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On The Distribution of Non - Zero Zeros of Generalized Mittag – Leffler Functions

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ABSTRACT
In this work, we derive some theorems involving distribution of non – zero zeros of generalized Mittag – Leffler functions of one and two variables. Some of the theorems obtained here for the distribution of zeros of the derivative of the generalized Mittag – Leffler function are applied to explore the infinitely divisible probability density function for the generalized Mittag – Leffler function
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I. INTRODUCTION

The classical Mittag – Leffler function of one parameter is given by [4]

$$E_\beta(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta n+1)}, z,\beta\in\mathbb{C},\Re(\beta)>0. \quad (1)$$

The Mittag – Leffler function given in Eqn. (1) reduces to the exponential function e^z when $\beta=1$, it interpolates between pure exponential e^z and geometric function $(1-z)^{-1}=\sum_{n=0}^{\infty}z^n, |z|<1$. The direct generalization of this function in the two parameter form is given by Wiman [16]

$$E_{\beta,\gamma}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta n+\gamma)}, z,\beta,\gamma\in\mathbb{C},\Re(\beta)>0,\Re(\gamma)>0. \quad (2)$$

Prabhakar [12] (in 1971) introduced a generalized Mittag - Leffler function $E_{\beta,\gamma}^{\delta}(z)$ involving three parameter of order $\rho=[\Re(\beta)]^{-1}$ in form

$$E_{\beta,\gamma}^{\delta}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\beta n+\gamma)n!}z^n, z,\beta,\gamma,\delta\in\mathbb{C},\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0. \quad (3)$$

Here, in our work we use a relation (see Mathai and Haubold [10]) in the form

$$\left(\frac{d}{dz}\right)^m E_{\beta,\gamma}^{\delta}(z)=(\delta)_m E_{\beta,\gamma+m\beta}^{\delta+m}(z). \quad (4)$$

Shukla and Prajapati [13] also introduced another generalization in the form

$$E_{\beta,\gamma}^{\delta,q}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}}{\Gamma(\beta n+\gamma)n!}z^n, z,\beta,\gamma,\delta\in\mathbb{C},\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0,q\in(0,1)\cup\mathbb{N}. \quad (5)$$

Here, $(\delta)_{qn}=\frac{\Gamma(\delta+qn)}{\Gamma(\delta)}$ and when $q\in\mathbb{N},(\delta)_{qn}=\prod_{r=1}^{q}\left(\frac{\delta+r-1}{q}\right)_n$. (6)

Recall that from Eqns. (1), (2), (3) and (5), we have the relations $E_{\beta,\gamma}^{\delta,1}(z)=E_{\beta,\gamma}^{\delta}(z)$, $E_{\beta,\gamma}^{1,1}(z)=E_{\beta,\gamma}(z)$, $E_{\beta,1}^{1,1}(z)=E_\beta(z)$. (7)

Application of the technique of Eqn. (4) in Eqn. (5), we obtain

$$\left(\frac{d}{dz}\right)^m E_{\beta,\gamma}^{\delta,q}(z)=(\delta)_{mq}E_{\beta,\gamma+m\beta}^{\delta+mq,q}(z). \quad (8)$$

Clearly, with the help of Eqns. (7) and (8), we get the equality $\left(\frac{d}{dz}\right)^m E_{\beta,\gamma}^{\delta,1}(z)=\left(\frac{d}{dz}\right)^m E_{\beta,\gamma}^{\delta}(z)$.

Now let $G(z)=E_{\beta,\gamma}^{\delta,q}(z), z,\beta,\gamma,\delta\in\mathbb{C},\Re(\beta)>0,\Re(\gamma)>0,\Re(\delta)>0,q\in(0,1)\cup\mathbb{N}, G(0)=\frac{1}{\Gamma(\gamma)}$. Then for $F(z)=\frac{(G(z))'}{G(z)}$, as on using Eqn. (8),we have

$$F(z)=\frac{(\delta)_q E_{\beta,\gamma+\beta}^{\delta+q,q}(z)}{E_{\beta,\gamma}^{\delta,q}(z)}. \quad (9)$$

II. INFINITE PRODUCT FORMULA FOR ONE VARIABLE GENERALIZED MITTAG – LEFFLER FUNCTION

To derive infinite product formula for generalized Mittag – Leffler function, we use following theorems:

The theorem of Laguerre (see Titchmarsh [15, p. 268])

Let $f(z)$ be a polynomial,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_p z^p, \tag{10}$$

all of whose zeros are real; and let $\phi(w)$ be an integral function of genus 0 or 1, which is real for real w, and all the zeros of are real and negative, that is

$$\phi(w) = a e^{kw} \prod_{n=1}^{\infty} \left(1 + \frac{w}{\alpha_n}\right) e^{-\frac{w}{\alpha_n}}, \alpha_n > 0, a > 0, k > 0, \text{ for all values of } n \in \mathbb{N}. \tag{11}$$

Then the polynomial

$$g(z) = a_0 \phi(0) + a_1 \phi(1) z + a_2 \phi(2) z^2 + \cdots + a_p \phi(p) z^p \tag{12}$$

has all its zeros real, and as many positive, zero and negative zeros as $f(z)$.

The generalization of the theorem of Laguerre, stated in Eqns. (10) – (12), is given as follows:

The theorem of Titchmarsh [15]

Let $\phi(w)$ be defined in Eqn. (11) and $f(z)$ is an integral function of the form

$$f(z) = e^{bz+d} \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right), \tag{13}$$

the numbers b and z_n being all positive. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, a_n = \frac{f^{(n)}(0)}{n!}, f^{(n)}(0) = \frac{d^n}{dz^n} f(z) \text{ at } z = 0. \tag{14}$$

Then $g(z) = \sum_{n=0}^{\infty} a_n \phi(n) z^n$ (15)

is an integral function, all of whose zeros are real and negative.

The theorem of Peresyolkova [11]

If there are three transformations mapping the set $\{(\sigma, \gamma): 0 < \sigma < 1, \gamma > 0\}$ into itself:

$$A: (\sigma, \gamma) \to \left(\frac{\sigma}{2}, \gamma\right); B: (\sigma, \gamma) \to \left(\frac{\sigma}{2}, \gamma + \frac{1}{\sigma}\right); C: (\sigma, \gamma) \to \begin{cases} (\sigma, \gamma - 1), for\ \gamma > 1; \\ (\sigma, \gamma - 1), for\ 0 < \gamma \leq 1. \end{cases} \tag{16}$$

Put $W_a = \left\{(\sigma, \gamma): \frac{1}{2} < \sigma < 1, \gamma \in \left[\frac{1}{\sigma} - 1, 1\right] \cup \left[\frac{1}{\sigma}, 2\right]\right\}, W_b = AW_a \cup BW_a.$ (17)

Denote by W_i by least set containing W_b and invariant with respect to A, B, C. Then the set W_i can be represented by

$$W_i = \cup \left(A^{k_{11}} B^{k_{12}} C^{k_{13}}, \ldots, A^{k_{n1}} B^{k_{n2}} C^{k_{n3}}\right) W_b, \tag{18}$$

where, the union is taken over all $n = 1,2,3, \ldots$ and over all $3n$ – tuples $(k_{11}, k_{12}, k_{13}; \ldots; k_{n1}, k_{n2}, k_{n3})$ of non negative integers. Then for $(\sigma, \gamma) \in W_i$, all zeros of $E_\sigma(z, \gamma)$, $where, E_\sigma(z, \gamma) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\gamma + \frac{n}{\sigma}\right)}$, are negative and simple.

In consequence of the theorem of Peresyolkova [11], given by Eqns. (16) – (18), we may state that:

Theorem 1.

Let there exists three transformations mapping the set $\left\{\left(\frac{1}{\beta}, \gamma\right): 0 < \beta < 2, \gamma > 0\right\}$ into itself:

$$A: \left(\frac{1}{\beta}, \gamma\right) \to \left(\frac{1}{2\beta}, \gamma\right); B: \left(\frac{1}{\beta}, \gamma\right) \to \left(\frac{1}{2\beta}, \gamma + \beta\right); C: \left(\frac{1}{\beta}, \gamma\right) \to \begin{cases} \left(\frac{1}{\beta}, \gamma - 1\right), for\ \gamma > 1; \\ \left(\frac{1}{\beta}, \gamma - 1\right), for\ 0 < \gamma \leq 1. \end{cases} \tag{19}$$

Put $W_a = \left\{\left(\frac{1}{\beta}, \gamma\right): 1 < \beta < 2, \gamma \in [\beta - 1, 1] \cup [\beta, 2]\right\}, W_b = AW_a \cup BW_a.$ (20)

Denote by W_i by least set containing W_b and invariant with respect to A, B, C. Then, the set W_i can be represented by

$$W_i = \cup \left(A^{k_{11}} B^{k_{12}} C^{k_{13}}, \ldots, A^{k_{n1}} B^{k_{n2}} C^{k_{n3}}\right) W_b, \tag{21}$$

where, the union is taken over all $n = 1,2,3, \ldots$ and over all $3n$ – tuples $(k_{11}, k_{12}, k_{13}; \ldots; k_{n1}, k_{n2}, k_{n3})$ of non negative integers. Then for $\left(\frac{1}{\beta}, \gamma\right) \in W_i$, all zeros of $E_{\beta,\gamma}(z)$, defined by Eqn. (2), are negative and simple.

Proof: In the theorem of Peresyolkova [11] replace σ by $\frac{1}{\beta}$, and then we prove our this theorem.

The complex zeros of Mittag – Leffler function were studied by Wiman [16] and found that the asymptotic curve along which the zeros are located for $0 < \beta < 2$ and showed that they fall on the negative real axis for $\beta \geq 2$.

Hilfer and Seybold [5] also studied the location of zeros of generalized Mittag Leffler function as function of β, for the case $1 < \beta < 2$ and noted that with increasing β more and more pairs of zeros collapse onto the negative real axis.
By above mentioned theory of Laguerre, (see Titchmarsh [15], Peresyolkova [11], Wiman [16], and Hilfer and Seybold [5]), we state that:

Theorem 2. Let $\phi(w)$ be defined in Eqn. (11), $\{a_n z^n\}_{n=0}^{\infty}$ be the bounded sequence, and there exists $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} \frac{1}{z} > 1$, thus $f(z)$ is the integral function defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n$,
Then, $g(z) = \sum_{n=0}^{\infty} a_n \phi(n) z^n$ (22)
is an integral function, all of whose zeros are real and negative.

Corollary 2. If $\delta > 0$, $a_n = \frac{(\delta)_{qn}}{n!} z^n$, $\phi(w) = \frac{1}{\Gamma(\beta w + \gamma)}$, and there be an integral function such that $f(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}}{n!} z^n$.
Then, for $\delta > 0$, $\frac{a_n}{a_{n+1}} \frac{1}{z} > 1$ as $n \to \infty$ when $0 < q \leq 1$and then for $\left(\frac{1}{\beta}, \gamma\right) \in W_i$, given in theorem 1, $g(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}}{n!} \frac{1}{\Gamma(\beta n + \gamma)} z^n$ be an integral function having zeros real and negative. (23)
(Here $\frac{a_n}{a_{n+1}} = \frac{\Gamma(qn+\delta)}{\Gamma(qn+q+\delta)} \frac{\Gamma(n+2)}{\Gamma(n+1)} \frac{1}{z}$ becomes greater than one as $n \to \infty$, when $0 < q \leq 1$, for this we apply the asymptotic formula due to Erde'lyi [2] given by
$\frac{\Gamma(s+a)}{\Gamma(s+b)} = s^{a-b} \left[1 + \frac{1}{2s}(a-b)(a+b-1) + O\left(\frac{1}{s^2}\right)\right]$ $as\ s \to \infty$), to get $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\Gamma(qn+\delta)}{\Gamma(qn+q+\delta)} \frac{\Gamma(n+2)}{\Gamma(n+1)} \frac{1}{z} \sim \lim_{n\to\infty} \frac{1}{(q)^q} (n)^{1-q} \frac{1}{z} > 1$, when $0 < q \leq 1$.) (24)

Theorem 3. For $z, \beta, \gamma, \delta \in \mathbb{C}, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, q \in (0,1) \cup \mathbb{N}$, there exists an infinite product formula of generalized Mittag – Leffler function in the form

$$E_{\beta,\gamma}^{\delta,q}(z) = \frac{1}{\Gamma(\gamma)} \exp\left[\frac{(\delta)_q \Gamma(\gamma)}{\Gamma(\gamma+\beta)} z\right] \prod_{k=1}^{\infty} \left(\left(1 + \frac{z}{z_k}\right) \exp\left[-\frac{z}{z_k}\right]\right) \qquad (25)$$

Proof: To prove this theorem, we consider the sequence of integrals in the form

$$I_n = \frac{1}{2\pi i} \int_{C_n} \frac{F(t)}{t(t-z)} dt, \forall n = 1,2, \ldots, i = \sqrt{(-1)}. \qquad (26)$$

$where, F(z) = \frac{(G(z))'}{G(z)}, G(z) = E_{\beta,\gamma}^{\delta,q}(z), \ \forall z, \beta, \gamma, \delta \in \mathbb{C}, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, q \in (0,1) \cup \mathbb{N}$. $\{C_n\}$ is the sequence of the circles about origin such that C_n includes all non – zero zeros (z_k) ($\forall k = 1,2, \ldots n$), of generalized Mittag – Leffler functions $E_{\beta,\gamma}^{\delta,q}(z)$, $all\ z_k$ are positive real. That is $E_{\beta,\gamma}^{\delta,q}(z_k) \neq 0, and\ E_{\beta,\gamma}^{\delta,q}(-z_k) = 0\ \forall k = 1,2,3, \ldots$ (Since, Craven and Csordas [1], Peresyolkova [11], Hilfer and Seybold [5] have proved that $E_{\beta,\gamma}^{1,1}(z)$ has negative real zeros and then due to theorem of Laguerre, (see Titchmarsh [15, p. 268]) and corollary 2, $E_{\beta,\gamma}^{\delta,q}(z)$ also has negative real zeros.)
Further, C_n is not passing through any zeros. Then, F is uniformly bounded on these circles. So, let $|F| \leq M$.Therefore, Eqn. (26) gives us $|I_n| \leq \frac{M}{R_n}, where, R_n$ denotes the radius of C_n.
Hence then, $\lim_{n\to\infty} I_n = 0\ as\ R_n \to \infty$ for $n \to \infty$. (27)
Now we use the techniques due to theorem of residue (see Titchmarsh [15, p. 111] , Hochstadt [6, p. 254], and Kumar and Srivastava [9]), and Eqn. (27) to obtain

$$F(z) = \frac{(\delta)_q \Gamma(\gamma)}{\Gamma(\beta+\gamma)} - \sum_{k=1}^{\infty} \frac{z}{z_k(z_k+z)} \qquad (28)$$

Then, making an appeal to the results in Eqns. (9) and (28), we find

$$G(z) = K \exp\left[\frac{(\delta)_q \Gamma(\gamma)}{\Gamma(\gamma+\beta)} z\right] \prod_{k=1}^{\infty} \left((z + z_k) \exp\left[-\frac{z}{z_k}\right]\right), K \text{ is any constant.} \qquad (29)$$

Further use the result of Eq. (9) that $G(0) = \frac{1}{\Gamma(\gamma)}$ in Eqn. (29) to find

$$K = \frac{1}{\Gamma(\gamma) \prod_{k=1}^{\infty}(z_k)}. \qquad (30)$$

Finally, with the aid of Eqns. (29) and (30) we find Eqn. (25).

III. APPLICATIONS

In this section, we use our theorem 3 to obtain the distribution of zeros of derivative of generalized Mittag – Leffler function [13] and to find out the probability density function of this function. This approach may be used for developing and finding extensions of related functions for further researches in this field.

3.1 The distribution of the zeros of derivative of generalized Mittag – Leffler function

For the distribution of zeros of derivative of generalized Mittag – Leffler function [13], we derive following theorem:

Theorem 4. For $z, \beta, \gamma, \delta \in \mathbb{C}, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, q \in (0,1) \cup \mathbb{N}$, $\left(E_{\beta,\gamma}^{\delta,q}(z)\right)'$ has only real simple zeros except $z = 0$ and these interlace with zeros of $E_{\beta,\gamma}^{\delta,q}(z), z = x + iy, i = \sqrt{(-1)}$.

Proof: We make an appeal to the theorem 3 to get

$$\frac{\left(E_{\beta,\gamma}^{\delta,q}(z)\right)'}{E_{\beta,\gamma}^{\delta,q}(z)} = \frac{(\delta)_q \Gamma(\gamma)}{\Gamma(\beta+\gamma)} - \sum_{k=1}^{\infty} \frac{z}{z_k(z_k+z)} \tag{31}$$

The imaginary part of right hand side of Eqn. (31) is given by

$$-\sum_{k=1}^{\infty} \frac{y}{(x+z_k+y^2)}. \tag{32}$$

Obviously, if in Eqn. (32), we put $y = 0$, it vanishes and hence all zeros of $\left(E_{\beta,\gamma}^{\delta,q}(z)\right)'$ are real and simple.

Further from Eqn. (31), for real z, we get

$$\frac{d}{dz} \frac{\left(E_{\beta,\gamma}^{\delta,q}(z)\right)'}{E_{\beta,\gamma}^{\delta,q}(z)} = -\sum_{k=1}^{\infty} \frac{1}{(z_k+z)^2} \tag{33}$$

From, Eqn. (33), we get

$$\frac{d}{dz} \frac{\left(E_{\beta,\gamma}^{\delta,q}(z)\right)'}{E_{\beta,\gamma}^{\delta,q}(z)} < 0. \tag{34}$$

Here, the Eqns. (31) and (34) show that the function $\frac{\left(E_{\beta,\gamma}^{\delta,q}(z)\right)'}{E_{\beta,\gamma}^{\delta,q}(z)}$ has infinity of poles, but is of negative slope.

Hence between any two zeros of the function $E_{\beta,\gamma}^{\delta,q}(z)$ there must be precisely one of $\left(E_{\beta,\gamma}^{\delta,q}(z)\right)'$.

3.2 The infinitely divisible probability density function for the generalized Mittag – Leffler function

Takano [14] has described the infinite divisibility of normed conjugate product of gamma function and Kumar, Pathan and Chandel [8] have obtained infinitely divisible probability density functions of Bessel and Legendre functions through their infinite product formula of Hochstadt [6] and Kumar and Srivastava [9], respectively, and then used them in finding out the distributions of the densities involving non – zero zeros of Bessel and Legendre functions and applied them to obtain their infinite divisibility.

To obtain the probability density functions, we make an appeal to the Eqn. (25) of theorem 3 and find that in the form

$$E_{\beta,\gamma}^{\delta,q}(iz) E_{\beta,\gamma}^{\delta,q}(-iz) = \left(\frac{1}{\Gamma(\gamma)}\right)^2 \prod_{k=1}^{\infty} \left(\left(1 + \left(\frac{z}{z_k}\right)^2\right)\right) \tag{35}$$

Now, from the Eqn. (35), we get the infinitely divisible probability density function

$$\left(\frac{1}{\Gamma(\gamma)}\right)^2 \frac{1}{E_{\beta,\gamma}^{\delta,q}(iz) E_{\beta,\gamma}^{\delta,q}(-iz)} = \prod_{k=1}^{\infty} \left(\left(\frac{(z_k)^2}{(z)^2 + (z_k)^2}\right)\right) \tag{36}$$

Now consider the function

$$H_n(z) = \frac{A}{\prod_{k=1}^{n}(z^2 + {z_k}^2)}, A \text{ is any constant}, -\infty < z < \infty. \tag{37}$$

We know that

$$\frac{A}{\prod_{k=1}^{n}(z^2+{z_k}^2)} = A \sum_{k=1}^{n} \frac{1}{\prod_{j=1, j\neq k}^{n}(-{z_k}^2 + {z_j}^2)(z^2+{z_k}^2)} \tag{38}$$

Also, we have for $z > \pm\sqrt{\rho}$ (See Hochstadt [6, p. 268])

$$\int_0^{\infty} exp\left[-t\sqrt{\{(z^2 + {z_k}^2)^2 - \rho^2\}}\right] J_0(t\rho)\, dt = \frac{1}{(z^2+{z_k}^2)}$$

(39)

Here $J_0(.)$ is zero order Bessel function and $J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2.4^2} - \frac{z^6}{2^2.4^2.6^2} + \cdots$.

Then make an appeal to the Eqns. (38) and (39) in Eqn. (37) to get

$$H_n(z) = A\int_0^\infty J_0(t\rho)\sum_{k=1}^{n}\frac{1}{\prod_{j=1,j\neq k}^{n}(-z_k{}^2+z_j{}^2)}exp\left[-t\sqrt{\{(z^2+z_k{}^2)^2-\rho^2\}}\right]dt \quad (40)$$

From Eqn. (40), let us choose that

$$h_n(t) = A\sum_{k=1}^{n}\frac{1}{\prod_{j=1,j\neq k}^{n}(-z_k{}^2+z_j{}^2)}exp\left[-t\sqrt{\{(z^2+z_k{}^2)^2-\rho^2\}}\right], \forall z > \pm\sqrt{\rho}, \rho > 0 \; and \; t > 0. \quad (41)$$

Since for $n = 1, we\ have\ \prod_{j=1,j\neq k}^{n}\left(-z_k{}^2+z_j{}^2\right) = 1$, so that

$$h_1(t) = A\; exp\left[-t\sqrt{\{(z^2+z_1{}^2)^2-\rho^2\}}\right], \forall z > \pm\sqrt{\rho}, \rho > 0 \; and \; t > 0, \quad (42)$$

and

$$h_2(t) = A\left\{\frac{1}{(-z_1{}^2+z_2{}^2)}exp\left[-t\sqrt{\{(z^2+z_1{}^2)^2-\rho^2\}}\right]+\frac{1}{(-z_2{}^2+z_1{}^2)}exp\left[-t\sqrt{\{(z^2+z_2{}^2)^2-\rho^2\}}\right]\right\}$$

$$\forall z > \pm\sqrt{\rho}, \rho > 0 \; and \; t > 0. \quad (43)$$

Here the function $h_n(t)$ is always positive for all $z > \pm\sqrt{\rho}, \rho > 0 \; and \; t > 0$, hence it is a density function $, \forall z > \pm\sqrt{\rho}, \rho > 0 \; and \; t > 0$.

By the techniques of Takano [14], we may prove that its Laplace transformation is infinitely divisible and thus from Eqns. (37) and (40) $H_n(z)$ is infinitely divisible and

thus $\lim_{n\to\infty} H_n(z) = \lim_{n\to\infty}\frac{A}{\prod_{k=1}^{n}(z_k{}^2)}\lim_{n\to\infty}\prod_{k=1}^{n}\left(\left(\frac{(z_k)^2}{(z)^2+(z_k)^2}\right)\right)$ is also infinitely divisible. Now make an appeal to the Eqn. (36) and above $\lim_{n\to\infty} H_n(z)$, finally, we may say that the density function $\left(\frac{1}{\Gamma(\gamma)}\right)^2\frac{1}{E_{\beta,\gamma}^{\delta,q}(iz)E_{\beta,\gamma}^{\delta,q}(-iz)}$ be infinitely divisible.

3.3 Another Distribution function

Further, we make an appeal to the Eqn. (36) and find that

$$\log\left\{\left(\frac{1}{\Gamma(\gamma)}\right)^2\frac{1}{E_{\beta,\gamma}^{\delta,q}(z)E_{\beta,\gamma}^{\delta,q}(-z)}\right\} = -\sum_{k=1}^{\infty}\left\{\log\left(1-\left(\frac{z}{z_k}\right)^2\right)\right\}, \forall \frac{z}{z_k} < 1, k = 1,2,3, \ldots \quad (44)$$

IV. THE INFINITE PRODUCT FORMULA PRODUCT OF TWO VARIABLE GENERALIZED MITTAG – LEFFLER FUNCTION

On the other hand, Garg, Manohar and Kalla [3] have generalized the Mittag – Leffler - type function and defined a two variable generalized Mittag – Leffler function

$$E_1(x,y) = E_1\left(\begin{matrix}\gamma_1,\alpha_1;\gamma_2,\beta_1\\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3\end{matrix}\middle|\begin{matrix}x\\y\end{matrix}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma_1)_{\alpha_1 m}(\gamma_2)_{\beta_1 n}}{\Gamma(\delta_1+\alpha_2 m+\beta_2 n)}\frac{x^m}{\Gamma(\delta_2+\alpha_3 m)}\frac{y^n}{\Gamma(\delta_3+\beta_3 n)} \quad (45)$$

Here, $\gamma_1,\gamma_2,\delta_1,\delta_2,\delta_3,x,y \in \mathbb{C}, \min\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\} > 0$.

Recently, Kumar and Pathan [7] have defined another two variable generalized Mittag – Leffler function in the form

$$E_{\alpha,\beta}^{(\delta)}(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}x^m y^n}{m!n!\Gamma(\alpha m+\beta n+1)}, \forall x,y,\alpha,\beta,\delta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0. \quad (46)$$

We now generalized this Mittag – Leffler function of two variables defined in Eqn. (46) in the form

$$E_{\alpha,\beta;\gamma}^{(\delta)}(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}x^m y^n}{m!n!\Gamma(\alpha m+\beta n+\gamma)}, \quad (47)$$

$provided\ that\ x,y,\alpha,\beta,\delta,\gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta) > 0, \Re(\gamma) > 0.$

For evaluating the infinite product formula of generalized Mittag – Leffler function, defined in Eqn. (47), we present following theorem:

The theorem of Craven and Csords [1]:

Let $\alpha > 2 \; and \; \gamma > 0$. Then there exists a positive integer $m_0 \coloneqq m_0(\alpha,\gamma)$ such that $E_{\alpha,\gamma}^{(m)}(x) = \sum_{n=0}^{\infty}\frac{\Gamma(m+n+1)}{\Gamma(\alpha(m+n)+\gamma)}\frac{x^n}{n!}, m \geq m_0$ be of Laguerre – Polya class that it is entire function. Thus the sequence $\left\{\frac{\Gamma(m+n+1)}{\Gamma(\alpha(m+n)+\gamma)}\right\}_{n=0}^{\infty}$ is a meromorphic Laguerre multiplier sequence for all non negative integers m sufficiently large. (48)

Theorem 5. For $x,y,\alpha,\beta,\delta,\gamma \in \mathbb{C}, \Re(\alpha) > 2, \Re(\beta) \geq \Re(\alpha), \Re(\delta) > 0, \Re(\gamma) > 0$, and $\left\{(\delta)_n\frac{y^n}{n!}\right\}_{n=0}^{\infty}$ be the bounded sequence. Then the function $E_{\alpha,\beta;\gamma}^{(\delta)}(x,y)$ defined in Eqn. (47) be of Laguerre – Polya class.

Proof: Make an appeal to the Eqns. (47) and (48) we may write

$$E_{\alpha,\beta;\gamma}^{(\delta)}(x,y) = \frac{1}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{y^n}{n!}\sum_{m=0}^{\infty}\frac{\Gamma(\delta+n+m)x^m}{m!\Gamma\left(\alpha(m+\frac{\beta}{\alpha}n)+\gamma\right)} = \frac{1}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{y^n}{n!}E_{\alpha,\beta;\gamma}^{(\delta+n)}(x).$$

(Here, we have $E_{\alpha,\alpha;\gamma}^{(1+n)}(x) = E_{\alpha,\gamma}^{(n)}(x)$). (49)

Now make an application the theorem of Craven and Csordas [1] in Eqn. (49), the two variable generalized Mittag – Leffler function be of Laguerre – Polya class.

Again, making an appeal to the Eqns. (3) and (47), we have an integral function

$$E_{\alpha,\beta;\gamma}^{(\delta)}(x,y) = \sum_{n=0}^{\infty}(\delta)_n \frac{y^n}{n!} E_{\alpha,\beta n+\gamma}^{\delta+n}(x) \quad (50)$$

Finally, use the theorem 2 in Eqn. (50) the function $E_{\alpha,\beta;\gamma}^{(\delta)}(x,y)$ has negative real zeros.

Theorem 6. For $x, y, \alpha, \beta, \delta, \gamma \in \mathbb{C}, \Re(\alpha) > 2, \Re(\beta) \geq \Re(\alpha), \Re(\delta) > 0, \Re(\gamma) > 0$, and $\{(\delta)_n\}_{n=0}^{\infty}$ be the bounded sequence. Then the function $E_{\alpha,\beta;\gamma}^{(\delta)}(x,y)$ has the infinite product formula

$$E_{\alpha,\beta;\gamma}^{(\delta)}(x,y) = \sum_{n=0}^{\infty}(\delta)_n \frac{y^n}{n!} \frac{1}{\Gamma(\beta n+\gamma)} \exp\left[\frac{(\delta+n)\Gamma(\beta n+\gamma)}{\Gamma(\alpha+\beta n+\gamma)} x\right] \prod_{k=1}^{\infty}\left(\left(1+\frac{x}{x_k}\right)\exp\left[-\frac{x}{x_k}\right]\right). \quad (51)$$

Proof. Make an appeal to the Eqn. (7) and theorem 3 in Eqn. (50) to get the result (51).

V. CONCLUDING REMARKS

Notice that the so-called generalized Mittag-Leffler function $E_{\beta,\gamma}^{\delta}(z)$ of Prabhakar involving three parameter is indeed a familiar hypergeometric function 1^{F_k} , k being a positive integer. That means that our results of this paper can be extended to generalized hypergeometric functions. Infinite product formula of one variable generalized Mittag – Leffler function obtained in this paper used to evaluate the infinitely divisible probability density function depending upon non – zero zeros of generalized Mittag - Leffler function can be rewritten in the form of familiar generalized hypergeometric functions and their extensions.

Similarly, the distribution of the zeros of derivative of generalized Mittag – Leffler function and infinite product formula of two variable generalized Mittag – Leffler function given in this paper may be developed in terms of generalized hypergeometric functions of more than one variables.

It is also interesting to note that the results derived in this paper may be helpful for the computing of several related problems consisting of one and two variable generalized Mittag – Leffler functions through their non – zero zeros.

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