:

$$f(u_i) = \left[\frac{i^3}{6}\right] \text{ for } 1 \le i \le 5 \text{ and}$$
$$\left[\frac{i^3}{6}\right] - 1 \text{ for } 6 \le i \le n.$$

$f(v_1) = p_n$

$f(v_j) = P_{n+i} - f(u_{i+1})$, for $2 \le j \le n$, $1 \le i \le n$. The following is an example:



Pyramidal Sum Labeling Of $P_{10} \odot K_1$

Theorem 2.8: All Lobsters admit Pyramidal Sum Labeling.

Proof: Let G = (V, E) be a Lobster with p vertices and q edges. A Lobster is a Tree with the property that the removal of the end points leaves a Caterpillar. Let the different Stars in a Lobster be denoted by $K_{1,K_1}, K_{1,K_2}, K_{1,K_3}, \ldots, K_{1,K_m}$. Let the Path in a Lobster be denoted by P_n where n is the number of vertices in the Path. Let the vertices of the Path be denoted by $v_1, v_2, v_3, \ldots, v_n$. Let v_{i,j_i} denote the pendant vertices of each star where i = 1, 2,....., m and j_i = 1, 2, 3... k_i,

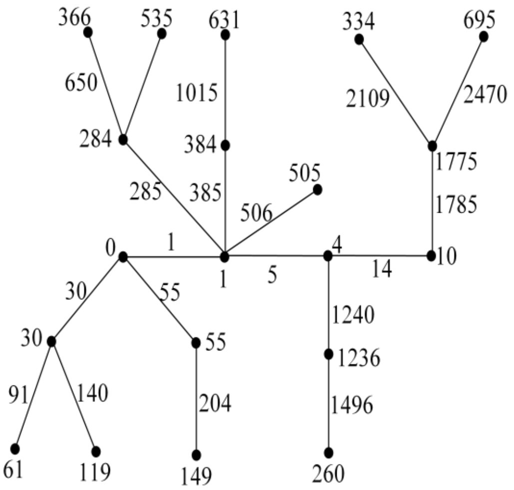
Define $f : V(G) \to \{0,1,2,\ldots,p_q\}$ as follows:

$$f(v_i) = \left[\frac{i^3}{6}\right], 1 \le i \le 5$$
$$\left[\frac{i^3}{6}\right] - 1, \; 6 \le i \le n$$

Define $f(v_{i,j_i}) = p_{n+K_1+K_2+\ldots+K_{l-1}+j_i-1}$ – Label of root vertices, where i = 1, 2, 3... n.

Such a labeling makes all the edges get distinct Pyramidal numbers.

Example:



Pyramidal Sum Labeling Of Lobster

Theorem 2.9: The Graph which is constructed by attaching the roots of different Stars to one vertex is a Pyramidal Sum graph.

Proof: Let $K_{1,n_1}, K_{1,n_2}, K_{1,n_3}, \ldots, K_{1,n_t}$ be t Stars that are attached by their central vertices to one vertex v. Label the vertex v as 0. Then label the central vertices of the Stars as $p_1, p_2 \ldots p_t$ respectively.

The end vertices of the first Star are labeled as $p_{t+1} - p_1, p_{t+2} - p_1, \ldots, p_{t+n_1} - p_1$,

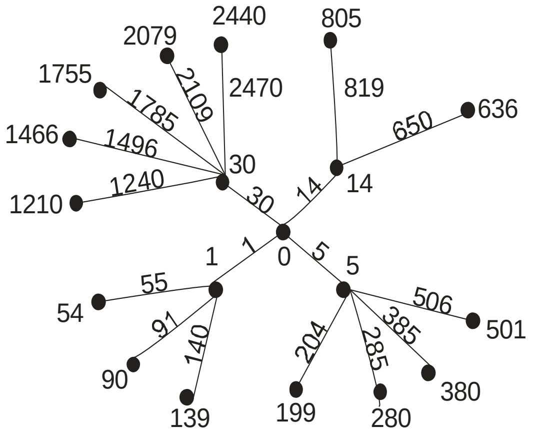
The end vertices of the second Star are labeled as, $p_{t+n_1+1} - p_2$, $p_{t+n_1+2} - p_2, \ldots$, $p_{t+n_1+n_2} - p_2$. .Proceeding like this,

The end vertices of the last Star are labeled as,

$p_{t+n_1+n_2+\cdots+n_{t-1}+1} - p_t, p_{t+n_1+n_2+\cdots+n_{t-1}+2} - p_t, \ldots p_{t+n_1+n_2+\cdots+n_{t-1}+n_t} - p_t$

Now the edge labels got by the sum of the labels of the end vertices are the distinct Pyramidal numbers.

Example: Consider the following graph constructed by attaching the roots of 4 Stars to one vertex v.



Pyramidal Sum Labeling Of Roots Of 4 Stars Attached To One Vertex V

III. UNPYRAMIDAL GRAPHS

Definition 3.1: The Graphs which do not admit Pyramidal Sum labeling are termed as Unpyramidal Graphs.

Remark 3.2: There does not exist consecutive integers which are Pyramidal numbers.

Remark 3.3: The difference between two consecutive Pyramidal numbers is a perfect square and the difference between any two Pyramidal numbers is atleast 4.

The following are some necessary conditions in any Pyramidal Sum Graph.

Result 3.4: In a Pyramidal Sum Graph the vertices with labels 0 and 1 are adjacent.

Proof: To get the edge label $p_1 = 1$ it is a must for the vertices with labels 0 and 1 to be adjacent. There is no other possibility. Hence in every Pyramidal Sum graph the vertices with labels 0 and 1 are adjacent.

Result 3.5: In a Pyramidal Sum Graph G, 0 and 1 cannot be the label of vertices of the same triangle contained in G.

Proof: Let x, y and z be the vertices of a triangle. If the vertices x and y are labeled with 0 and 1 and z is labeled with some $m \in \{0,1,2,\ldots,p_q\}$ where $m \neq 0$, $m \neq 1$. Then such a vertex labeling will give rise to the edge labels 1 , m+1 and m . In order to admit a Pyamidal Sum labeling m and m+1 must be Pyramidal numbers. But this is not possible as there does not exist consecutive integers which are Pyramidal numbers by Remark 3.2. Hence the result is proved.

Result 3.6: In a Pyramidal Sum graph 1 and 4 cannot be the labels of the vertices of the same Triangle contained in G.

Proof: Let x,y,z be the vertices of the triangle. Let the vertices x and y be labeled with 1 and 4 and Z be labeled with some $m \in \{0,1,2,\ldots,p_q\}$ where $m \neq 1$, $m \neq 4$. Such a vertex labeling will give rise to the edge labels 5, m+1, m+4 . In order to admit a Pyramidal Sum labeling m+1 and m+4 must be Pyramidal numbers. For any m, if $m+1$ is a Pyramidal number, definitely $m+4$ does not yield a Pyramidal number since it is not possible to get 3 as the difference between any two pyramidal numbers. Hence the result is proved.

Result 3.7: If a Graph G contains a cycle of length 3 then G is an Unpyramidal Graph.

Proof :. If G admits a Pyramidal Sum labeling, suppose two vertices of C_3 are labeled with labels 0 and 1 respectively then there is a Triangle having two of the vertices labeled with 0 and 1 which contradicts Result 3.5. Thus G does not admit a Pyramidal Sum labeling. If C_3 does not receive the labels 0 and 1, as the sum of any two Pyramidal numbers is not a Pyramidal number it is not possible.

Result 3.8: The Sum of any two Pyramidal numbers is not a Pyramidal number.

Proof: Suppose the result is not true. Then let p_x and p_y be two Pyramidal numbers such that their sum p_z is also a Pyramidal number. Therefore we have,

$p_x+p_y = p_z$............ (1)

By Remark 3.3

$p_2-p_1 = 2^2$

$p_3-p_2 = 3^2$

....

$p_n - p_{n-1} = n^2$

Hence the Pyramidal number p_x can be written as

$p_x - p_{x-1} = x^2$

Similarly for p_y and p_z we have

$p_y - p_{y-1} = y^2$, $p_z - p_{z-1} = z^2$

Adding all we get ,

$p_x + p_y + p_z - (p_{x-1} + p_{y-1} + p_{z-1})$

$= x^2+y^2+z^2$……(2)

From (1) we have , $p_{x-1} + p_{y-1} = p_{z-1}$..........(3)

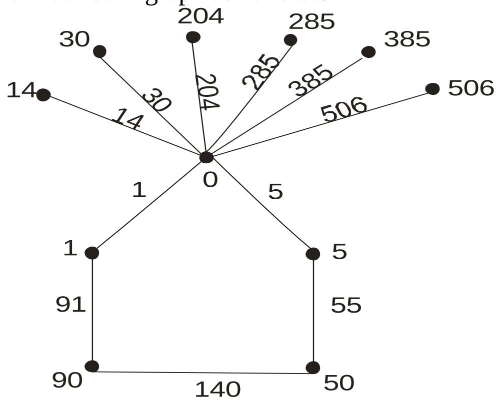
Applying (1) and (3) in (2) we get

$2p_z - 2p_{z-1} = x^2+y^2+z^2$

Therefore, $p_z - p_{z-1} = \frac{x^2+y^2+z^2}{2}$ which is not a perfect square for any positive integers x,y,z where x # 0, y # 0 , z # 0 and x, y, z ≥ 2.

But the difference between any two Pyramidal numbers is a perfect square . Hence this is a contradiction. Therefore the result follows.

Remark 3.9: All Cycles are Unpyramidal Graphs because of Result 3.8. But Tte Cycle C_5 alone can be embedded as an Induced Subgraph of a Pyramidal Sum graph as follows:

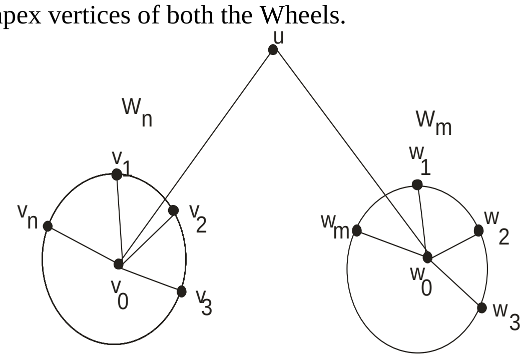


Embedding Of C_5

Remark 3.10: Result 3.7 implies that the Complete Graphs (n ≥ 3), Wheel Graphs, Triangular Crocodiles, Dutchwindmill Graphs, Flower Graphs, Gear Graphs, Helm Graphs, Fan Graphs, Parachute Graphs are Unpyramidal Graphs.

Theorem 3.11: The Graph $<W_n :W_m>$ obtained by joining the apex vertices of Wheels W_n and W_m to a new vertex x is an Unpyramidal graph.

Proof: Let G = $<W_n :W_m>$. Let us denote the apex vertex of W_n by v_0 and the vertices adjacent to v_0 of the Wheel W_n by $v_1, v_2, ..., v_n$. Similarly denote the apex vertex of other Wheel W_m By w_0 and the vertices adjacent to W_0 of the Wheel W_m by $w_1, w_2,...,w_m$. Let 'u' be the new vertex adjacent to the apex vertices of both the Wheels.



The Graph $<W_n :W_m>$

Let f:V$(G) \rightarrow \{0,1,2,...,p_q\}$ be one of the possible Pyramidal Sum Labeling . As 0 and 1 are the labels of any two adjacent vertices of the graph G, we have the following cases:

Case 1: If 0 and 1 are the labels of adjacent vertices in W_n or W_m , then there is a Triangle having two of the vertices labeled with 0 and 1 which is a contradiction to Result 3.5.

Case 2: If f(u) = 0 then one of the vertices v_0, w_0 is labeled with 1. Without loss of generality assume that $f(v_0)$ = 1.To get the edge label p_2 = 5 we discuss the following possibilities:

(A) One of the vertices from $v_1, v_2,.......v_n$ is labeled with 4. Without loss of generality assume that $f(v_i)$ = 4 for some i∈ { 1,2,...,n } . Hence we get a Triangle having two of its vertices labeled with 1 and 4 which is a contradiction to Result 3.6.

(B) If $f(w_0)$= 5, To get the next edge label p_3 = 14 We have the following Subcases.

Subcase 2.1: Assume that $f(v_i)$ = 13 for some i ∈ { 1,2,...,n } . In this Case we get a Triangle whose vertex labels are 1,13 and x. Then x + 13 and x + 1 will be the edge labels of two edges with difference 12 which is not a perfect square.

Subcase 2.2: Assume that 2 and 12 or 3 and 11 or 6 and 8 are the labels of two adjacent vertices from one of the two Wheels. so there exists a Triangle whose vertex labels are either 1,2,12 or 5,2,12 and 1,3,11 or 5,3,11 and 1,6,8 or 5,6,8. In these cases we get an edge label which is not a Pyramidal number.

Subcase 2.3: Assume that $f(w_i)$ = 9 for Some i∈ { 1,2,3,...,m } . In this case we get a Triangle whose vertex labels are 5,9 and x . Then x+9 and x+5 are the labels of two edges which are not Pyramidal numbers for any x.

Case 3: If f(u) = 1, then one of the vertices v_0 and w_0 must be labeled with 0 . Without loss of generality assume that $f(v_0)$ = 0 . To have an edge label 5 we have the following cases:

Subcase 3.1**:** If $f(v_i)$ = 5 for some i ∈ { 1,2,…,n } then there is a Triangle having vertex labels as 0,5,x.. Therefore we have two edge labels as x + 5 and x which are not Pyramidal numbers for any x.

Subcase 3.2: Assume that $f(w_0)$ = 4. To obtain the edge label p_3 = 14 We discuss the following possibilities:

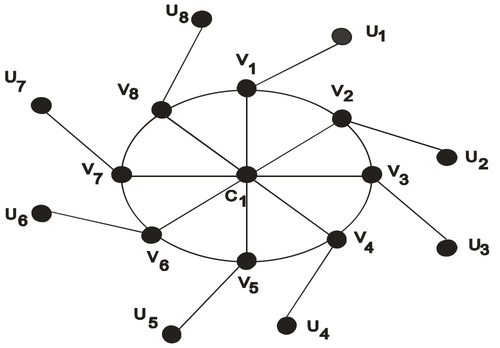
(A) If $f(v_i)$ = 14 for some i ∈{ 1,2,…,n }, then there is a Triangle having vertex labels 0,14,x. Therefore we have two edge labels as x and x + 14 having difference 14 which is not possible.

(B) If $f(w_i)$ = 10 for some i ∈ { 1,2,...,m }, then there is a Triangle with vertex labels 10,4,x . The edge labels are 14, 4 + x , 10 + x whose difference is 6 . As discussed above this is not possible .

(C) Assume that 2 and 12 , 3 and 11 , 5 and 9 , 6 and 8 are the labels of two adjacent vertices from one of the two Wheels. So there is a Triangle whose vertices are either 0,2,12 and 4,2,12 (Or) 0,3,11 and 4,3,11 (Or) 0,5,9 and 4,5,9 (Or) 1,6,8 and 4,6,8 . In all the above possibilities there exists an edge label which is not a Pyramidal number. Hence the Graph G = $<W_n: W_m>$is an Unpyramidal Graph.

Theorem 3.12: The Helm Graph H_n is an Unpyramidal Graph.

Proof: The Helm Graph H_n is obtained from a Wheel $W_n = C_n+K_1$ by attaching a Pendant edge at each vertex of C_n. Let c_1 be the apex vertex of the Wheel and $v_1,v_2,\ldots,v_n$ be the vertices adjacent to c_1 in the clockwise direction. Let the Pendant vertices adjacent to $v_1,v_2,...,v_n$ be denoted by $u_1,u_2,....,u_n$ respectively.



The Helm H_8

If H_n admits a Pyramidal Sum labeling then we have the following cases:

Case 1: Suppose $f(c_1) = 0$. Since the labels 0 and 1 are always adjacent in a Pyramidal Sum graph we assign the label 1 to exactly one of the vertices $v_1,v_2,...,v_n$. In this case there is a Triangle with vertex labels as 0 and 1 which is a contradiction to Result 3.5.

Case 2: Suppose one of the vertices from $v_1,v_2,...,v_n$ is labeled with 0. Without loss of generality assume that $f(v_1) = 0$. Here we consider two subcases:

Subcase 2.1: If one of the vertices from c_1,v_2,v_n is labeled with1, then in each case there is a Triangle with vertex labels as 0 and 1 which is a contradiction to Result 3.5..

Subcase 2.2: If $f(u_1) = 1$ then we discuss the possibility of the next edge label. The label $p_2 = 5$ can be obtained by using the vertex labels 0,5 or 1,4. As u_1 is a pendant vertex having label 1 the combination 1,4 is not possible. Therefore one of the vertices v_2,v_n,c_1 must be labeled with 5. Thus there is a Triangle with two vertices labeled as 0 and 5. Let the third vertex be labeled with x such that $x \neq 0$, $x \neq 5$. To admit a Pyramidal Sum labeling x and x+5 must be distinct Pyramidal numbers. But here the difference is 5 which is not a perfect square. Hence it is not possible for the edges to receive the labels as Pyramidal numbers.

Case 3: Suppose one of the vertices from $u_1,u_2,...,u_n$ is labeled with 0 . Without loss of generality assume that $f(u_1) = 0$. Since 0 and 1 are always adjacent in a Pyramidal Sum labeling we have $f(v_1) = 1$. Now, the edge label $p_2 = 5$ can be obtained by using the vertex labels 0,5 or 1,4. As u_1 is a pendant vertex the combination 0,1 is not possible. Therefore one of the vertices from v_2,v_n,c_1 must be labeled with 4. Hence there is a Triangle with vertex labels as 1 and 4 which is a contradiction to Remark 3.6. Hence in each of the possibilities discussed above, H_n does not admit Pyramidal Sum labeling. Hence H_n is an Unpyramidal Graph.

IV. CONCLUSION

As all Connected acyclic graphs satisfy the condition of Pyramidal Sum Labeling, Pyramidal numbers can be applied in problems connected with Trees. In the above Labeling, the Pyramidal numbers are brought into existence. As the difference between any two Pyramidal nmbers is a perfect square and the difference is sufficiently large, Pyramidal numbers can be used as frequencies in Distance labelings such as L(3,2,1), L(4,3,2,1) and in Radio labelings to make the frequencies of the Transmitters sufficiently large for better transmission.

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