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Elementary way of proving Fermat's theorem

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ABSTRACT
The paper gives the simple way of proving Fermat's last theorem. The method is based on the study of the properties of natural numbers and the analysis of restrictions on the supposed solutions.
Keywords: number theory, natural numbers, Fermat's theorem, Descartes' rule

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I. INTRODUCTION

Great (last) Fermat's theorem was formulated over 300 years ago. In view of the significance of the problem in many areas of mathematic, large, but unsuccessful efforts have been made to prove it. Finally, in [4], a theorem was proved that was accepted by mathematicians and based on its connection with the theory of modular elliptic curves. The proof is too complicated, so attempts were made to find a simple proof in the framework of the number theory. In particular, in [1 – 3] we investigate alternative methods for proving of this theorem. The theorem of P. Fermat, as is known, asserts that equation

$$x^p + y^p = z^p \tag{1}$$

has no positive integer solutions for $p > 2$. In this paper, we prove this theorem in a way in the framework of elementary number theory.

II. RESTRCTIONS ON POSSIBLE SOLUTIONS OF THE EQUATION AND ADMISSIBLE TRANSFORMATIONS

To prove the theorem, we consider restrictions on the possible solutions of equation (1). Let us formulate the first restriction. Put for definiteness that $x < y$, i.e. x always means the smallest number on the left. Since the numbers x, y, z are all different, we have the following inequality:

$$x^p + y^p < (x+y)^p . \tag{2}$$

If a is a positive integer, then *a fortiori*

$$x^p + y^p < (x+y+a)^p . \tag{3}$$

From inequalities (2), (3) and the form of equation (1), *the first restriction* for numbers as possible solutions of equation (1) follows:

$$\max(x, y) < z < (x+y) . \tag{4}$$

The second restriction is associated with the obvious requirement that the number $x^p + y^p$ ends in the same digit as the number z^p . In particular, it follows that the left and right sides of equation (1) must be of the same parity. Let us formulate *the third restriction*. If the following relation holds

$$x^2 + y^2 \le z^2 , \tag{5}$$

then x, y, z are not solutions of the basic equation (1). In this case, strict inequalities hold

$$x^3 + y^3 < z^3 ,$$
$$x^4 + y^4 < z^4 ,...$$
$$x^p + y^p < z^p . \tag{6}$$

Indeed, multiplying (5) by z and using the left-hand side of inequality (4), we have $z^3 \ge zx^2 + zy^2 > x^3 + y^3$. Multiplying this inequality by z and using (4), we obtain $z^4 > z(x^3 + y^3) > x^4 + y^4$ etc. *The fourth restriction* is the equality of the exponents of all components in equation (1). The restrictions formulated are necessary conditions for numbers to be solutions of the basic equation (1). They are quite strong and allow us to select the proposed solutions for equation (1). *The second and fourth restrictions are called basic*, since their fulfillment is an unconditional requirement. The first and third restrictions are auxiliary and can be provided through transformations (see below). Now consider the permissible transformations that keep safe the restrictions. As the starting point we take the "elementary" bases of degree – the smallest natural numbers from the first tens that satisfy the basic restrictions. Such transformations include:

1. Multiplication of all bases of degree in (1) by a positive integer $l = 2, 3, \ldots$. Since the starting point is "elementary" bases, division is excluded.
2. An increase of one, two or all three bases by a number $a = 10k$ that is a multiple of 10, where $k = 1, 2, \ldots$. Since the starting point is "elementary" bases, the first increase is 10.
3. If two triplets of numbers x, y, z and x', y', z' satisfy the second restriction, then the triplet of numbers (bases) $\{x'' = x^p + x'^p$, $y'' = y^p + y'^p$, $z'' = z^p + z'^p\}$ satisfies this restriction.

We call triplets obtained from elementary triplets through transformations, derived triplets. The first transformation is useful for obtaining from the known solution all solutions of the same class. For example, it can be used to obtain solutions of equation (1) for $p = 2$ (see below). In our proof it is not used. This transformation does not change the "status" of the triplet, i.e. if the triplet is not a solution of (1), then after this transformation it will not be a solution of (1). Therefore, in proving the theorem without loss of generality, it is sufficient to consider only *prime* triplets, namely, those in which the bases do not have a common divisor different from 1. The second transformation is used to provide the first restriction (4) if it does not hold for "elementary" bases, but the second restriction holds. This transformation is the main one in the proof. The third transformation can be applied only if some solution is known, for example, when solving equation (1) for $p = 2$. In our proof it is not used. Thus, the main "generator" of allowed combinations of triplets of numbers is the second transformation. By induction on the number 10 it is easy to prove that the second transformation allows us to go over all the numbers that are admissible by restrictions. Let's take a detailed look at the second restriction, for which we analyze the degrees of "elementary" numbers (bases) from 1 to 10, starting with degree 3. We have replaced 10 by 0, so as not to violate the uniformity of the representation (see below). The results are presented in table 1.

Table 1: Admissible ends of powers for elementary bases

Number	Last digit of number	Number	Last digit of number	Number	Last digit of number	Number	Last digit of number
2^3	8	3^3	7	4^3	4	5^3	5
2^4	6	3^4	1	4^4	6	5^4	5 (repeat)
2^5	2	3^5	3	4^5	4 (repeat)		
2^6	4	3^6	9				
2^7	8 (repeat)	3^7	7 (repeat)				
Number	Last digit of number	Number	Last digit of number	Number	Last digit of number	Number	Last digit of number
6^3	6	7^3	3	8^3	2	9^3	9
6^4	6 (repeat)	7^4	1	8^4	6	9^4	1
		7^5	7	8^5	8	9^5	9 (repeat)
		7^6	9	8^6	4		
		7^7	3 (repeat)	8^7	2 (repeat)		
Number	Last digit of number	Number	Last digit of number				
0^3	0	1^3	1				
0^4	0 (repeat)	1^4	1(repeat)				

It follows from the data in table 1 that the repetition period of the last digit for bases 2, 3, 7 and 8 is 4, for bases 4 and 9 the period is 2, for bases 5, 6, 0 and 1, the period is 1. Now we consider combinations of powers of different bases, taking into account the basic restrictions. The analysis is performed in the following order. First, we consider combinations of numbers with period 4, i.e., powers of 2 are combined consistently with the degrees of the numbers 3, 4,..., 0, 1, then the number 3 is combined with the remaining, the number 7– with the remaining, the number 8 – with the remaining. After that, the bases with a period of 2 are combined, i.e. number 4 – with the numbers 5, 6, 9, 0 and 1, then the number 9 – with 5, 6, 0 and 1. Lastly, the numbers with period 1 are combined, i.e. number 5 with the numbers 6, 0 and 1, the number 6 – with 0 and 1, the number 0 – with 1. Note that some of the combinations can be immediately excluded due to violation of the second restriction. The results of the analysis are given in table 2.

Table 2:Combination of ends for "elementary" bases permissible under basic restrictions

$2^{3+2k}+3^{3+2k}=5^{3+2k}$	$3^{3+4k}+4^{3+4k}=1^{3+4k}$	$7^{3+4k}+4^{3+4k}=3^{3+4k}$	$8^{3+4k}+4^{3+4k}=6^{3+4k}$
$2^{4+2k}+4^{4+2k}=0^{4+2k}$	$3^{5+4k}+4^{5+4k}=7^{5+4k}$	$7^{5+4k}+4^{5+4k}=1^{5+4k}$	$8^{5+4k}+4^{5+4k}=2^{5+4k}$
$2^{5+4k}+4^{5+4k}=6^{5+4k}$	$3^{6+4k}+4^{6+4k}=5^{6+4k}$	$7^{6+4k}+4^{6+4k}=5^{6+4k}$	$8^{6+4k}+4^{6+4k}=0^{6+4k}$
$2^{3+k}+5^{3+k}=7^{3+k}$	$3^{3+k}+5^{3+k}=8^{3+k}$	$7^{3+k}+5^{3+k}=2^{3+k}$	$8^{3+4k}+5^{3+4k}=3^{3+4k}$
$2^{4+4k}+5^{4+4k}=1^{4+4k}$	$3^{4+4k}+5^{4+4k}=6^{4+4k}$	$7^{4+4k}+5^{4+4k}=6^{4+4k}$	$8^{4+4k}+5^{4+4k}=1^{4+4k}$
$2^{6+4k}+5^{6+4k}=3^{6+4k}$	$3^{6+4k}+5^{6+4k}=2^{6+4k}$	$7^{4+4k}+5^{4+4k}=4^{4+4k}$	$8^{4+2k}+5^{4+2k}=7^{4+2k}$
$2^{3+4k}+6^{3+4k}=4^{3+4k}$	$3^{3+4k}+6^{3+4k}=7^{3+4k}$	$7^{4+2k}+5^{4+2k}=8^{4+2k}$	$8^{3+k}+5^{3+k}=3^{3+k}$
$2^{5+4k}+6^{5+4k}=8^{5+4k}$	$3^{5+4k}+6^{5+4k}=9^{5+4k}$	$7^{3+4k}+6^{3+4k}=9^{3+4k}$	$8^{6+4k}+5^{6+4k}=7^{6+4k}$
$2^{6+4k}+6^{6+4k}=0^{6+4k}$	$3^{6+4k}+6^{6+4k}=5^{6+4k}$	$7^{5+4k}+6^{5+4k}=3^{5+4k}$	$8^{3+4k}+6^{3+4k}=2^{3+4k}$
$2^{3+4k}+7^{3+4k}=1^{3+4k}$	$3^{3+2k}+7^{3+2k}=0^{3+2k}$	$7^{6+4k}+6^{6+4k}=5^{6+4k}$	$8^{5+4k}+6^{5+4k}=4^{5+4k}$
$2^{5+4k}+7^{5+4k}=9^{5+4k}$	$3^{3+4k}+8^{3+4k}=9^{3+4k}$	$7^{3+2k}+8^{3+2k}=5^{3+2k}$	$8^{6+4k}+6^{6+4k}=0^{6+4k}$
$2^{3+2k}+8^{3+2k}=0^{3+2k}$	$3^{5+4k}+8^{5+4k}=1^{5+4k}$	$7^{3+4k}+9^{3+4k}=8^{3+4k}$	$8^{3+4k}+9^{3+4k}=1^{3+4k}$
$2^{3+4k}+9^{3+4k}=3^{3+4k}$	$3^{3+4k}+9^{3+4k}=6^{3+4k}$	$7^{5+4k}+9^{5+4k}=6^{5+4k}$	$8^{5+4k}+9^{5+4k}=7^{5+4k}$
$2^{5+4k}+9^{5+4k}=1^{5+4k}$	$3^{5+4k}+9^{5+4k}=2^{5+4k}$	$7^{6+4k}+9^{6+4k}=0^{6+4k}$	$8^{6+4k}+9^{6+4k}=5^{6+4k}$
$2^{6+4k}+9^{6+4k}=5^{6+4k}$	$3^{6+4k}+9^{6+4k}=2^{6+4k}$	$7^{4+4k}+0^{4+4k}=1^{4+4k}$	$8^{4+2k}+0^{4+2k}=2^{4+2k}$
$2^{3+4k}+1^{3+4k}=9^{3+4k}$	$3^{6+4k}+9^{6+4k}=0^{6+4k}$	$7^{6+4k}+0^{6+4k}=3^{6+4k}$	$8^{4+4k}+0^{4+4k}=6^{4+4k}$
$2^{4+4k}+0^{4+4k}=4^{4+4k}$	$3^{4+2k}+0^{4+2k}=7^{4+2k}$	$7^{6+4k}+4^{6+4k}=5^{6+4k}$	$8^{3+4k}+1^{3+4k}=7^{3+4k}$
$2^{4+2k}+0^{4+2k}=8^{4+2k}$	$3^{4+4k}+0^{4+4k}=9^{4+4k}$	×	$8^{5+4k}+1^{5+4k}=9^{5+4k}$
$2^{5+4k}+1^{5+4k}=3^{5+4k}$	$3^{3+4k}+1^{3+4k}=2^{3+4k}$	$5^{3+k}+6^{3+k}=1^{3+k}$	$8^{6+4k}+1^{6+4k}=5^{6+4k}$
$2^{6+4k}+1^{6+4k}=5^{6+4k}$	$3^{5+4k}+1^{5+4k}=4^{5+4k}$	$5^{3+k}+6^{3+k}=1^{3+k}$	$8^{4+4k}+5^{4+4k}=9^{4+4k}$
×	$3^{6+4k}+1^{6+4k}=0^{6+4k}$	$5^{3+k}+1^{3+k}=6^{3+k}$	×
$4^{3+4k}+5^{3+4k}=9^{3+4k}$	×	$5^{4+4k}+6^{4+4k}=3^{4+4k}$	$6^{3+4k}+1^{3+4k}=3^{3+4k}$
$4^{4+4k}+5^{4+4k}=3^{4+4k}$	$9^{3+k}+5^{3+k}=4^{3+k}$	$5^{4+4k}+6^{4+4k}=7^{4+4k}$	×
$4^{4+4k}+5^{4+4k}=7^{4+4k}$	$9^{4+2k}+5^{4+2k}=6^{4+2k}$	×	$0^{4+4k}+1^{4+4k}=3^{4+4k}$
$4^{3+4k}+6^{3+4k}=0^{3+4k}$	$9^{4+4k}+5^{4+4k}=2^{4+4k}$		$0^{4+4k}+1^{4+4k}=7^{4+4k}$
$4^{3+4k}+9^{3+4k}=7^{3+4k}$	$9^{4+4k}+5^{4+4k}=8^{4+4k}$		$0^{4+2k}+1^{4+2k}=9^{4+2k}$
$4^{4+4k}+0^{4+4k}=2^{4+4k}$	$9^{3+2k}+6^{3+2k}=5^{3+2k}$		×
$4^{4+4k}+0^{4+4k}=8^{4+4k}$	$9^{4+2k}+0^{4+2k}=1^{4+2k}$		
$4^{3+4k}+1^{3+4k}=5^{3+4k}$	$9^{4+4k}+0^{4+4k}=3^{4+4k}$		
×	$9^{4+4k}+0^{4+4k}=7^{4+4k}$		
	$9^{3+2k}+1^{3+2k}=0^{3+2k}$		
	×		

Note. The relationships given in table 2 do not mean actual equality and they are symbolic notation representing the fulfillment of the basic restrictions necessary for equation (1), namely, the coincidence of the exponents of all components and the coincidence of the last digit, to which the left and right sides of equation (1) end; the first number in the exponent is the smallest permissible exponent; k = 1, 2, 3,….

The data of table 2 we use to select the permissible combinations of elementary bases by means of the first and third restrictions. We have replaced 0 and 1 by 10 and 11 respectively, i.e. the smallest bases that make sense. Here we give selective results of the analysis of combinations for base 2. We have $2^{3+4k}+3^{3+4k}=5^{3+4k}$ (here and below k = 1, 2, ...) – the combination is impossible because of condition (4); $2^{5+4k}+3^{5+4k}=5^{5+4k}$ is impossible because of condition (4); $2^{3+4k}+4^{3+4k}=8^{3+4k}$ – impossible by virtue of (4); $2^{4+4k}+4^{4+4k}=10^{4+4k}$ – impossible by (4); $2^{5+4k}+4^{5+4k}=6^{5+4k}$ is impossible by virtue of (4); $2^{6+4k}+4^{6+4k}=10^{6+4k}$ is impossible by virtue of (4) and so on. Similarly, analysis is performed for other elementary bases (see table 2). The results of the analysis show that the "candidates" for the solution of equation (1) satisfying the basic restrictions are the following combinations: $3^{6+4k}+4^{6+4k}=5^{6+4k}$, $3^{4+4k}+5^{4+4k}=6^{4+4k}$, $3^{3+4k}+6^{3+4k}=7^{3+4k}$, $3^{3+4k}+8^{3+4k}=9^{3+4k}$, $3^{6+4k}+9^{6+4k}=10^{6+4k}$, $7^{4+4k}+5^{4+4k}=8^{4+4k}$, $7^{6+4k}+5^{6+4k}=8^{6+4k}$, $7^{3+4k}+6^{3+4k}=9^{3+4k}$, $7^{6+4k}+9^{6+4k}=10^{6+4k}$, $7^{4+4k}+10^{4+4k}=11^{4+4k}$, $8^{3+4k}+9^{3+4k}=11^{3+4k}$, $8^{6+4k}+10^{6+4k}=12^{6+4k}$, $5^{4+4k}+6^{4+4k}=7^{4+4k}$, $5^{4+4k}+8^{4+4k}=9^{4+4k}$, $9^{4+2k}+10^{4+2k}=11^{4+2k}$ and several others (for completeness of consideration, we have included several bases from the second ten). Applying conditions (5), (6) to these combinations, we see that most of them can not be a solution of equation (1) for p > 2. After selection according to the conditions (5), (6) remain combinations: $7^{4+4k}+5^{4+4k}=8^{4+4k}$, $5^{4+4k}+8^{4+4k}=9^{4+4k}$ $7^{6+4k}+5^{6+4k}=8^{6+4k}$, $7^{3+4k}+6^{3+4k}=9^{3+4k}$, $5^{4+4k}+6^{4+4k}=7^{4+4k}$ $7^{6+4k}+9^{6+4k}=10^{6+4k}$, $7^{4+4k}+10^{4+4k}=11^{4+4k}$, $8^{3+4k}+9^{3+4k}=11^{3+4k}$, $9^{4+2k}+10^{4+2k}=11^{4+2k}$, $8^{6+4k}+10^{6+4k}=12^{6+4k}$, $8^{3+4k}+14^{3+4k}=16^{3+4k}$, which should be considered separately, since for them the conditions (5), (6) are not satisfied. How to make sure, that not a combination of elementary bases is lost. We must consider elementary bases (triplets) for which the strict inequality exists in relation (5), and all other restrictions are satisfied. The elementary triplet (3, 4, 5) corresponds to the equality of the left and right sides in (5). Therefore, interest is represented by bases for which the relations $3<x<y<z\le 10$ are satisfied, which immediately gives the desired result. We have five elementary prime triplets: (5, 6, 7), (5, 7, 8), (5, 8, 9), (6, 7, 9) and (7, 9, 10). This result shows how effective simple restrictions are used. We note that such an analysis, taking into account possible transformations, allows us to obtain solutions of the quadratic equation for p =2. For "elementary" bases, such solutions are obtained as an additional result of our study (see below). It is easy to verify that the theorem is true for elementary prime triplets. Indeed. For triplet (5, 6, 7) we have $5^2+6^2>7^2$, but $5^3+6^3<7^3$, therefore all other powers of these bases will give the same result (zero is passed). For triplet (5, 7, 8) we have $5^2+7^2>8^2$, but $5^3+7^3<8^3$ and $5^4+7^4<8^4$(zero is passed). For triplet (6, 7, 9) we obtain $6^2+7^2>9^2$, but $6^3+7^3<9^3$ (zero is passed). For triplet (5, 8, 9) we have $5^2+8^2>9^2$, but $5^3+8^3<9^3$(zero is passed). For triplet (7, 9, 10) we obtain $7^2+9^2>10^2$, $7^3+9^3>10^3$, but $7^4+9^4<10^4$ (zero is passed). Similarly, we verify the validity of the theorem for the remaining triplets. For triplet (7, 10, 11) we have $7^2+10^2>11^2$, $7^3+10^3>11^3$, but $7^4+10^4<11^4$ (zero is passed). For triplet (8, 9, 11) we have $8^2+9^2>11^2$, but $8^3+9^3<11^3$ (zero is passed). For triplet (9, 10, 11) we obtain $9^2+10^2>11^2$, $9^3+10^3>11^3$, $9^4+10^4>11^4$, but $9^5+10^5<11^5$ (zero is passed). For triplet (8, 10, 12) we have $8^2+10^2>12^2$, but $8^3+10^3<12^3$ (zero is passed). For triplet (8, 14, 16) we have $8^2+14^2>16^2$, but $8^3+14^3<16^3$ (zero is passed) and so on (see tables 3 – 6). We note that triplets (8, 10, 12), (8, 14, 16) reduce to elementary triplets (4, 5, 6), (4, 7, 8) for which the theorem is obviously satisfied by virtue of the second restriction.

III. PROOF OF THE THEOREM AND STUDY OF THE PROPERTIES OF ASSUMED SOLUTIONS

Let us prove in general case that equation (1) cannot have solutions among natural numbers. According to the rule of Descartes, increasing the variable by a number a decreases all roots by the same number. As we have shown in the foregoing analysis, for elementary bases (triplets) there are no solutions of (1) among the natural numbers for p > 2, therefore with increasing bases by the number a = 10k, *a fortiori* there will be no such solutions. Indeed, equation (1) admits a representation $(z-u)^p+(z-v)^p-z^p=0$, where x, y, z are elementary bases, and $x=z-u$, $y=z-v$. We will consider z as a variable, and u and v as parameters. The resulting equation considered with respect to z, of course, has roots. If these roots are not natural numbers, then the theorem is proved. Suppose that this equation has at least one natural root (if there are several, then we take the largest one). We denote it by z_0, then the values of $x_0 = z_0 - u$ and $y_0 = z_0 - v$ are uniquely determined. All three numbers z_0, x_0 and y_0 cannot be simultaneously natural numbers, as follows from our previous analysis. If we increase z by $a = 10k$, then $z \to z + 10k$, $x \to z + 10k - u = x + 10k$, $y \to z + 10k - v = y + 10k$. It should be kept in mind that when z changes by a, the quantities u and v do not change (see below). In this case, the root will decrease by the same number $10k$, i.e. $z_0 \to z_0 - 10k$, and likewise $x_0 \to x_0 - 10k$, $y_0 \to y_0 - 10k$. Therefore, the root will not be a natural number. If, conversely, we reduce z by $10k$, then z will go beyond the set of natural numbers. Let us explain this with a concrete example. Let $u = 2$, $v = 1$; the permitted degree $p = 6$, then we have an equation $(z-2)^6 + (z-1)^6 - z^6 = 0$ in which z is an elementary base and varies within [1, 10]. The obvious solution is $z_0 = 1$, then $x_0 = -1$, $y_0 = 0$, i.e. three numbers z_0, x_0 and y_0 are not natural numbers at the same time. With increasing z for example by a =10 (here k = 1), it (and also x and y) will go into the second ten, and the equation is transformed into $(z+10-2)^6 + (z+10-1)^6 - (z+10)^6 = 0$. The root z_0 will decrease by 10, i.e. $z_0 \to z_0 - 10 = -9$, and likewise $x_0 \to x_0 - 10 = -11$, $y_0 \to y_0 - 10 = -10$, so solutions will not be a natural numbers. With decreasing z by $a = 10$ it (and also x and y) goes beyond the limits of the set of natural numbers, i.e. will not satisfy the conditions of the theorem. Thus, synchronous (simultaneous) change of bases by the value $a = 10k$ does not change the class of solutions. *Fermat's theorem is proved.* However, we will continue to investigate the properties of the assumed solutions, since this makes it possible to understand the reasons why equation (1) does not have natural solutions. Let us consider the role of transformations in more detail. The results of the analysis for elementary bases show that for several of them the conditions (4), (5) are not satisfied. To achieve it, we use the second transformation, i.e. an increase in all or some of the bases by a number $a = 10k$, so that conditions (4), (5) were satisfied. A multiple of 10 is added so that the basic condition is not violated. We call a triplet (x, y, z), where $x < y$, regular if it satisfies conditions (4), (5) and irregular otherwise. Several cases are possible. If an elementary triplet is regular, then increasing three bases by 10, we obtain the regular triplet. For example, for p = 4 we have (5, 6, 7) → (15, 16, 17) → (25, 26, 27) etc. But only one of the bases cannot be increased by 10, since condition (4) will be violated. Similarly, if initial elementary triplet is irregular and condition (4) is satisfied, then increasing all three bases by 10, we obtain regular triplet. For example, for p = 4 we have (5, 6, 9) → (15, 16, 19) → (25, 26, 29) etc. If initial elementary triplet is irregular, and $z = x + y$, i.e. condition (4) is not satisfied, then increasing all three bases by 10, we obtain regular triplet. For example, for p = 5 we have (3, 4, 7) → (13, 14, 17) → (23, 24, 27) etc. If initial elementary triplet is irregular, and $x < z < y$, then a regular triplet is obtained by increasing bases x and z by 10 or x and y by 10, but z by 20. For example, for p = 6 we have (4, 7, 5) → (7, 14, 15), (14, 17, 25) → (24, 27, 35) etc. If initial elementary triplet is irregular, and $z < x < y$, then a regular triplet is obtained by increasing bases x and y by 10, but z by 20. For example, we have (5, 7, 4) → (15, 17, 24) → (25, 27, 34) etc. Finally, if $z > (x + y)$, then a regular triplet is obtained by increasing all three bases by 10. For example, for p = 6 we have (2, 3, 7) → (12, 13, 17) → (22, 23, 27) etc. So, in the second or the third tens all triplets become regular and further all regular triplets are obtained by synchronously increasing all the bases by a number $a = 10k$, but, of course, sequentially, i.e. first by 10, then by 20, etc. In this case conditions (4), (5) are satisfied and, at the same time, the basic condition is not violated, i.e. the ends of numbers on the left and right sides of equation (1) do not change. The analysis of the bases in the second tens can be carried out directly according to the example of analysis for elementary bases and does not, in principle, add anything new to the solution of the main problem. In the second tens, the same relationships are used as for elementary bases, namely, if $x^2 + y^2 \le z^2$, then such bases are excluded from consideration, if $x^2 + y^2 > z^2$, then the sign of the difference $x^l + y^l - z^l$ is verified for the first admissible degree (or for degree 3 – for the reduction of calculations). If this sign is positive, then l increases by one, if negative, then the test is ended, and these bases can be excluded from consideration. Further the analysis is carried out in the third tens (from 20 to 30), etc. The process quickly "converges" in the sense that the number of bases analyzed decreases, and after a few steps there are only bases for which the inequality $x^2 + y^2 > z^2$ holds (see below). The method of induction easily proves that in this way all the admissible bases are obtained. Nevertheless, such an analysis may seem laborious, so let us give a general relationship that simplifies the analysis. We have for any fixed p the following relation

$$(x+a)^p+(y+a)^p-(z+a)^p = (x^p+y^p-z^p)+C_p^1(x^{p-1}+y^{p-1}-z^{p-1})a+ C_p^2(x^{p-2}+y^{p-2}-z^{p-2})a^2+...+C_p^{p-2}\cdot (x^2+y^2-z^2)a^{p-2}+C_p^{p-1}(x+y-z)\cdot a^{p-1}+a^p \tag{7}$$

In the expression (7), the elementary bases are taken as the initial ones, or the smallest ones for which the left-hand side of condition (4) is satisfied, so that all bases are regular even with their increase by 10. Note that if the base in the right-hand side increases by a to increase by a only one base in the left-hand side of equation (1), then (7) is transformed to the form

$$(x+a)^p+y^p-(z+a)^p=(x^p-z^p)+ C_p^1(x^{p-1}-z^{p-1})a+C_p^2(x^{p-2}-z^{p-2})a^2+...+ C_p^{p-2}(x^2-z^2)a^{p-2}+C_p^{p-1}(x-z)a^{p-1} \tag{7a}$$

All the terms on the right-hand side of (7a) are less than zero, so equality is impossible, and such bases cannot be solutions of equation (1). However, with the subsequent increase of both numbers x and y by a, additional combinations can be obtained (see below). Let us introduce the characteristic quantities in the framework of our approach. Designate $x + y - z = b$, $u = z - x$, $\nu = z - y$, $\Delta = x - y = u - \nu$, $x^2 + y^2 - z^2 = c$, $x^3 + y^3 - z^3 = d$, $x^4 + y^4 - z^4 = h$, $x^5 + y^5 - z^5 = f$. All numbers are natural, and for definiteness we take, as above, that $x < y$. We give the following relationships that are convenient for subsequent calculations:

$$d \equiv x^3+y^3-z^3=b^3-3uv(z+b), \tag{8}$$

$$h \equiv x^4+y^4-z^4=b^4-4uv\cdot [b^2+z(z+b)]+2u^2v^2, \tag{9}$$

$$c \equiv x^2+y^2-z^2=b^2-2uv, \tag{10}$$

$$b \equiv x+y-z=z-(u+v), \tag{11}$$

$$f \equiv x^5+y^5-z^5=b^5-5u\nu\cdot [b^3+3bz^2+z(z-b)^2-u\nu(z+b)], \tag{12}$$

$$g \equiv x^6+y^6-z^6=(b+u)^6+ (b+v)^6-(b+u+v)^6, \tag{13}$$

etc.

Relation (10) is obtained if we express x and y through z in expression $x^2 + y^2 - z^2 = c$ and perform simple transformations. The relations (8), (9), (12), (13) are obtained similarly. It is easy to derive a general relationship. For this we designate $x^{p-1}+y^{p-1}=h_{p-1}+z^{p-1}$; using (11), we multiply this equation on the left by $x + y$, and on the right by $z + b$. Carrying out the grouping, we obtain the recurrence relation

$$h_p \equiv x^p+y^p-z^p=z^{p-1}b+h_{p-1}\cdot (z+b)-(z-u)(z-\nu)(h_{p-2}+z^{p-2}), \tag{14}$$

where $h_1=b$, $h_2=c$, etc.; $p > 2$.

Let us see how the characteristic quantities introduced above change if the elementary bases increase by a number a. Relations (8) – (13) are convenient since they allow us to express the quantities of interest through a quantity b, variation of which proportional to a, and the quantities u and ν which do not depend on the value of a. From (8) – (13) it follows that the value b will increase by a, the value of c will increase by $a(2b+a)$, d increases by $a(3c+3ba+a^2)$, the quantity h increases by $a(4d+6ca+4ba^2+a^3)$ etc., the values u, ν and Δ will not change. Consider the following cases: $b<0$, $b=0$ and $b>0$. We can say that the "fate" of the natural solutions of equation (1) is already determined for p = 1 and p = 2. The bases grouped according to the selected cases are given in table 3, and the results of the analysis for different b are given in tables 4 – 6. The results of the analysis for bases with $b<0$ (this case occurs only up to the first increase in elementary bases by 10) are presented in table 4. We note that all the characteristic quantities in this case are negative, and we are interested in their transition through zero. When the elementary bases are increased by the number $a=10k$, the characteristic values change as follows. Let's put a = 10 (k = 1). Since we consider prime elementary bases, $|b|<10$, b varies from -6 to -2 so then b will become positive for all triplets and remains positive with further increase of a. The value of c changes by 20 at $b=-4$, by 60 at $b=-2$, and by -20 at $b=-6$. When a = 20 value of c increases by 240 at $b=-4$, by 320 at $b=-2$, and by 160 at $b=-6$. When a = 30 value of c increases by 660 at $b=-4$, by 780 at $b=-2$, and by 540 at $b=-6$, i.e. becomes positive for all triplets. The value of d changes as follows. For a = 10, d varies from -1400 ($b=-2, uv=32$) to -5420 ($b=-2, uv=99$). For a = 20, d varies from number 2000 ($b=-2, uv=32$) to number -6040 ($b=-2, uv=99$). For a = 30, d varies from number $16200\,(b=-2\,,uv=32)$ to number $4140\,(b=-2\,,uv=99)$. As a result, we obtain that the value *d* becomes positive when *a* = 30($b=-2$), when *a* = 40($b=-4$) and when *a* = 50($b=-6$). Similarly, an analysis is carried out for the quantities *h*, *f* and *g*. The quantity *g* corresponds to the largest permitted degree 6 (taking into account the repetition period of the last digit). In particular, the quantity *h* becomes positive when *a* = 30 ($b=-2\,,uv=32$) and when *a* = 50 ($b=-2\,,uv=99$). Let us consider the case *b* = 0. The results of the analysis are given in table 5. Here, all calculations are greatly simplified. The final result is as follows. The value of *c* becomes positive, when *a* = 20, for the whole range of its values, changing the sign from minus to plus when *a* changes from 10 to 20. The value of *d* becomes positive, when *a* = 30, for the whole range of values, changing the sign from minus to plus when *a* changes from 10 to 30 (depending on the values of *u* and *v*). The quantity *h* becomes positive, when *a* = 50, changing the sign from minus to plus when *a* changes from 20 to 50 (depending on the values of *u* and *v*). Similarly, an analysis is carried out for the quantities *f* and *g*. Let us consider the case *b* > 0. The results are given in table 6. The final result is as follows. The quantity *b* takes the values 2, 4, 6, 8, 10 and when the bases are increased by *a* = 10 its range is from 12 to 20. The quantity *c* becomes positive when *a* = 10. We note that for the case *b* > 0 some of the initial values of *c* are zero or positive. For the quantities *d* and *h* we give estimates for the "worst" case *b* = 2. For the remaining *b*, these quantities become positive even at *a* = 10 and *a* = 20. For *a* = 10, the change in *d* is from 1600 ($b=2\,,uv=2$) to -224 ($b=2\,,uv=66$). Its change is more than zero for $172>6uv$ and less than zero for $172<6uv$; the equality to zero is impossible for the admissible values of the quantities, since 172 is not divisible by 3. For *a* = 20, the change in *d* is from 10400 ($b=2\,,uv=2$) to 2720 ($b=2\,,uv=66$). For *a* = 30, the change in *d* is 20880 ($b=2\,,uv=66$), i.e. all values of *d* become positive. For the value of *h*, the following results are obtained. For *b* = 2 and *a* = 10, the variation of *h* is from number **16640** ($uv=2\,,u+v=3$) to number $-14560\,(uv=66\,,u+v=17)$. For *a* = 20, the change will be from **221280** ($uv=2\,,u+v=3$) to -415200 ($uv=66\,,u+v=17$). For *a* = 30, the change will be from **1022820** ($uv=2\,,u+v=3$) to -163200 ($uv=66\,,u+v=17$). Finally, for *a* = 40, the change is 295200($uv=66\,,u+v=17$), and all the values of *h* become positive. Note that the change cannot be zero for the combinations of quantities under consideration. Selective calculations were carried out according to the general relations (7), (8) – (13), and for the verification tables of degrees were used. Their purpose is to determine the scale and trends of the change in characteristic quantities. The transition of the characteristic quantities through zero is hidden from our view and does not occur for integer, positive values for *p* > 2 for the allowed exponents. From table 6, as a side (additional) result, solutions of the quadratic equation (1) for *p* = 2 are obtained. In the range of elementary bases and the closest to them, we have four families of such solutions: {(3, 4, 5), (6, 8, 10), (9, 12, 15)}, {(8, 15, 17)}, {(5, 12, 13)}, {(7, 24, 25)}. Summarize our previous consideration. We designate $F(p,x,y,z)=x^p+y^p-z^p$. Obviously, if $b\le 0$, then *F*(*p*, *x*, *y*, *z*) < 0 (for *p* > 2) by virtue of (4) – (6). For any fixed *p* > 2, increasing the initial elementary bases by *a* = 10*k*, one can observe a change in the sign of the quantity *F*(*p*, *x*, *y*, *z*) from minus to plus due to the fact that, as follows from (7), the quantity a^p will predominate over the rest of members. On the other hand, for any fixed base (or, which is the same, for any *a* = 10*k*), increasing the exponent *p* > 2, one can observe a change in the sign of the quantity *F*(*p*, *x*, *y*, *z*) from plus to minus due to the fact that, as follows from (7), z^p will predominate over the rest of members (as *z* > max(*x*, *y*)). For *p* > 2, the equality of *F*(*p*, *x*, *y*, *z*) to zero on the set of natural numbers is impossible for any fixed *a* = 10*k* and for any fixed *p*.

Table 3: Combinations of prime "elementary" and derived bases permissible under basic conditions for different *b*

$b<0$			
$2^{4+4k}+5^{4+4k}=9^{4+4k}$	$3^{6+4k}+5^{6+4k}=12^{6+4k}$	$4^{3+4k}+9^{3+4k}=17^{3+4k}$	$7^{6+4k}+6^{6+4k}=15^{6+4k}$
$2^{4+4k}+5^{4+4k}=11^{4+4k}$	$3^{3+4k}+9^{3+4k}=16^{3+4k}$	$7^{3+4k}+4^{3+4k}=13^{3+4k}$	$7^{3+4k}+9^{3+4k}=18^{3+4k}$
$2^{6+4k}+5^{6+4k}=13^{6+4k}$	$3^{4+4k}+10^{4+4k}=17^{4+4k}$	$7^{6+4k}+4^{6+4k}=15^{6+4k}$	$4^{4+2k}+5^{4+2k}=11^{4+2k}$
$2^{3+4k}+7^{3+4k}=11^{3+4k}$	$3^{6+4k}+10^{6+4k}=17^{6+4k}$	$7^{4+4k}+5^{4+4k}=16^{4+4k}$	$8^{6+4k}+5^{6+4k}=17^{6+4k}$

$2^{3+4k}+9^{3+4k}=13^{3+4k}$	$3^{4+4k}+10^{4+4k}=19^{4+4}$	$9^{4+2k}+5^{4+2k}=16^{4+2k}$	$8^{4+4k}+5^{4+4k}=17^{4+4k}$
$2^{6+4k}+9^{6+4k}=15^{6+4k}$	$5^{4+k}+6^{4+k}=13^{4+k}$	$7^{4+4k}+5^{4+4k}=14^{4+4k}$	$9^{4+4k}+5^{4+4k}=18^{4+4k}$
$3^{3+4k}+4^{3+4k}=11^{3+4k}$	$4^{4+4k}+5^{4+4k}=13^{4+4k}$		
$b=0$			
$2^{3+2k}+3^{3+2k}=5^{3+2k}$	$3^{5+4k}+6^{5+4k}=9^{5+4k}$	$7^{5+4k}+4^{5+4k}=11^{5+4k}$	$8^{3+k}+5^{3+k}=13^{3+k}$
$2^{3+k}+5^{3+k}=7^{3+k}$	$3^{3+2k}+7^{3+2k}=10^{3+2k}$	$7^{3+k}+5^{3+k}=12^{3+k}$	$8^{5+4k}+9^{5+4k}=17^{5+4k}$
$2^{5+4k}+7^{5+4k}=9^{5+4k}$	$3^{5+4k}+8^{5+4k}=11^{5+4k}$	$7^{5+4k}+6^{5+4k}=13^{5+4k}$	$9^{3+k}+5^{3+k}=14^{3+k}$
$2^{5+4k}+9^{5+4k}=11^{5+4k}$	$3^{3+k}+10^{3+k}=13^{3+k}$	$7^{3+2k}+8^{3+2k}=15^{3+2k}$	$9^{3+2k}+6^{3+2k}=15^{3+2k}$
$3^{5+4k}+4^{5+4k}=7^{5+4k}$	$4^{3+k}+5^{3+k}=9^{3+k}$	$7^{5+4k}+9^{5+4k}=16^{5+4k}$	$9^{3+k}+10^{3+k}=19^{3+k}$
$3^{3+k}+5^{3+k}=8^{3+k}$	$5^{3+k}+6^{3+k}=11^{3+k}$	$7^{3+k}+10^{3+k}=17^{3+k}$	
$b>0$			
$c<0$			
$3^{4+4k}+5^{4+4k}=6^{4+4k}$	$13^{3+4k}+9^{3+4k}=16^{3+4}$	$6^{6+4k}+13^{6+4k}=15^{6+4k}$	$8^{6+4k}+11^{6+4k}=15^{6+4k}$
$3^{3+4k}+6^{3+4k}=7^{3+4k}$	$6^{4+4k}+15^{4+4k}=17^{4+4}$	$8^{3+4k}+11^{3+4k}=17^{3+4k}$	$4^{4+4k}+5^{4+4k}=7^{4+4k}$
$3^{3+4k}+8^{3+4k}=9^{3+4k}$	$3^{6+4k}+9^{6+4k}=10^{6+4k}$	$8^{6+4k}+9^{6+4k}=15^{6+4k}$	$14^{3+4k}+9^{3+4k}=17^{3+4k}$
$3^{3+4k}+11^{3+4k}=12^{3+4k}$	$7^{4+2k}+10^{4+2k}=13^{4+2}$	$8^{4+4k}+5^{4+4k}=11^{4+4k}$	$10^{4+2k}+13^{4+2k}=17^{4+2}$
$6^{3+4k}+11^{3+4k}=13^{3+4k}$	$9^{4+4k}+5^{4+4k}=12^{4+4k}$	$9^{4+4k}+10^{4+4k}=17^{4+4k}$	$5^{4+2k}+6^{4+2k}=9^{4+2k}$
$10^{4+4k}+13^{4+4k}=19^{4+4}$	$7^{3+4k}+16^{3+4k}=19^{3+4}$	$10^{6+4k}+13^{6+4k}=17^{6+4}$	$3^{3+4k}+18^{3+4k}=19^{3+4k}$
$c=0$			
$3^{6+4k}+4^{6+4k}=5^{6+4k}$	$5^{6+4k}+12^{6+4k}=13^{6+4k}$	$7^{6+4k}+24^{6+4k}=25^{6+4}$	$8^{4+2k}+15^{4+2k}=17^{4+2k}$
$c>0$			
$7^{4+2k}+5^{4+2k}=8^{4+2k}$	$7^{6+4k}+9^{6+4k}=10^{6+4k}$	$7^{4+4k}+10^{4+4k}=11^{4+4k}$	$9^{4+2k}+10^{4+2k}=11^{4+2k}$
$13^{3+4k}+16^{3+4k}=17^{3+4}$	$7^{3+4k}+6^{3+4k}=9^{3+4k}$	$5^{4+4k}+8^{4+4k}=9^{4+4k}$	$5^{4+4k}+11^{4+4k}=12^{4+4k}$
$13^{4+4k}+15^{4+4k}=16^{4+4}$	$17^{3+4k}+9^{3+4k}=18^{3+4}$	$10^{4+4k}+11^{4+4k}=13^{4+4}$	$13^{3+4k}+15^{3+4k}=18^{3+4k}$
$13^{6+4k}+14^{6+4k}=15^{6+4}$	$8^{3+4k}+9^{3+4k}=11^{3+4k}$	$6^{3+4k}+7^{3+4k}=9^{3+4k}$	$14^{3+2k}+15^{3+2k}=19^{3+2}$
$13^{3+4k}+18^{3+4k}=19^{3+4}$	$9^{3+4k}+12^{3+4k}=13^{3+4k}$	$13^{5+4k}+14^{5+4k}=17^{5+4}$	$12^{5+4k}+13^{5+4k}=15^{5+4}$

$9^{4+2k}+15^{4+2k}=16^{4+2k}$	$5^{4+4k}+6^{4+4k}=7^{4+4k}$	$15^{3+k}+11^{3+k}=16^{3+k}$	$12^{6+4k}+15^{6+4k}=17^{6+4}$
$15^{4+4k}+16^{4+4k}=17^{4+4}$	$9^{4+4k}+10^{4+4k}=13^{4+4}$		

Note. The table gives prime triplets from the first and second tens, so as to show the full range of possibilities. Irregular triplets for which $b \le 0$, and also triplets, for which $b > 0$, but $c \le 0$, serve as the basis for obtaining regular triplets.

Table 4: Calculation of the characteristic values for *b* < 0

v	*u*	*b*	*c*	*x*	*y*	*z*	*d*	*h*	*f*	*g*	*p*
4	7	-2	-52	2	5	9	-596	-5920	-55892	-515752	4
6	9	-4	-92	2	5	11	-1197	- 14000	-157894	-1755870	4
8	11	-6	-140	2	5	13	-2100	-28000	-370000	-4800000	6
4	9	-2	-71	2	7	11					3
4	11	-2	-84	2	9	13					3
6	13	-4	-140	2	9	15					6
6	7	-2	-80	4	5	11					4
8	9	-4	-128	4	5	13	-2000	-28000	-367000	-4800000	4
8	11	-4	-192	4	9	17					3
7	8	-4	-96	3	4	11					3
7	9	-4	-110	3	5	12					6
7	13	-4	-166	3	9	16					3
7	14	-4	-180	3	10	17					4; 6
9	16	-6	-252	3	10	19					4
7	11	-2	-151	5	9	16					4
9	13	-4	-218	5	9	18	-1900	-31000	-480000	-7000000	4
6	9	-2	-104	4	7	13					3
8	11	-4	-160	4	7	15					6
7	9	-2	-122	5	7	14					4
8	9	-2	-140	6	7	15	-2900	-47000	-735000	-11240000	6
9	11	-2	-196	7	9	18					3
9	12	-4	-200	5	8	17					4; 6
9	11	-4	-182	5	7	16					4
7	8	-2	-108	5	6	13					4

Note. Here and below *p* is the smallest exponent, allowed (permissible) by the basic restrictions; for *d*, *h*, *f*, *g* are given separate values indicating the order of quantities.

Table 5: Calculation of the characteristic values for *b* = 0

v	*u*	*b*	*c*	*x*	*y*	*z*	*d*	*h*	*p*
2	3	0	-12	2	3	5	-90	-578	3; 5
2	5	0	-20	2	5	7	-210		3; 5; 6
2	7	0	-28	2	7	9	-243	-4144	5
2	9	0	-36	2	9	11	-594		5
3	4	0	-24	3	4	7	-252		5
3	5	0	-30	3	5	8	-360		3; 4; 5; 6
3	7	0	-42	3	7	10	-630		3; 5
3	8	0	-48	3	8	11	-792		5
3	10	0	-60	3	10	13	-1170		3; 4; 5; 6
4	5	0	-40	4	5	9	-540		3
4	7	0	-56	4	7	11	-924		5
5	7	0	-70	5	7	12	-1260		3; 4; 5; 6
6	7	0	-84	6	7	13	-507		5
7	8	0	-112	7	8	15	-2520		3; 5
7	9	0	-126	7	9	16	-3024		5
7	10	0	-140	7	10	17	-3570		3; 4; 5; 6

v	*u*	*b*	*c*	*x*	*y*	*z*	*d*	*h*	*p*
5	8	0	-80	5	8	13	-1560		3; 4; 5; 6
8	9	0	-144	8	9	17	-3672		5
8	11	0	-176	8	11	19	-5016		5
4	11	0	-88	4	11	15	-1980		3
5	9	0	-90	5	9	14	-1890		3; 4
9	10	0	-180	9	10	19	-5130		3; 4; 5; 6
5	6	0	-30	5	6	11	-990		3
5	11	0	-110	5	11	16	-2640		3
5	12	0	-120	5	12	17	-3060		5
2	15	0	-60	2	15	17	-1530		3

Table 6: Calculation of the characteristic values for b > 0

v	*u*	*b*	*c*	*x*	*y*	*z*	*d*	*h*	*p*
1	3	2	-2	3	5	6	-64	-590	4
1	4	2	-4	3	6	7	-100	-1024	3
1	6	2	-8	3	8	9	-53	-2209	3
3	7	6	-6	9	13	16	-1170	-30414	3
1	7	2	-10	3	9	10	-244	-3358	6
6	9	4	-92	10	13	19	-3662	-91760	4
1	9	2	-14	3	11	12	-370	-6314	3
3	6	4	-20	7	10	13	-854	-16160	4; 6
3	6	2	-32	5	8	11	-694	-9920	4
6	7	2	-80	8	9	15	-2134	-39968	6
6	9	2	-104	8	11	17	-3070	-64784	3
4	7	4	-40	8	11	15	-1532	-31888	6
2	3	2	-8	4	5	7	-154	-1520	4
3	7	2	-38	5	9	12	-874	-13550	4
3	8	6	-12	9	14	17	-1440	-38544	3
7	8	2	-108	9	10	17	-3184	-66960	4
2	7	4	-20	6	11	13	-650	-12624	3
1	4	8	-56	9	12	13	260	-1264	3
3	11	2	-62	5	13	16	-1774	-36350	4
1	13	2	-22	3	15	16	-694	-14830	4
1	14	2	-24	3	16	17	-790	-17904	3
4	11	2	-84	6	13	17	-2500	-53664	3
2	9	4	-20	6	13	15	-962	-20768	6
1	16	2	-28	3	18	19	-1000	-25264	3
6	11	2	-128	8	13	19	-4150	-97664	3
3	7	6	-6	9	13	16	-1170	-30414	3
3	12	2	-68	5	14	17	-2044	-44480	4
2	13	2	-48	4	15	17	-1474	-32640	4
2	11	2	-40	4	13	15	-1114	-21818	6
1	12	2	-20	3	14	15	-604	-12128	6
1	12	4	-8	5	16	17	-692	-17360	4
2	11	4	-28	6	15	17	-1322	-31600	4
3	4	2	-20	5	6	9	-388	-4640	4; 6
1	2	2	0	3	4	5	-34	-288	6
2	9	6	0	8	15	17	-1026	-28800	4; 6
1	8	4	0	5	12	13	-344	-7200	6
1	18	6	0	7	24	25	-1458	-56448	6
3	4	12	120	15	16	19	612	-14160	4; 6
2	3	14	184	16	17	19	2150	18736	3
3	5	10	70	13	15	18	-260	-25790	3
4	5	10	60	14	15	19	-740	-41280	3

v	*u*	*b*	*c*	*x*	*y*	*z*	*d*	*h*	*p*
1	7	4	2	5	11	12	-272	-5470	4
3	4	10	76	13	14	17	28	-16544	5
2	3	10	88	12	13	15	550	-1328	5
1	2	4	12	5	6	7	-2	-480	4
1	3	4	10	5	7	8	-44	-1070	4; 6
2	3	4	4	6	7	9	-170	-2864	3
1	9	8	46	9	17	18	-190	-14894	3
1	7	6	22	7	9	10	72	-1038	6
1	4	6	28	7	10	11	12	-2240	4
2	3	6	24	8	9	11	-90	-3984	3
1	7	8	50	9	15	16	8	-8350	4
3	4	6	12	9	10	13	-468	-12000	4
1	5	10	90	11	15	16	610	-270	3
1	8	6	20	7	14	15	-288	-9808	6
2	5	10	80	12	15	17	190	-12160	6
1	2	12	140	13	14	15	1566	16352	6
1	4	4	8	5	8	9	-92	-1840	4
1	2	8	60	9	10	11	398	1920	4
1	6	12	132	13	18	19	1170	3216	3
1	4	12	136	13	16	17	1379	10576	3
1	3	12	138	13	15	16	1476	13650	4

Note. For the triplet (13, 14, 15), calculations in accordance with (12 –14) give f = 149742, g = 965720, h_7 = -2697354; for the triplet (9, 10, 11) calculations give f = -2002; for the triplet (16, 17, 19) calculations give f = -7666; for the triplet (13, 18, 19) calculations give f = -215238; for the triplet (13, 16, 17) calculations give f = 12, g = -2533544; for the triplet (13, 15, 16) calculations give f = 82092, g = -559782.

IV. DISCUSSION OF THE RESULTS

Now let us see whether it is possible to prove the validity of Fermat's theorem by induction on *p*. From our investigation it follows that the theorem holds for *p* = 3. In order to verify this, it is sufficient for a fixed *p* to observe the sign change from minus to plus with increasing elementary bases (their number is finite) for *a* = 10*k*. For each base the value of *a* depends on *b*, *u*, ν and is situated in the range from 10 to 30, as it follows from our foregoing analysis. If for some *p* the value of *F*(*p*, *x*, *y*, *z*) is less than zero, then the induction transition can be proved. Indeed, we have

$$F(p+1,x,y,z) = x\cdot x^p + y\cdot y^p - z\cdot z^p = x(x^p + y^p - z^p) + u(y^p - z^p) - vy^p \quad . \quad (15)$$

Since all the terms on the right side of (15) are negative, this implies the desired statement, i.e. $F(p+1,x,y,z) < 0$. Now let $F(p,x,y,z) > 0$, then nothing can be proved with respect to $F(p+1,x,y,z)$, since the proof by induction (like any in number theory) is based on the continuity property, and here we have a discontinuity. The exponent (*p* + 1) can turn out to be just such that the sign changes from plus to minus. From the perspective of our study, this means that we are in the range of basis where the sign of $F(p,x,y,z)$ has changed from minus to plus by increasing the base at a fixed *p*, but the sign has not changed from plus to minus due to an increase in the exponent *p*. Therefore, to prove by induction, we must assume that we are in the range of values of *p* where such a transition occurred, and then the proof is possible. In any case, the use of Descartes' rule makes it possible to overcome the difficulties noted. Without its application, a direct verification of the validity of Fermat's theorem using the proposed analysis would be possible only on a bounded set of natural numbers. Thus, it can be argued that: 1) if equation (1) of higher degrees does not have natural solutions among elementary bases, then it does not have them at all; 2) the existence of "natural" solutions of equation (1) is defined for *p* = 1 and *p* = 2. For all elementary bases and all the degrees allowed for them by the conditions of the theorem, and even earlier (for *p* = 3 or *p* = 4), we are initially in the negative range (see tables 4 – 6), where the left side of equation (1) is less than the right one, so that the increase in the exponent is nothing gives, and its decrease takes us beyond the conditions of Fermat's theorem. The increase of elementary bases also does not give anything, since it shifts the roots of equation (1) beyond the limits of the set of natural numbers.

V. CONCLUSION

Consider the geometric interpretation of the results obtained. For $p = 1$, equation (1) always has a solution, i.e. the sum of two integer segments is always an integer segment. For $p = 2$, equation (1) has a solution only in some cases, i.e. the sum of the areas of two squares with integer sides is only sometimes equal to the area of the square with integer sides. For $p = 3$, equation (1) has no solution, i.e. the volume of a cube with integer sides is never the sum of the volumes of two cubes with integer sides. This is true *a fortiori* for hypercube.

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