RESEARCH ARTICLE OPEN ACCESS

Effect of Modulation on the Onset of Thermal Convection in a Viscoelastic Fluid-Saturated Nanofluid Porous Layer

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Abstract
The stability of a viscoelastic fluid-saturated by a nanofluid in a horizontal porous layer, when the boundaries of the layer are subjected to periodic temperature modulation, is analyzed. The Darcy-Brinkman-Oldroyd-B fluid model is employed and only infinitesimal disturbances are considered. The model used for the nanofluids incorporates the effect of Brownian motion. The thermal conductivity and viscosity are considered to be dependent on the nanoparticle volume fraction. Three cases of the oscillatory temperature field were examined (a) symmetric, so that the wall temperatures are modulated in phase, (b) asymmetric corresponding to out-of phase modulation and (c) only the bottom wall is modulated. Perturbation solution in powers of the amplitude of the applied field is obtained. The effect of the frequency of modulation on the stability is clearly shown. The stability of the system characterized by a correction Rayleigh number is calculated as a function of the viscoelastic parameters, the concentration Rayleigh number, porosity, Lewis number, heat capacity ratio, Vadász number, viscosity and conductivity variation parameters and frequency of modulation. It is found that the onset of convection can be delayed or advanced by the factors represented by these parameters. The nanofluid is found to have more stabilizing effect when compared to regular fluid. The effect of all three types of modulation is found to be destabilizing as compared to the unmodulated system.

I. Nomenclature

c nanofluid specific heat at constant pressure
c_p specific heat of the nanoparticle material
$(\rho c)_m$ effective heat capacity of the porous medium
d_p nanoparticle diameter
$\boldsymbol{g}$ gravitational acceleration
D_B Brownian diffusion coefficient (m^2/s)
h_p specific enthalpy of the nanoparticle material
H dimensional layer depth (m)
j_p diffusion mass flux for the nanoparticles
$\mathrm{j}_{p,T}$ thermophoretic diffusion
k thermal conductivity of the nanofluid
k_B Boltzman's constant
k_m effective thermal conductivity of the porous medium
k_p thermal conductivity of the particle material
Le Lewis number
N_A modified diffusivity ratio
N_B modified particle-density increment
p^* pressure
p dimensionless pressure $\left(p^* K/\mu\alpha_m\right)$
q energy flux relative to a frame moving with the nanofluid velocity v
R thermal Rayleigh- Darcy number
Rn concentration Rayleigh number
t^* time
t dimensionless time $\left(t^*\alpha_m/\sigma H^2\right)$
T^* nanofluid temperature
T dimensionless temperature $\left(\left(T^*-T\right)/\left(T_h^*-T\right)\right)$
T_c^* temperature at the upper wall
T_h^* temperature at the lower wall
T_R reference temperature
(u,v,w) dimensionless Darcy velocity components $\left(\left(u^*,v^*,w^*\right)H/\alpha_m\right)$
v nanofluid velocity
v_D Darcy velocity $\varepsilon\mathrm{v}$
v_D^* dimensionless Darcy velocity $\left(u^*,v^*,w^*\right)$
Va Vadász number
(x,y,z) dimensionless Cartesian coordinate $\left(\left(x^*,y^*,z^*\right)/H\right)$
$\left(x^*,y^*,z^*\right)$ Cartesian coordinates

Greek letters
α_m thermal diffusivity of the porous medium, $\left(k_m/\left(\rho c_p\right)_f\right)$
$\tilde{\beta}$ proportionality factor
γ conductivity variation parameter

γ_a Non dimensional acceleration coefficient
λ_1 stress relaxation coefficient
λ_2 strain retardation coefficient
ε porosity of the medium
ε_t amplitude of the modulation
μ viscosity of the fluid
ν viscosity variation parameter
ρ fluid density
ρ_p nanopraticle mass density
σ parameter
ϕ^* nanoparticle volume fraction
ϕ relative nanoparticle volume fraction $\left(\left(\phi^*-\phi\right)/\left(\phi_h^*-\phi_c^*\right)\right)$
Ω dimensional frequency
ω dimensionless frequency $\left(\Omega H^2/k\right)$
ϕ phase angle ($\phi=0$, symmetric modulation; $\phi=\pi$,antisymmetric modulation; $\phi=-i\infty$, only lower wall temperature modulation)

II. INTRODUCTION

Interest in sustainable energy has created significant demand for new thermal storage and thermal management technologies, including technologies that employ nanofluids (which are suspensions of nanoparticles in liquids). There is also interest in increasing the efficiency of existing heat transfer processes via improvements in the transport properties of heat transfer media such as nanofluids. The ability to tune the properties of nanofluids offers many advantages in this respect. For example, a 39% increase in the heat transfer coefficient has been reported by Xuan et al. [1] when an aqueous nanofluid containing 2% (v/v) copper nanoparticles was employed in place of water in forced convective heat transfer experiments in a horizontal tube. Similarly, pool boiling experiments with an aqueous nanofluid containing 1.25% (v/v) alumina nanoparticles have yielded Wen et al. [2] a 40% enhancement in the heat transfer coefficient when compared with the experiments conducted with pure water.

Maxwell [3] was the first presenter of a theoretical basis to predict a suspension's effective conductivity about 140 years ago and his theory was applied from millimeter to micrometer sized particles suspensions but Choi and Eastman [4] introduced the novel concept of nanofluids by applying the unique properties of nanofluids at the annual Mechanical Engineering meeting of American Society in 1995. Goldstein et al. [5] added the condition that the particles must be in colloidal suspension. Choi and his colleagues carried out experiments on heat transport in systems with CuO nanoparticles in water and Al_2O_3 particles in ethylene glycol and water. They found that the particles improve the heat transport by as much as 20%, and they interpreted their result in terms of an improved thermal conductivity k/k_0 which they named the effective conductivity [4].

A nanofluid is a fluid produced by dispersion of metallic or non-metallic nanoparticles or nanofibres with a typical size of less than 100 nm in a liquid. These nanofluids can be employed to cool the pipes exposed to such high temperature of the order 100-350^{0}C, while extracting the geothermal energy. Further when drilling, they can also be used as coolants for the machinery and equipment working in high friction and high temperature environment. In the petroleum industry also, nanofluids can be used as coolants or as drilling fluids. Also in the above fields, we come across porous media in the form of rocks inside the earth's crust, which is being affected by the rotational component of the earth's spin on its axis.

Buongiorno and Hu [6] suggested the possibility of using nanofluids in advanced nuclear systems. Another recent application of the nanofluid is in the delivery of nano-drug as suggested by Kleinstreuer et al. [7] and Eastman et al. [8] and conducted a comprehensive review on thermal transport in nanofluids to conclude that a satisfactory explanation for the abnormal enhancement in thermal conductivity and viscosity of nanofluids needs further studies. Buongiorno [9] conducted a comprehensive study to account for the unusual behavior of nanofluids based on Inertia, Brownian diffusion, thermophoresis, diffusophoresis, Magnus effects, fluid drainage, and gravity settling, and proposed a model incorporating the effects of Brownian diffusion and the thermophoresis. With the help of these equations, studies were conducted by Tzou [10] and more recently by Nield and Kuznetsov [11].

Quite recently non-Newtonian fluids housed in fluid-based systems, with and without porous matrix, have been extensively used in application situations and hence warrant the attention they have been duly getting. In the asthenosphere and the deeper mantle it is well known now that viscoelastic behavior is an important rheological process [12]. The other application areas of viscoelastic fluid saturated porous media are flow through composites, timber wood, snow systems and rheology of food transport. The problem housed in a porous medium suggests an elastohydrodynamical model for geophysical applications and the likes of it [13-15].

Herbertt [16] was the first to study natural convection in a viscoelastic fluid using the Oldroyd [17] model and showed that the elasticity of the fluid influenced the onset of marginal convection only in the presence of initial finite elastic stress. Green [18], and Vest and Arpaci [19] studied the analogous problem of Rayleigh-Benard convection in Jeffrey and Maxwell fluids, respectively, and concluded that the onset of marginal convection is independent of viscoelastic parameters, but the condition for the onset of oscillatory convection is influenced by these parameters. Eltayeb [20] studied linear and nonlinear Rayleigh-Benard convection in a visco-elastic fluid using the Oldroyd model, and showed that for Prandtl number tending to infinity the results obtained by Green [18] using Jeffrey's model, violate the criterion required for the onset of oscillatory convection. Further, he showed that the criterion for onset of oscillatory convection in the Maxwell fluid used by Vest and Arpaci [19] is always satisfied for larger values of the Prandtl number.

One of the effective methods to control convection is by maintaining a non uniform temperature gradient. Such a temperature gradient may be generated by (i) an appropriate heating or cooling at the boundaries [21], (ii) injection of fluid at one boundary and removal of the same at the other boundary [22, (iii) an appropriate distribution of heat sources [23], and (iv) radiative heat transfer [24]. These methods are mainly concerned with only space-dependent temperature gradients. However, in many of the practical situations cited earlier, the non uniform temperature gradient finds its origin in transient heating or cooling at the boundaries, so the basic temperature profile depends explicitly on position and time. This has to be determined by solving the energy equation under suitable time-dependent temperature boundary conditions, called thermal modulation.

Rudraiah et al. [25] studied the effect of modulation on the onset of thermal convection in a viscoelastic fluid saturated sparsely packed porous layer. Malashetty et al. [26] analyzed the combined effect of anisotropy of the porous medium and time dependent wall temperature on the onset of convection in a horizontal porous layer saturated with Oldroyd fluid.

Although the problem of Rayleigh-Bénard problem has been extensively investigated for non-Newtonian fluids, relatively little attention has been devoted to the thermal convection of nanofluids. The corresponding problem in the case of the effects of conductivity and viscosity ratio has also not received much attention until recently. Kuznetsov and Nield [27] investigated the onset of Double-Diffusive nanofluid convection in a layer of saturated porous medium. Agarwal et al. [28] studied the non-linear convective transport in a binary nanofluid saturated porous layer. Nield and Kuznetsov [29] studied the linear stability theory for the porous medium saturated by nanofluid with thermal conductivity and viscosity dependent on the nanoparticle volume fraction.

In the present study, the effect of thermal modulation on the onset of convection in a Oldryod-B fluid saturated with nanofluid porous medium is investigated. The boundary temperature modulation alters the basic temperature distribution from linear to nonlinear which helps in effective control of convective instability. The difficulty in dealing with such instability problems is that one has to solve time-dependent stability equations with variable coefficients, and to our knowledge no work has been initiated for such fluids in this direction. The resulting eigenvalue problem is solved by regular perturbation technique with amplitude of the temperature modulation as a perturbation parameter. In particular, it is shown that the onset of convection can be advanced by a proper tuning of the frequency of the boundary temperature modulation.

III. MATHEMATICAL FORMULATION

We consider an infinite horizontal porous layer saturated with a viscoelastic nanofluid, confined between the planes $z^*=0$ and $z^*=H$, with the vertically downward gravity force acting on it. A Cartesian frame of reference is chosen with the origin in the lower boundary and the z-axis vertically upwards. The Boussinesq approximation, which states that the variation in density is negligible everywhere in the conservation except in the buoyancy term, is assumed to hold.

For an Oldroyd-B fluid, the extra-stress tensor $\mathbf{T}$ is given by the constitutive equation [30]

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{S},\ \mathbf{S} + \lambda_1 \frac{\mathbf{DS}}{\mathbf{Dt}} = \mu\left(\mathbf{A} + \lambda_2 \frac{\mathbf{DA}}{\mathbf{Dt}}\right) \qquad (1)$$

where p is the hydrostatic pressure, $\mathbf{I}$ the identity tensor, μ the viscosity of the fluid, and $\mathbf{S}$ the extra stress tensor, λ_1 and λ_2 are constant relaxation and retardation times respectively. $\mathbf{A} = \nabla \boldsymbol{q} + \nabla \boldsymbol{q}^T$ is the strain-rate tensor, $\boldsymbol{q}$ is the velocity vector, ∇ is the gradient operator, and

$$\frac{\mathbf{DS}}{\mathbf{Dt}} = \left(\frac{\partial}{\partial t} + \boldsymbol{q}.\nabla\right)\mathbf{S} - \mathbf{S}\left(\nabla \boldsymbol{q}\right) - \left(\nabla \boldsymbol{q}\right)^T \mathbf{S} \qquad (2)$$

$$\frac{\mathbf{DA}}{\mathbf{Dt}} = \left(\frac{\partial}{\partial t} + \boldsymbol{q}.\nabla\right)\mathbf{A} - \mathbf{A}\left(\nabla \boldsymbol{q}\right) - \left(\nabla \boldsymbol{q}\right)^T \mathbf{A} \qquad (3)$$

It should be noted that this model includes the classical viscous Newtonian fluid as a special case for $\lambda_1 = \lambda_2 = 0$, and to be the Maxwell fluid when $\lambda_2 = 0$.

It is well known that in flow of viscous Newtonian fluid at a low speed through a porous medium the pressure drop caused by the frictional drag is directly proportional to velocity, which is the Darcy's law. By analogy with Oldroyd-B constitutive relationships, the following phenomenological model, which relates pressure drop and velocity for a viscoelastic fluid in a porous medium has been given by [31]

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\nabla p = -\frac{\mu}{K}\left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\boldsymbol{q}_D \qquad (4)$$

where K is permeability, $\boldsymbol{q}_D$ is Darcian velocity, which is related to the usual (i.e. volume averaged over a volume element consisting of fluid only in the pores) velocity vector $\mathbf{q}$ by $\boldsymbol{q}_D = \varepsilon \boldsymbol{q}$, ε is porosity of the porous medium. We note that when $\lambda_1 = \lambda_2 = 0$, Equation (4) simplified to Darcy's law for flow of viscous Newtonian fluid through a porous medium. Thus Equation (4) can be regarded as an approximate form of an empirical momentum equation for flow of Oldroyd-B fluid through a porous medium.

Under consideration of the balance of forces acting on a volume element of fluid, the local volume average balance of linear momentum is given by

$$\rho_0 \frac{d\boldsymbol{q}}{dt} = -\nabla p + \rho\,\mathrm{g} + \nabla.\mathbf{S} + \mathbf{r} \tag{5}$$

where $\frac{d}{dt}$ is the material time derivative, $\mathbf{r}$ is Darcy resistance for an Oldroyd-B fluid in the porous medium. Since the pressure gradient in Equation (4) can also be interpreted as a measure of the resistance to flow in the bulk of the porous medium, and $\mathbf{r}$ is a measure of the flow resistance offered by the solid matrix, thus $\mathbf{r}$ can be inferred from Equation (4) to satisfy the following equation:

$$\left(1+\lambda_1 \frac{\partial}{\partial t}\right)\mathbf{r} = -\frac{\mu\varepsilon}{K}\left(1+\lambda_2 \frac{\partial}{\partial t}\right)\boldsymbol{q} \tag{6}$$

Substituting Equation (6) into Equation (5), we obtain

$$\left(1+\lambda_1 \frac{\partial}{\partial t}\right)\left(\rho_0 \frac{d\boldsymbol{q}}{dt} + \nabla p - \rho\mathbf{g} - \nabla.\mathbf{S}\right) = -\frac{\mu\varepsilon}{K}\left(1+\lambda_2 \frac{\partial}{\partial t}\right)\boldsymbol{q} \tag{7}$$

For Darcy model, ignoring the advection term $\boldsymbol{q}.\nabla\boldsymbol{q}$ and the viscous term $\nabla.\mathbf{S}$, Equation (7) can be simplified to (after dropping the suffix D on $\boldsymbol{q}$ for simplicity)

$$\left(1+\lambda_1 \frac{\partial}{\partial t}\right)\left(\frac{\rho_0}{\varepsilon}\frac{\partial\boldsymbol{q}}{\partial t} + \nabla p - \rho\mathbf{g}\right) = -\frac{\mu}{K}\left(1+\lambda_2 \frac{\partial}{\partial t}\right)\boldsymbol{q} \tag{8}$$

The conservation equations take the form

$$\nabla^*.\mathbf{v}_\mathrm{D}^* = 0\,. \tag{9}$$

Here $\mathbf{v}_\mathrm{D}^*$ is the nanofluid Darcy velocity. We write $\mathbf{v}_\mathrm{D}^* = \left(u^*, v^*, w^*\right)$.

The conservation equation for the nanoparticles, in the absence of thermophoresis and chemical reactions, takes the form

$$\frac{\partial\phi^*}{\partial t^*} + \frac{1}{\varepsilon}\mathbf{v}_D^*.\nabla\phi^* = \nabla^*.\left[D_B \nabla^*\phi^*\right] \tag{10}$$

where ϕ^* is the nanoparticle volume fraction, ε is the porosity, and D_B is the Brownian diffusion coefficient. We use the Darcy model for a porous medium, then the momentum equation for Oldryod-B nanofluid following Equation (5) can be written as

$$\left(1+\tilde{\lambda}_1 \frac{\partial}{\partial t^*}\right)\left(\frac{\rho}{\varepsilon}\frac{\partial\mathbf{v}_\mathrm{D}^*}{\partial t^*} + \nabla^* p^* - \rho\,\mathrm{g}\right) = \frac{\mu_\mathrm{eff}}{K}\left(1+\tilde{\lambda}_2 \frac{\partial}{\partial t^*}\right)\mathbf{v}_\mathrm{D}^* \tag{11}$$

Here ρ is the overall density of the nanofluid, which we now assume to be given by

$$\rho = \phi^*\rho_\mathrm{p} + \left(1-\phi^*\right)\rho_0\left[1-\beta_\mathrm{T}\left(T^* - T_0^*\right)\right] \tag{12}$$

where ρ_p is the particle density, ρ_0 is a reference density for the fluid, and β_T is the thermal volumetric expansion. The thermal energy equation for a nanofluid can be written as

$$\left(\rho c\right)_\mathrm{m}\frac{\partial T^*}{\partial t^*} + \left(\rho c\right)_\mathrm{f}\mathbf{v}_\mathrm{D}^*.\nabla^* T^* = k_\mathrm{m}\nabla^{*2}T^* + \varepsilon\left(\rho c\right)_\mathrm{p}\left[D_\mathrm{B}\nabla^*\phi^*.\nabla T^*\right] \tag{13}$$

The conservation of nanoparticle mass requires that

$$\frac{\partial\phi^*}{\partial t^*} + \frac{1}{\varepsilon}\mathbf{v}_\mathrm{D}^*.\nabla^*\phi^* = D_\mathrm{B}\nabla^{*2}\phi^*\,. \tag{14}$$

Here c is the fluid specific heat (at constant pressure), k_m is the overall thermal conductivity of the porous medium saturated by the nanofluid, and c_p is the nanoparticle specific heat of the material constituting the nanoparticles (following Nield and Kuznetsov [29]).

Thus,

$$k_m = \varepsilon k_{eff} + (1-\varepsilon)k_s\,, \tag{15}$$

where ε is the porosity, k_{eff} is the effective conductivity of the nanofluid (fluid plus nanoparticles), and k_s is the conductivity of the solid material forming the matrix of the porous medium.

We now introduce the viscosity and the conductivity dependence on nanoparticle fraction. Following Tiwari and Das[32], we adopt the formulas, based on a theory of mixtures,

$$\frac{\mu_{eff}}{\mu_f} = \frac{1}{(1-\phi^*)^{2.5}} \tag{16}$$

$$\frac{k_{eff}}{k_f} = \frac{(k_p + 2k_f) - 2\phi^*(k_f - k_p)}{(k_p + 2k_f) + \phi^*(k_f - k_p)} \tag{17}$$

Here k_f and k_p are the thermal conductivities of the fluid and the nanoparticles, respectively.

Equation (16) was obtained by Brinkman [33], and Equation (17) is the Maxwell-Garnett formula for a suspension of spherical particles that dates back to Maxwell [34].

In the case where ϕ^* is small compared with unity, we can approximate these formulas by

$$\frac{\mu_{eff}}{\mu_f} = 1 + 2.5\phi^* \tag{18}$$

$$\frac{k_{eff}}{k_f} = \frac{(k_p + 2k_f) - 2\phi^*(k_f - k_p)}{(k_p + 2k_f) + \phi^*(k_f - k_p)} = 1 + 3\phi^*\frac{(k_p - k_f)}{(k_p + 2k_f)} \tag{19}$$

We assume that the volumetric fractions of the nanoparticles are constant on the boundaries. Thus, the boundary conditions are

$$w^* = 0,\quad \phi^* = \phi_0^* \ \text{ at } \ z^* = 0 \tag{20}$$

$$w^* = 0,\quad \phi^* = \phi_1^* \ \text{ at } \ z^* = H \tag{21}$$

For thermal modulation, the external driving force is modulated harmonically in time by varying the temperature of lower and upper horizontal boundary. Accordingly, we take

$$T(z,t) = T_0 + \frac{\Delta T}{2}\left[1+\varepsilon_t \cos(\Omega t)\right] \text{ at } z^* = 0 \tag{22}$$

$$T(z,t) = T_0 + \frac{\Delta T}{2}\left[1-\varepsilon_t \cos(\Omega t+\phi)\right] \quad \text{at } z^* = H \tag{23}$$

where ε_t represents a small amplitude of modulation (which is used as a perturbation parameter to solve the problem), Ω the frequency of modulation and ϕ the phase angle.
We consider three types of modulation, viz.,
Case (a): Symmetric (in phase, $\phi = 0$)
Case (b): asymmetric (out of phase, $\phi = \pi$) and
Case (c): only lower wall temperature is modulated while the upper one is held at constant temperature $(\phi = -i\infty)$.

A. Basic State

The basic state of the fluid is quiescent and is given by

$$\rho_b \vec{g} + \nabla p_b = 0 \tag{24}$$

$$(\rho c)_m \frac{\partial T^*}{\partial t^*} = k_m \nabla^2 T^* \tag{25}$$

$$\frac{d^2 \phi_b^*}{dz^2} = 0 \tag{26}$$

The solution of Eq. (25) satisfying the thermal conditions as given in Eqs. (22) and (23) is

$$T_b = T_1(z) + \varepsilon_t T_2(z,t)$$

where

$$T_1(z) = T_R + \frac{\Delta T}{2}\left(1 - \frac{2z}{H}\right) \tag{27}$$

$$T_2(z,t) = \mathrm{Re}\left[\left\{b(\lambda) e^{\lambda z/H} + b(-\lambda) e^{-\lambda z/H}\right\} e^{-i\omega t}\right] \tag{28}$$

with $\lambda = (1-i)\left(\frac{(\rho c)_m \, \omega H^2}{2k_m}\right)^{1/2}$,

$$b(\lambda) = \frac{\Delta T}{2}\left(\frac{e^{-i\phi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right) \tag{29}$$

and Re stands for real part. We do not record the expressions of p_b and ρ_b as these are not explicitly required in the remaining part of the paper.

B. Linear Stability Analysis

Let the basic state be distributed by an infinitesimal perturbation. We now have

$$\mathbf{v} = \mathbf{v}', \; p = p_b + p', \; T = T_b + T', \quad \phi = \phi_b + \phi' \tag{30}$$

where prime indicates that the quantities are infinitesimal perturbations. Substituting Eq. (26) in Eqs. (9)-(15) and linerising by neglecting products of primed quantities, we have,

$$(1+\lambda_1 s)(\nabla p - RT\hat{e}_z + Rn\,\phi\,\hat{e}_z + \gamma_a s\,\mathbf{v}) + \tilde{\mu}(1+\lambda_2 s)\,\mathbf{v} = 0 \tag{31}$$

$$\frac{\partial T'}{\partial t} + w'\frac{\partial T_b}{\partial z} = \tilde{k}\frac{\partial^2 T}{\partial z^2} + \frac{N_B}{Le}\left(\frac{\partial T_b}{\partial z} + \frac{\partial T'}{\partial z} + \frac{\partial \phi'}{\partial z}\frac{\partial T_b}{\partial z}\right) \tag{32}$$

$$\frac{1}{\sigma}\frac{\partial \phi'}{\partial t} + \frac{1}{\varepsilon} w' = \frac{1}{Le}\nabla^2 \phi' \tag{33}$$

$$w' = 0, \; T' = 0, \; \phi' = 0 \text{ at } z = 0,1 \tag{34}$$

After using the transformations

$(x,y,z) = (x^*, y^*, z^*)/H$, $t = t^*\alpha_m/\sigma H^2$,
$(u,v,w) = (u^*, v^*, w^*)\,H/\alpha_m$, $p = p^* K/\mu_f \alpha_m$,
$\phi = \frac{\phi^* - \phi_0^*}{\phi_1^* - \phi_0^*}$, $T = \frac{T^* - T_c^*}{T_h^* - T_c^*}$, $\omega = \frac{\sigma \Omega H^2}{\alpha_m}$,
$\alpha_m = \frac{k_m}{(\rho c_p)_f}$, $\sigma = \frac{(\rho c_p)_m}{(\rho c_p)_f}$, $\tilde{\mu} = \frac{\mu_{eff}}{\mu_f}$, $\tilde{k}_p = \frac{k_p}{k_f}$,
$\tilde{k}_s = \frac{k_s}{k_f}$, $\tilde{k} = \frac{k_m}{k_f}$.

The dimensionless group that appear are $\Pr = \frac{\mu_f}{\rho\,\alpha_m}$ is the Prandtl number, $Da = \frac{K}{H^2}$ is the Darcy number, $Va = \frac{\varepsilon^2 \Pr}{Da}$ is the Vadász number, $\lambda_1 = \frac{\tilde{\lambda}\alpha_m}{\sigma H^2}$ is the relaxation parameter and $\lambda_2 = \frac{\tilde{\lambda}_2 \alpha_m}{\sigma H^2}$ is the retardation parameter. $\gamma_a = \frac{\varepsilon}{\sigma Va}$ is the acceleration coefficient, $Le = \frac{\alpha_m}{D_m}$ is the nanofluid Lewis number, $R = \frac{R_0\, g\, K(1-\phi_0^*)\beta_T\, \Delta T^*\, H}{\mu_f\, \alpha_m}$ is the nanoparticle Rayleigh number, $N_B = \frac{(\rho c)_p}{(\rho c)_f}(\phi_1^* - \phi_0^*)$ is a modified particle-density increment.
In deriving Eq. (31) the term proportional to the product of ϕ and T (Oberbeck-Boussinesq approximation) is neglected. This assumption is likely to be valid in the case of small temperature gradients in a dilute suspension of nanoparticles. For the case of regular fluid (not a nanofluid), the parameters Rn and N_B are zero.

We eliminate pressure by operating on Eq. (31) with $\hat{e}_z$ *curl curl* and using the identity *curl curl* $\equiv$ *grad div* $-\nabla^2$ results in

$$\left\{(1+\lambda_1 s)\, s\,\gamma_a + (1+\lambda_2 s)\,\tilde{\mu}\right\}\nabla^2 w' = (1+\lambda_1 s)\left\{R\nabla_H^2 - Rn\nabla_H^2 \phi'\right\} \tag{35}$$

Here ∇_H^2 is the two-dimensional Laplacian operator on the horizontal plane. Combining Eqs. (32)-(34), we obtain equations for the vertical component of velocity w in the form (dropping prime)

(36)

$$\left[\frac{\partial}{\partial t}-\nabla^2\gamma\right]\left[\frac{1}{\sigma}\frac{\partial}{\partial t}-\frac{\nabla^2}{Le}\right]$$
$$\left[\nu\left(1+\lambda_2 s\right)+s\,\gamma_a\left(1+\lambda_1 s\right)\right]\nabla^2 w-$$
$$\frac{\left(1+\lambda_1 s\right)Rn}{\varepsilon}\left[\frac{\partial}{\partial t}-\nabla^2\gamma\right]\nabla_1^2 w+$$
$$\left(1+\lambda_1 s\right)R\,\frac{\partial T_b}{\partial z}\left[\frac{1}{\sigma}\frac{\partial}{\partial t}-\frac{\nabla^2}{Le}\right]\nabla_1^2 w=0$$

where $\nu=1+1.25\left(\phi_1^*+\phi_0^*\right)$, and

$$\gamma=\varepsilon+(1-\varepsilon)\,\tilde{k}_s+\frac{3\left(\phi_1^*+\phi_0^*\right)\varepsilon}{2}\left(\frac{\tilde{k}_p-1}{\tilde{k}_p+2}\right).$$

It is worth noting that the factor ν comes from the mean value of $\tilde{\mu}(z)$ over the range [0, 1] and the factor γ is the mean value of $\tilde{k}(z)$ over the same range. That means that when evaluating the critical Rayleigh number it is a good approximation to base that number on the mean values of the viscosity and conductivity based in turn on the basic solution for the nanofluid fraction.

The boundary condition (34) can also be expressed in terms of w in the form

$$w=\frac{d^2 w}{dz^2}=0 \quad \text{at} \quad z=0,1 \tag{37}$$

Using Eq. (27), the dimensionless temperature gradient appearing in Eq. (32) may be written as

$$\frac{\partial T_b}{\partial z}=-1+\varepsilon f \tag{38}$$

where $f=\mathrm{Re}\left[\left\{A(\lambda)e^{\lambda z}+A(-\lambda)e^{-\lambda z}\right\}e^{-i\omega t}\right]$ for

$$A(\lambda)=\frac{\lambda}{2}\left(\frac{e^{-i\varphi}-e^{-\lambda}}{e^{\lambda}-e^{-\lambda}}\right) \text{and } \lambda=(1-i)\left(\frac{\sigma\omega}{2}\right)^{1/2} \tag{39}$$

IV. Method of Solution

We seek the eigen functions w and eigen values Ra of Eq. (36) for the basic temperature gradient given by Eq. (38) that departs from the linear profile $\frac{\partial T_b}{\partial z}=-1$ by quantities of order ε_t. We therefore assume the solution of Eq. (36) in the form

$$(w,R)=(w_0,R_0)+\varepsilon_t\,(w_1,R_1)+\varepsilon_t^2\,(w_2,R_2)+... \tag{40}$$

Substituting Eq. (40) into Eq. (36) and equating the coefficients of various powers of ε_t on either side of the resulting equation, we obtain the following system of equations up to the order of ε_t^2 :

$$Lw_0=0 \tag{41}$$

$$Lw_1=(1+\lambda_1 s)\left[\left(\frac{R_0\,\omega G}{\sigma}\nabla_1^2+\frac{R_0\,\nabla^2 f}{Le}\right)\nabla_1^2-\frac{R_1}{Le}\nabla^2\,\nabla_1^2\right]w_0 \tag{42}$$

$$Lw_2=(1+\lambda_1 s)\left[R_0\left(\frac{\omega G}{\sigma}+\frac{f}{Le}\nabla^2\right)-\frac{R_1}{Le}\nabla^2\right]\nabla_1^2 w_1+$$
$$(1+\lambda_1 s)R_1\left(\frac{\omega G}{\sigma}+\frac{f}{Le}\nabla^2-\frac{R_2}{Le}\nabla^2\right)\nabla_1^2 w_0 \tag{43}$$

where

$$L=\left(\frac{\partial}{\partial t}-\nabla^2\gamma\right)\left(\frac{1}{\sigma}\frac{\partial}{\partial t}-\frac{\nabla^2}{Le}\right)\left(\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nu+\left(1+\lambda_1\frac{\partial}{\partial t}\right)\gamma_a\frac{\partial}{\partial t}\right)\nabla^2-\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{Rn}{\varepsilon}\left(\frac{\partial}{\partial t}-\nabla^2\gamma\right)\nabla_1^2+$$
$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{R_0}{Le}\nabla^2\nabla_1^2$$

.

and w_0, w_1, w_2 are required to satisfy the boundary conditions of Equation (37).

We now assume the solutions for Eq. (41) in the form $w_0=w_0(z).\exp\left[i(lx+my)\right]$, where $w_0(z)=w_0^n(z)=\sin(n\pi z)n$ = 1, 2, 3, … and *l*, *m* are the wave numbers in the $x-y$ plane such that $l^2+m^2=\alpha^2$. The corresponding eigen values are given by

$$R_0=\frac{\left(n^2\pi^2+\alpha^2\right)^2\nu\gamma}{\alpha^2}-\frac{Rn\;Le\,\gamma}{\varepsilon} \tag{44}$$

For a fixed value of the wave number α, the least eigen value occurs at $n=1$ and is given by

$$R_0=\frac{\left(\pi^2+\alpha^2\right)^2\nu\gamma}{\alpha^2}-\frac{Rn\;Le\gamma}{\varepsilon} \tag{45}$$

R_{0c} assumes the minimum value when $\alpha^2=\pi^2$

$$R_{0c}=4\pi^2\nu\gamma-\frac{Rn\;Le\,\gamma}{\varepsilon} \tag{46}$$

These are the values reported by Horton and Rogers [35] in the absence of concentration Rayleigh number Rn .

The equation for w_1 then takes the form

$$Lw_1=R_0\alpha^2\left(1-\lambda_1\,i\,\omega\right)\left(\frac{\omega}{\sigma}G+\frac{\left(D^2-\alpha^2\right)f}{Le}\right)\sin\pi z \tag{47}$$

where $D=\frac{d}{dz}$.

Thus

$$D^2 f\sin\pi z=\left(\lambda^2-\pi^2\right)f\sin\pi z+2\lambda\pi\,f\,'\cos\pi z \tag{48}$$

with

$$f\,'=R.P.\left[\left(A(\lambda)e^{\lambda z}-A(-\lambda)e^{-\lambda z}\right)e^{-i\omega t}\right]$$

Using Eq. (48), Eq. (47) becomes

(49)

$$Lw_1 = R_0\alpha^2\left(-1+\lambda_1 i\,\omega\right)\begin{pmatrix}\dfrac{\omega}{\sigma}G\sin\pi z - \\ L_1 f\sin\pi z+\dfrac{2\lambda\pi f'}{Le}\cos\pi z\end{pmatrix}$$

where $L_1 = \dfrac{i\,\omega+\pi^2+\alpha^2}{Le}$.

We solve Eq. (49) for w_1 by expanding the right hand side of it in Fourier series expansion and inverting the operator L. For this we need the following Fourier series expansions

$$g_{nm}(\lambda)=2\int_0^1 e^{\lambda z}\sin(m\pi z)\sin(n\pi z)\,dz = \frac{-4\,nm\pi^2\lambda\left[1+(-1)^{n+m+1}e^z\right]}{\left[\lambda^2+(n+m)^2\pi^2\right]\left[\lambda^2+(n-m)^2\pi^2\right]} \tag{50}$$

$$f_{nm}(\lambda)=2\int_0^1 e^{\lambda z}\cos(m\pi z)\cos(n\pi z)\,dz = \frac{2\lambda\left[\lambda^2+(n+m)^2\pi^2\right]\left[1+(-1)^{n+m+1}e^z\right]}{\left[\lambda^2+(n+m)^2\pi^2\right]\left[\lambda^2+(n-m)^2\pi^2\right]} \tag{51}$$

so that

$$e^{\lambda z}\sin(m\pi z)=\sum_{n=1}^{\infty}g_{nm}(\lambda)\,\sin(n\pi z) \tag{52}$$

$$e^{\lambda z}\cos(m\pi z)=\sum_{n=1}^{\infty}f_{nm}(\lambda)\,\cos(n\pi z) \tag{53}$$

Now $L(\omega,n)= A+i\,\omega B$ (54)

where

$$A=\left[\begin{array}{l}\omega^2\gamma_a\left(n^2\pi^2+\alpha^2\right)^2\left(\dfrac{1}{Le}+\dfrac{\gamma}{\sigma}\right)+\lambda_1\omega^2\left(-\dfrac{\omega^2\gamma_a}{\sigma}\left(n^2\pi^2+\alpha^2\right)+\left(n^2\pi^2+\alpha^2\right)^3\dfrac{\gamma_a}{Le}-\dfrac{Rn}{\varepsilon}\alpha^2\right)\\ -\nu\left(-\dfrac{\omega^2\left(n^2\pi^2+\alpha^2\right)}{\sigma}+\left(n^2\pi^2+\alpha^2\right)^3\dfrac{\gamma}{Le}\right)-\nu\lambda_2\omega^2\left(-\left(n^2\pi^2+\alpha^2\right)^2\left(\dfrac{1}{Le}+\dfrac{\gamma}{\sigma}\right)\right)\end{array}\right]$$

and

$$B=\left[\begin{array}{l}\left(-\dfrac{\omega^2\gamma_a}{\sigma}\left(n^2\pi^2+\alpha^2\right)+\left(n^2\pi^2+\alpha^2\right)^3\dfrac{\gamma\gamma_a}{Le}-\dfrac{Rn}{\varepsilon}\alpha^2\right)-\lambda_1\,\omega^2\gamma_a\left(n^2\pi^2+\alpha^2\right)^2\left(\dfrac{1}{Le}+\dfrac{\gamma}{\sigma}\right)-\\ \nu\left(-\left(n^2\pi^2+\alpha^2\right)^2\left(\dfrac{1}{Le}+\dfrac{\gamma}{\sigma}\right)\right)+\nu\lambda_2\left(-\dfrac{\omega^2\left(n^2\pi^2+\alpha^2\right)}{\sigma}+\left(n^2\pi^2+\alpha^2\right)^3\dfrac{\gamma}{Le}\right)\end{array}\right]$$

It is easily seen that

$$L\left[\sin(n\pi z)e^{-i\omega t}\right] = L(\omega,n)\,\sin(n\pi z)\,e^{-i\omega t}$$

$$L\left[\cos(n\pi z)e^{-i\omega t}\right] = L(\omega,n)\,\cos(n\pi z)\,e^{-i\omega t}$$

and Eq. (49) now become

$$Lw_1=\left(-1+\lambda_1 i\omega\right)\alpha^2R_0\left[\begin{array}{l}\dfrac{\omega}{\sigma}I.P.\displaystyle\sum_{n=1}^{\infty}A_n(\lambda)\,\sin n\pi z\,e^{-i\omega t}-L_1\,R.P.\displaystyle\sum_{n=1}^{\infty}A_n(\lambda)\,\sin n\pi z\,e^{-i\omega t}\\ +\dfrac{2\lambda\pi}{Le}R.P.\displaystyle\sum_{n=1}^{\infty}B_n(\lambda)\cos n\pi z\,e^{-i\omega t}\end{array}\right] \tag{55}$$

so that

$$w_1=\left(-1+\lambda_1 i\omega\right)\alpha^2R_0\left[\begin{array}{l}\dfrac{\omega}{\sigma}I.P.\displaystyle\sum_{n=1}^{\infty}\dfrac{A_n(\lambda)}{L(\omega,n)}\sin n\pi z\,e^{-i\omega t}-L_1\,R.P.\displaystyle\sum_{n=1}^{\infty}\dfrac{A_n(\lambda)}{L(\omega,n)}\sin n\pi z\,e^{-i\omega t}\\ +\dfrac{2\lambda\pi}{Le}R.P.\displaystyle\sum_{n=1}^{\infty}\dfrac{B_n(\lambda)}{L(\omega,n)}\cos n\pi z\,e^{-i\omega t}\end{array}\right] \tag{56}$$

where $A_n(\lambda)=A(\lambda)\,g_{n1}(\lambda)+A(-\lambda)\,g_{n1}(-\lambda)$, and $B_n(\lambda)=A(\lambda)\,f_{n1}(\lambda)+A(-\lambda)\,f_{n1}(-\lambda)$.

To simplify Eq. (43) for w_2 we need

$$Lw_2=(1+\lambda_1 s)\begin{bmatrix} R_0\left(\frac{\omega G}{\sigma}+\frac{\nabla^2 f}{Le}\right)\nabla_1^2 w_1 \\ -R_2\frac{\nabla^2}{Le}.\nabla_1^2 w_0\end{bmatrix} \tag{57}$$

The equation for w_2 then can be written as

$$Lw_2=(1-\lambda_1 i\omega)\left[R_0\left(\frac{\omega G}{\sigma}-L_n f\right)w_1+\frac{2Df\,Dw_1}{Le}\right] -R_2\frac{\alpha^2}{Le}\left(\pi^2+\alpha^2\right) \tag{58}$$

where

$$L_n=\frac{i\,\omega+n^2\,\pi^2+\alpha^2}{Le}.$$

We shall not require the solution of this equation but merely use it to determine R_2.

The solvability condition requires that the time-independent part of the right hand side of (58) must be orthogonal to $\sin(\pi z)$. Multiplying Eq. (58) by $\sin(\pi z)$ and integrating between 0 and 1 we obtain

$$R_2=\frac{2LeR_0\,(1-2i\lambda\omega)}{\nabla^2}\int_0^1\overline{\left(\frac{\nabla^2 f}{Le}+\frac{\omega G}{\sigma}\right)w_1}\,\sin(\pi z)\,dz \tag{59}$$

where an upper bar denotes the time average.

We have the Fourier series expansions

$$f\,\sin\pi z=R.P.\sum A_n(\lambda)\sin n\pi z\,e^{-i\omega t},$$
$$Df\,\sin\pi z=R.P.\sum\lambda C_n(\lambda)\sin n\pi z\,e^{-i\omega t} \tag{60}$$

where $C_n(\lambda)=A(\lambda)\,g_{n1}(\lambda)-A(-\lambda)\,g_{n1}(-\lambda)$

Using Eq. (60) in Eq. (59) we obtain

$$R_2=\frac{LeR_0^2\,\alpha^2}{2\left(\pi^2+\alpha^2\right)}\begin{bmatrix}\left(-\frac{\omega^2}{\sigma^2}-\overline{L}_n\,L_1\right)R.P.\sum\frac{|A_n|^2}{|L(\omega,\,n)|^2}L^*(\omega,n)(1-2i\lambda_1\omega)(-1+i\lambda_1\omega) \\ +\frac{4\,n\,\pi^2\,\lambda_1}{Le^2}R.P.\sum\overline{\lambda_1\,C_n}\,\frac{B_n}{|L(\omega,\,n)|^2}L^*(\omega,n)(1-2i\lambda_1\omega)(-1+i\lambda_1\omega)\end{bmatrix} \tag{61}$$

where $L^*(\omega,n)$ is the complex conjugate of $L(\omega,n)$, $|A_n(\lambda)|^2=\frac{16n^2\pi^4\omega^2}{\left(\omega^2+(n+1)^4\,\pi^4\right)\left(\omega^2+(n-1)^4\,\pi^4\right)}$. The critical value of R_2, denoted by R_{2c}, is obtained at the wave number given by equation $\alpha_c=\pi$ for the following three different cases. We evaluate R_{2c} for the following cases.

(a) when the oscillating temperature field is symmetric so that the wall temperatures are modulated in phase (with $\phi=0$),

(b) when the wall temperature field is antisymmetric corresponding to out-of-phase modulation (with $\phi=\pi$)

(c) when only the temperature of the bottom wall is modulated, the upper wall being held at a constant temperature (with $\phi=-i\,\infty$).

V. Results and Discussion

The problem of linear convection in a sparsely packed Oldryod-B fluid saturated with nanofluid layer subject to different periodic temperature boundary conditions is investigated. The solution is obtained on the assumption that the amplitude of the applied temperature modulation is small. The expression for the critical correction Rayleigh number R_{2c} is computed as a function of frequency of the modulation for different parameter values and the results are depicted in Figs. 1-18. The sign of R_{2c} characterizes the stabilizing or destabilizing effect of modulation. A positive R_{2c} indicates that the modulation effect is stabilizing while a negative R_{2c} indicates that the modulation effect is destabilizing compared to the system in which the modulation is absent.

The variation of critical correction thermal Rayleigh number R_{2c} with frequency ω for symmetric modulation for different governing parameters (Figs 1-8) for both regular $(Rn=0)$ and nanofluids $(Rn\neq 0)$ show that for small frequencies the critical correction Rayleigh number is negative indicating that the effect of symmetric modulation is destabilizing while for moderate and large values of ω its effect is stabilizing. Figure 1 reveal that an increase in the value of λ_1 is to increase the magnitude of R_{2c}. On the other hand the effect of modulation diminishes as the stress relaxation parameter λ_1 becomes smaller and smaller. The peak negative or positive value of R_{2c} is found to increase with λ_1 for both regular and nanofluids. The effect of strain retardation parameter λ_2 is found to be opposite of the stress relaxation parameter λ_1 as seen in Fig. 2. Therefore one can conclude that the stress relaxation parameter is more pronounced in aiding the onset of convection compared to the effect of strain retardation parameter for both regular and nanofluids. Similar results were observed by Malashetty et al. [26] for regular fluid.

Figure 3 shows the variation of R_{2c} with ω for different values of concentration Rayleigh number Rn. As Rn increases the magnitude of correction thermal Rayleigh number R_{2c} decreases indicating that the effect of Rn is to delay the onset of convection. However, R_{2c} is negative for small frequencies indicating that the symmetric modulation has destabilizing effect while for moderate and large values of frequency its effect is stabilizing. This is a similar result obtained by Umavathi [36] for Newtonian fluid. The effect of porosity ε for symmetric modulation is shown in Fig. 4. This figures reveal that as ε increases the value of $|R_{2c}|$ becomes small indicating that larger values of ε decreases the effect of modulation. Here also it is observed that as ω increases R_{2c} increases to its maximum value initially and then starts decreasing with further increase in ω. When ω is very large all the curves for different porosity coincide and $|R_{2c}|$ approaches to zero for both regular and nanofluid. Figure 5 depicts the variation of R_{2c} with frequency ω for different values of Lewis number Le for the case of symmetric modulation. Lewis number shows the similar effect as that of porosity ε. That is to say that, an increase in the value of Lewis number decreases the value of $|R_{2c}|$ indicating that the effect of increasing Le is to reduce the effect of thermal modulation for both regular and nanofluid which is a similar result observed Umavathi[36]. The effect of thermal capacity ratio σ is to increase $|R_{2c}|$ for both regular and nanofluids as seen in Fig. 6. Here also as ω increases R_{2c} increases to its maximum initially and then starts decreasing with further increase in ω. When ω is very large all the curves for different thermal capacity ratio σ coincide and $|R_{2c}|$ approaches to zero for both regular and nanofluids. The effect of Vadasz number Va shows the similar nature as that of thermal capacity ratio σ as seen in Fig. 7. That is to say that, increasing the value of Vadasz number is to decrease the thermal modulation for both regular and nanofluid. The effect of viscosity variation parameter υ and conductivity variation parameter γ is shown in Figs. 8 and 9 respectively for symmetric modulation. Their effect is found to be similar to the effect of stress relaxation parameter λ_1, thermal capacity ratio σ and Vadasz number Va. That is to say that, υ and γ delay the onset of convection for both regular and nanofluids.

The results obtained for the case of asymmetric modulation are presented in Figs. 10-18. All these figures show that for all parameters small frequencies has destabilizing effect while for moderate and large values of frequency their effects are stabilizing for both regular and nanofluid. It is seen from Fig. 10 that an increase in the value of λ_1 is to increase the magnitude of R_{2c}. The effect of strain retardation parameter λ_2 is to decrease the magnitude of R_{2c} as seen in Fig.11. The effect of concentration Rayleigh number Rn, porosity ε, Lewis number Le, thermal capacity ratio σ, Vadasz number Va, viscosity and conductivity variation parameters υ and γ show the same effect as in the case of symmetric modulation and hence a detailed explanation is not required.

The nature of all the graphs for lower wall temperature modulation is found to be qualitatively similar to the asymmetric modulation and therefore we omit a graphical representation of the same.

VI. Conclusions

The effect of thermal modulation on the onset of convection for Oldroyd fluid saturated with nanofluids porous layer is studied using a linear stability analysis and the following conclusion are drawn.

1. The increase in stress relaxation parameter enhances the effect of modulation while increase in strain retardation parameter suppresses the effect of modulation for all three types of modulation.
2. The concentration Rayleigh and Vadasz number suppresses the effect of modulation. The porosity suppresses the effect of modulation for regular fluid whereas it enhances the effect of modulation for nanofluids. The thermal capacity ratio, Vadasz number, viscosity and conductivity ratio enhances the effect of modulation for both regular and nanoflid.
3. The peak values of thermal correction Rayleigh number is obtained for regular fluid when compared to nanofluids for all thermal modulations.
4. The effect of all three types of modulation namely, symmetric, asymmetric, and only lower wall temperature modulations is found to be destabilizing as compared to the unmodulated systerm.
5. The effect of thermal modulation disappears for large frequency in all the cases.
6. The onset of convection is delayed for nanofluids when compared to regular fluid.
7. The effect of stress and strain relaxation parameter for symmetric modulation for regular fluid were also obtained by Malashetty et al. [26].

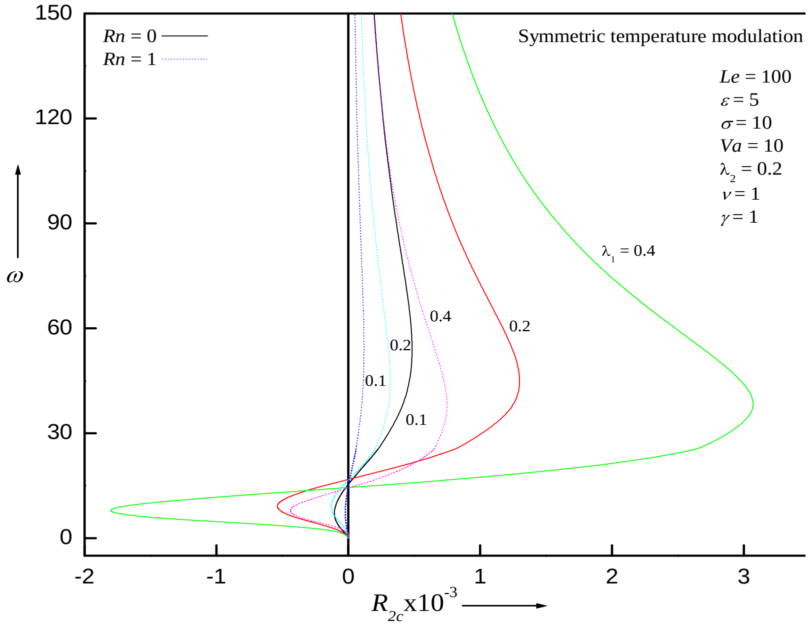

Fig. 1 Variation of R_{2c} with ω for different values of Rn and λ_1

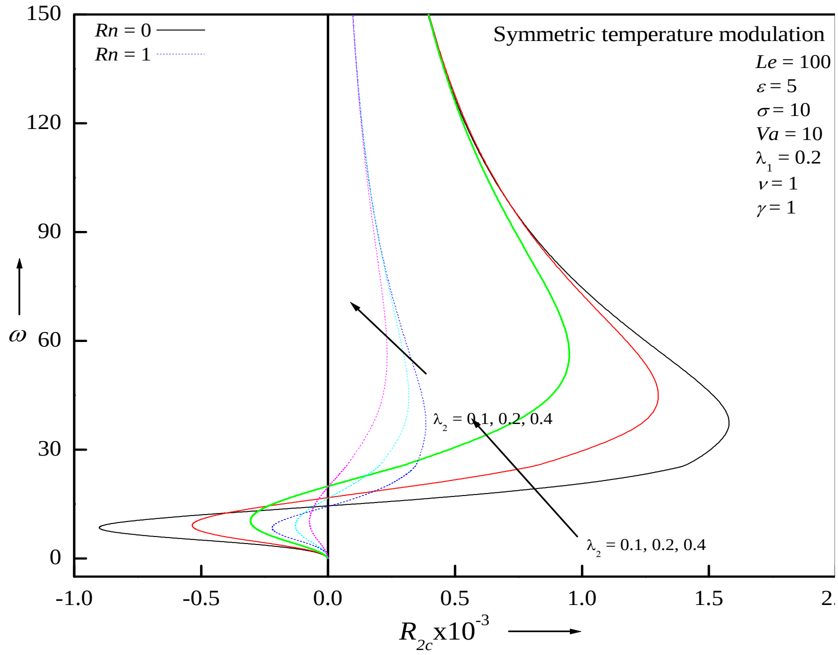

Fig. 2 Variation of R_{2c} with ω for different values of Rn and λ_2

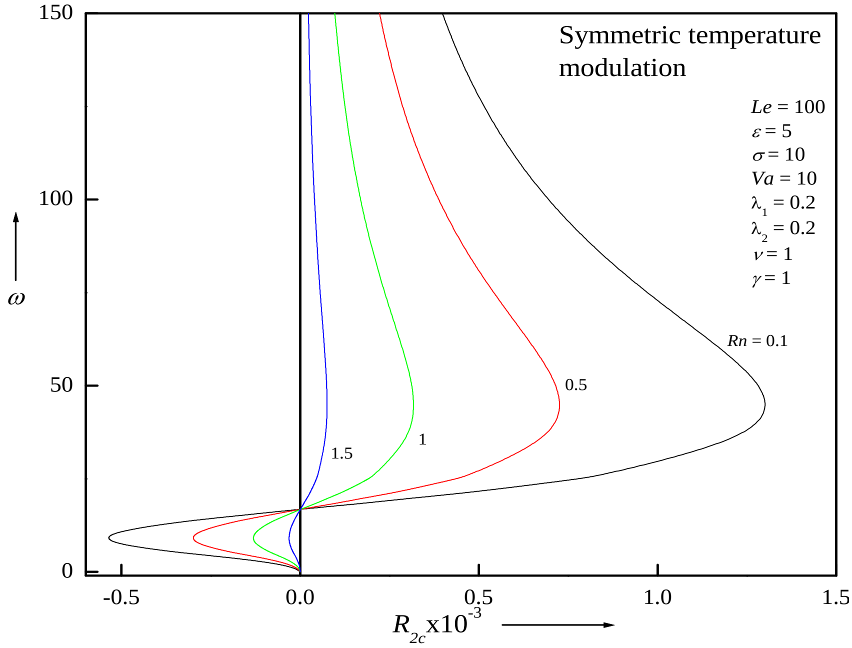

Fig. 3 Variation of R_{2c} with ω for different values of Rn

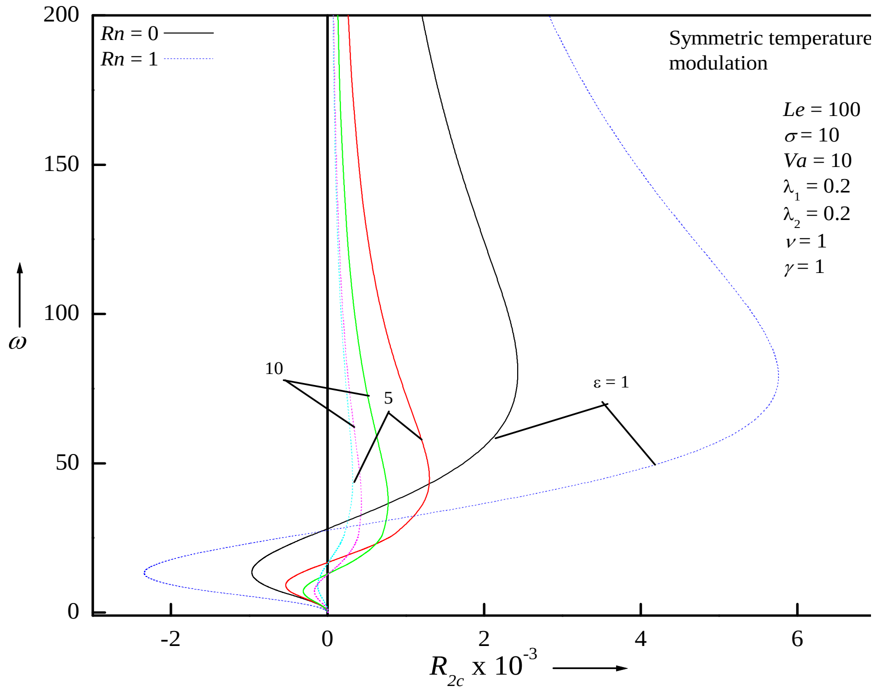

Fig. 4 Variation of R_{2c} with ω for different values of Rn and ε

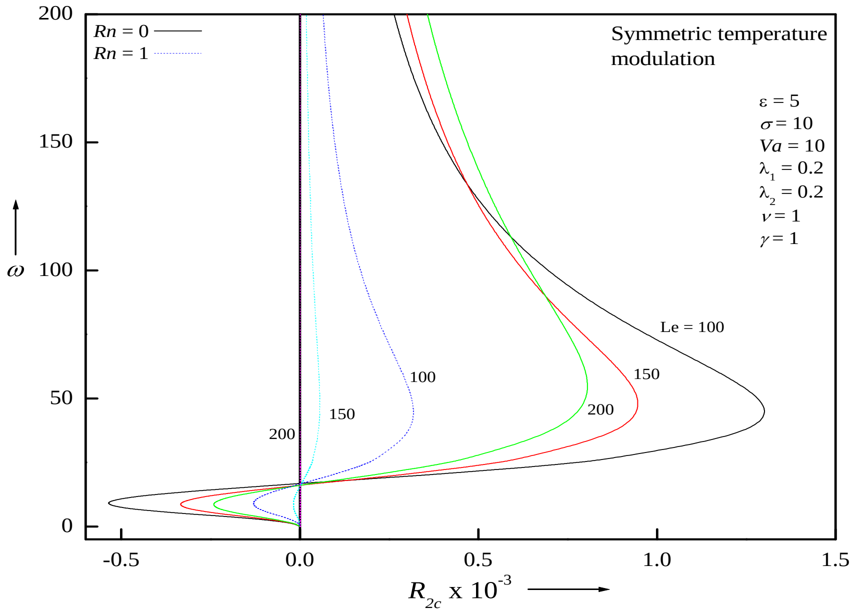

Fig. 5 Variation of R_{2c} with ω for different values of *Rn* and Le

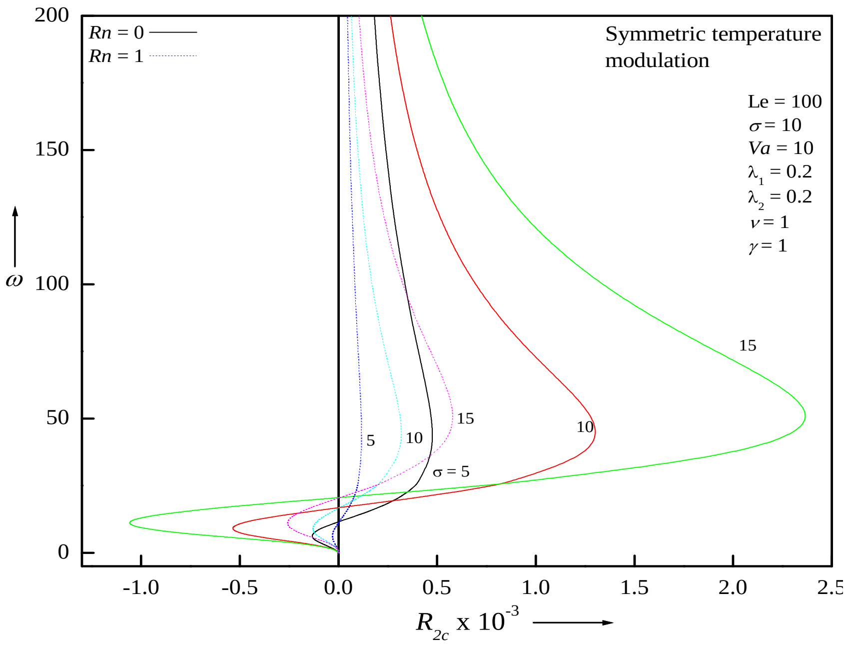

Fig. 6 Variation of R_{2c} with ω for different values of *Rn* and ε

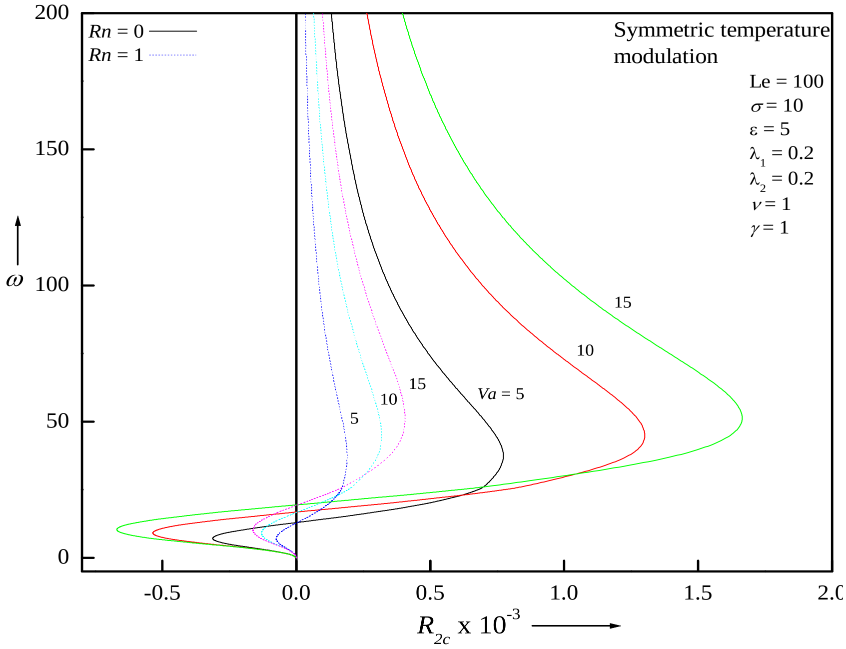

Fig. 7 Variation of R_{2c} with ω for different values of *Rn* and *Va*

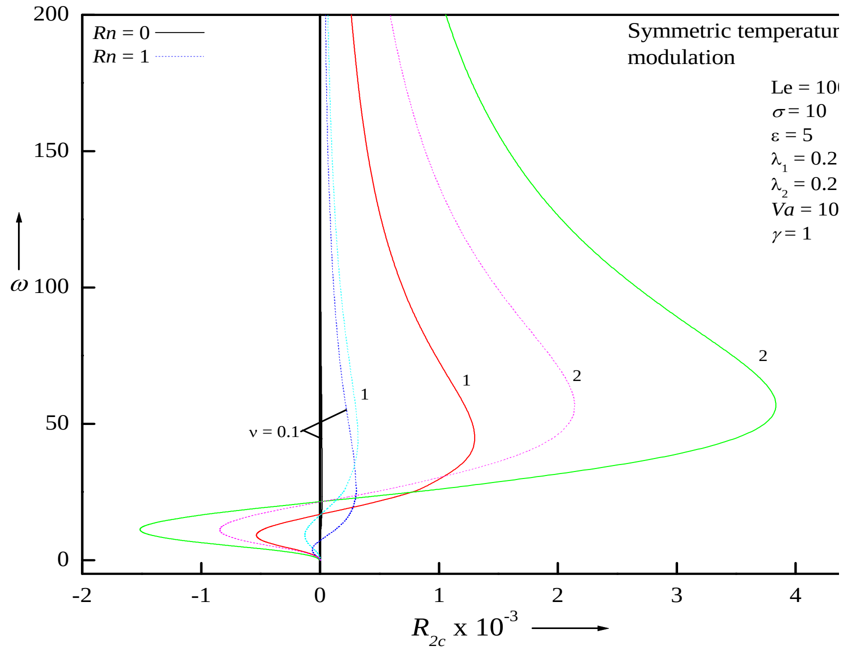

Fig. 8 Variation of R_{2c} with ω for different values of *Rn* and ν

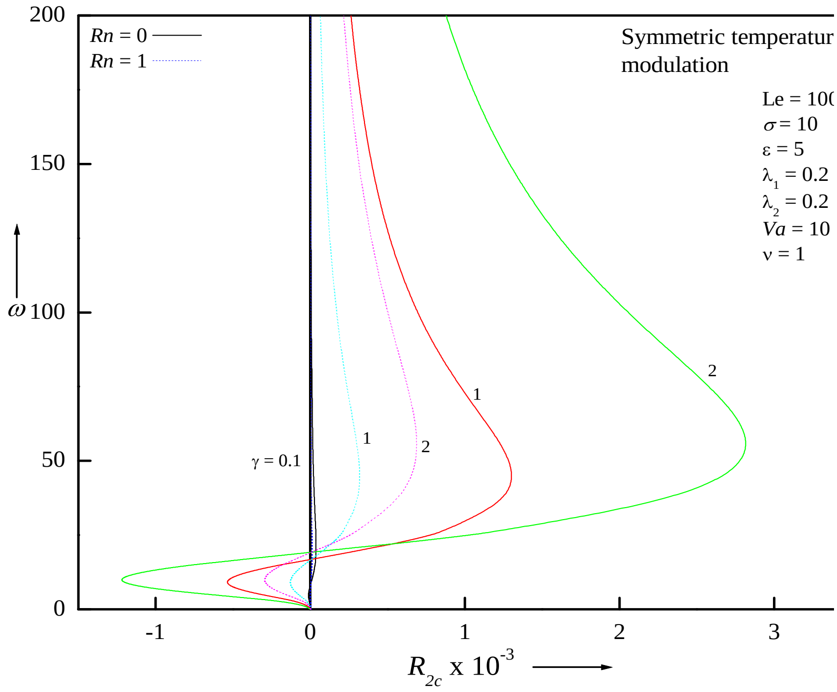

Fig. 9 Variation of R_{2c} with ω for different values of Rn and γ

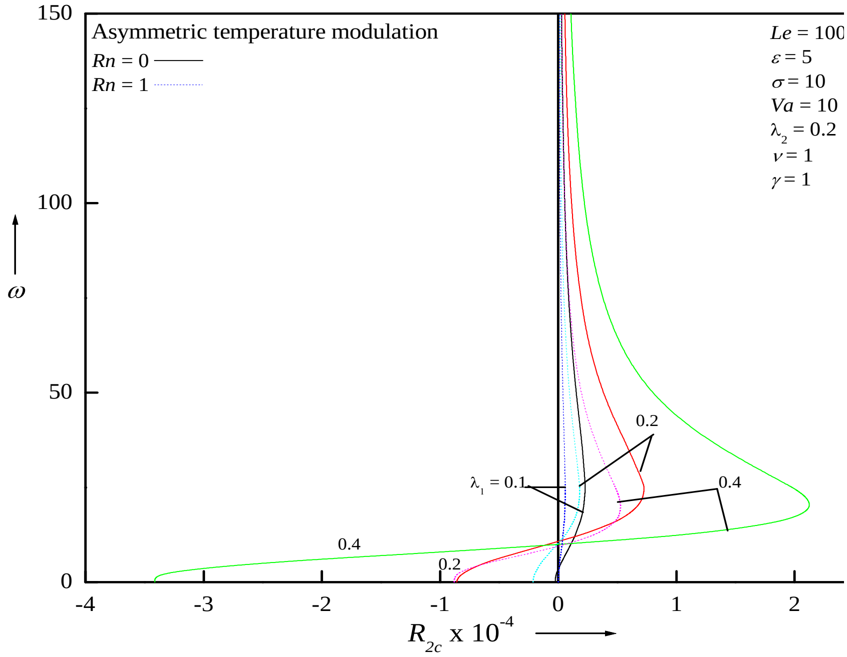

Fig. 10 Variation of R_{2c} with ω for different values of Rn and λ_1

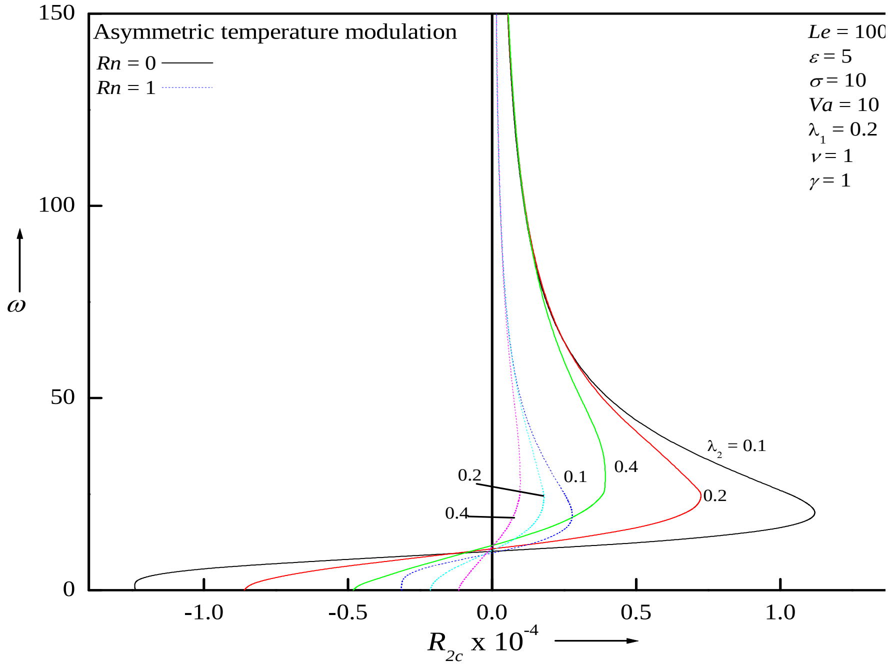

Fig. 11 Variation of R_{2c} with ω for different values of Rn and λ_2

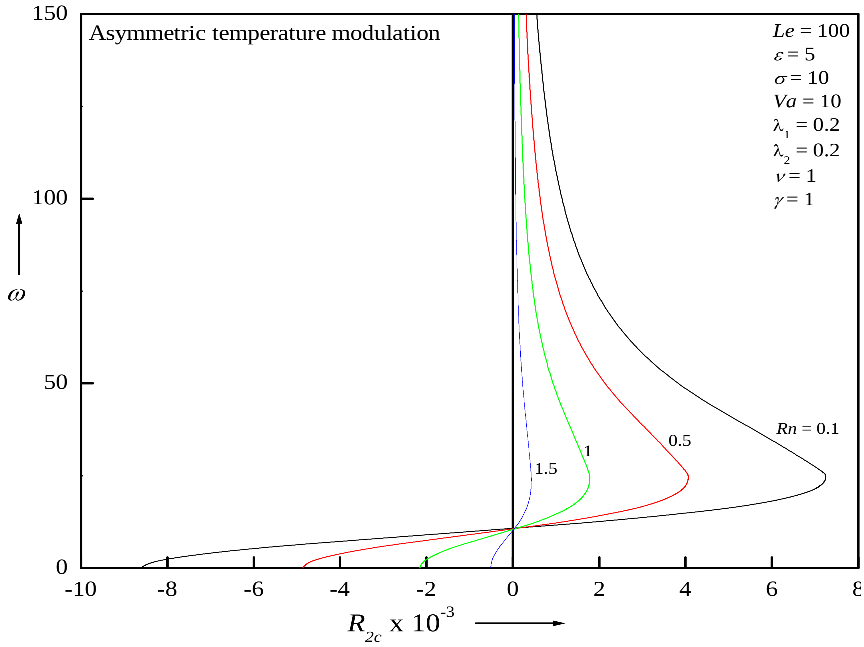

Fig. 12 Variation of R_{2c} with ω for different values of Rn

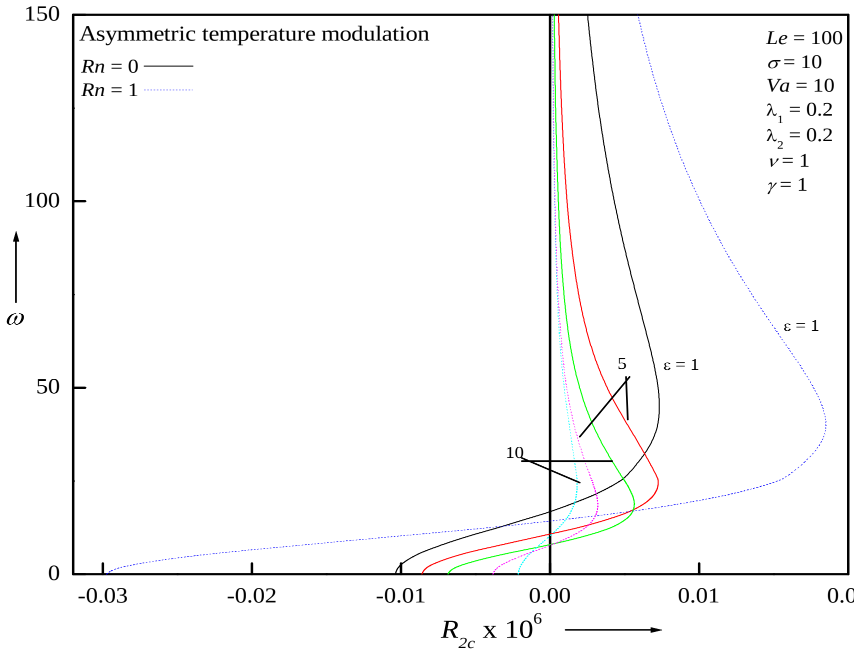

Fig. 13 Variation of R_{2c} with ω for different values of Rn and ε

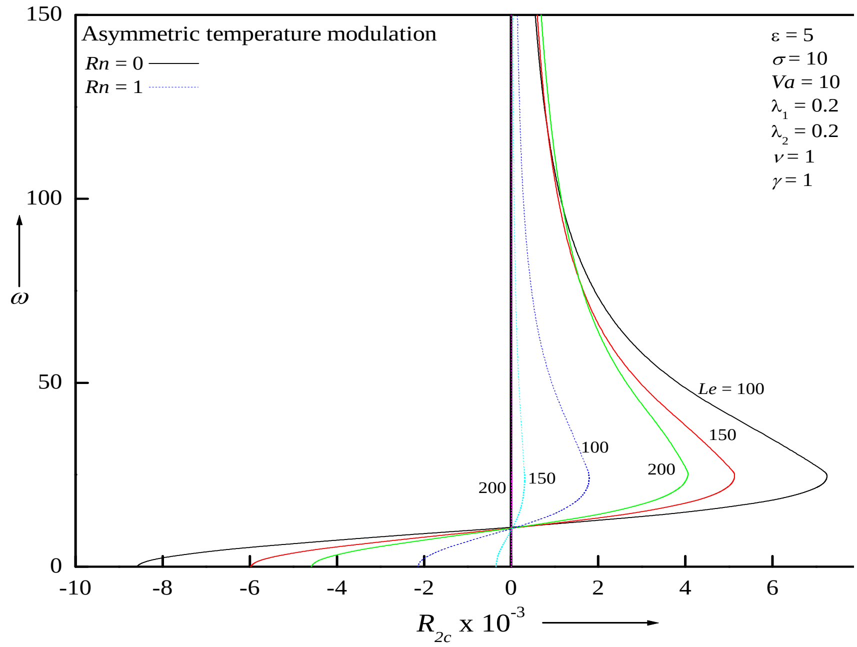

Fig. 14 Variation of R_{2c} with ω for different values of Rn and Le

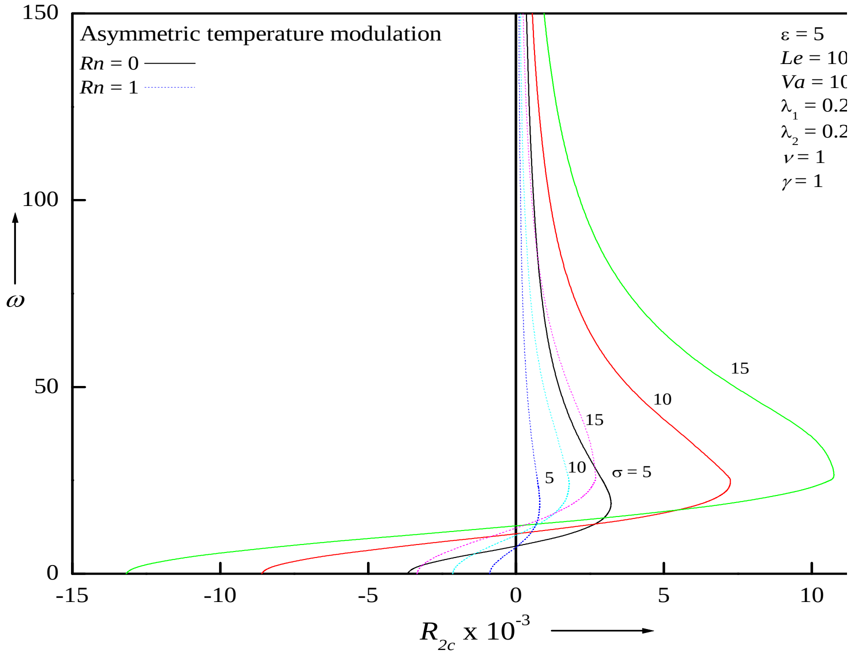

Fig. 15 Variation of R_{2c} with ω for different values of Rn and σ

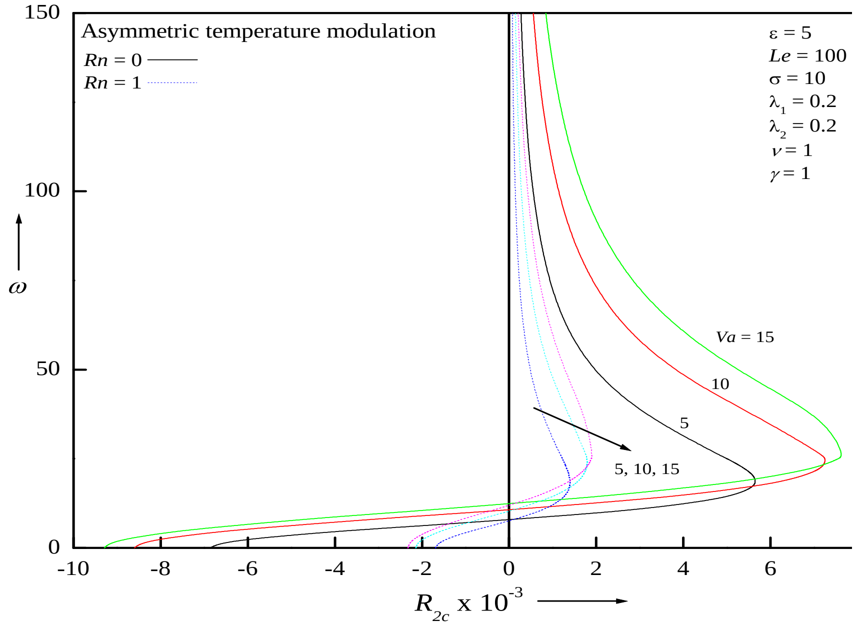

Fig. 16 Variation of R_{2c} with ω for different values of Rn and Va

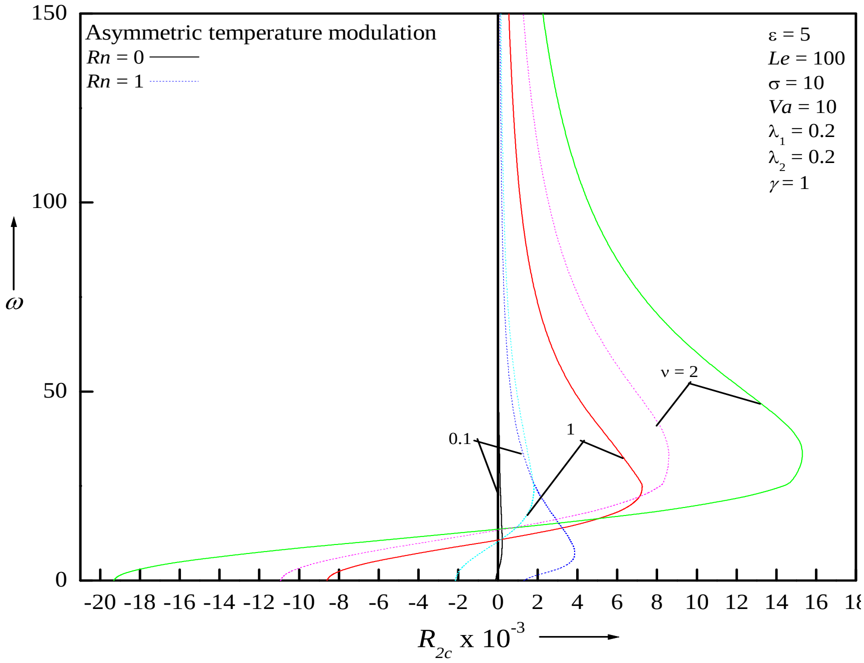

Fig. 17 Variation of R_{2c} with ω for different values of *Rn* and ν

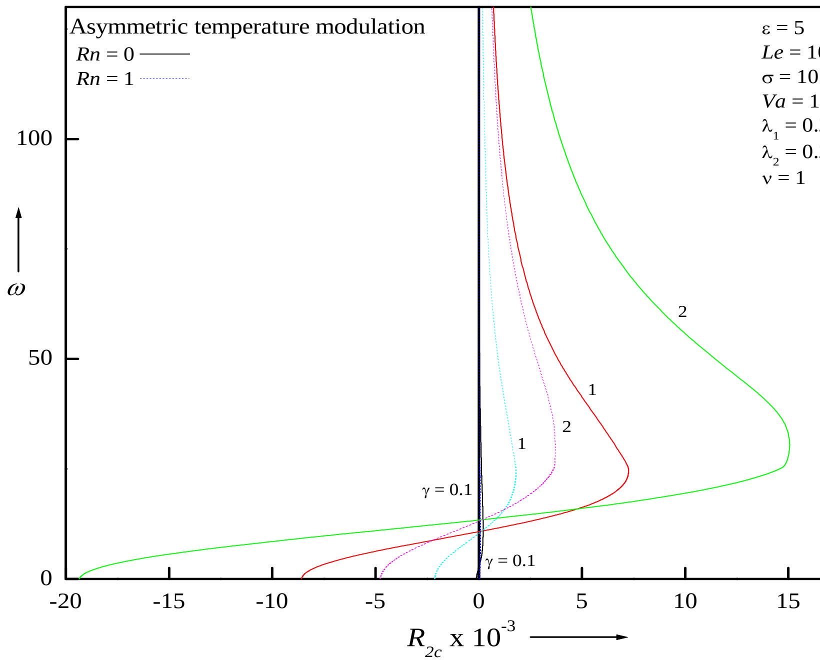

Fig. 18 Variation of R_{2c} with ω for different values of *Rn* and γ

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