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A Note on Modified Cosine Sums

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Abstract

The aim of this paper is to give necessary and sufficient conditions for the integrability and L^1–convergence of modified cosine sums introduced by Kumari and Ram[4]. We obtain a sharper result than that of Garrett, Rees and Stanojević [2, Theorem 3].

Keywords : Integrability, L^1 convergence, null sequence, class BV.

1. Introduction

Let us consider the cosine series

(1.1) $$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx .$$

where $\lim_{n\to\infty} a_n = 0$ and $\Sigma |\Delta a_n| < \infty$.

Let $S_n(x)$ denote the partial sum of (1.1) and $f(x) = \lim_{n\to\infty} S_n(x)$.

Kumari and Ram [4] introduced modified cosine sums

(1.2) $f_n(x)$ =

$$\frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

and studied its L^1–convergence.

The aim of this paper is to study first integrability of the limit of (1.2) and then its L^1-convergence .

2. Main Results

We shall prove the following theorems :

Theorem 1. Let $a_n \to 0$. Then

$f(x) = \lim_{n\to\infty} f_n(x)$ exists for $x \in (0, \pi]$ and

$f(x) \in L^1[0,\pi]$.

Theorem 2. Let $\{a_k\} \in BV$, f(x) exist for $x \in (0, \pi]$ and

$$\sum_{k=0}^{\infty} [\,|\Delta a_k| - t\Delta a_k\,] < \infty \ , \text{ where } t = 1 \text{ or } t = -1$$

Then $f \in [0, \pi]$ if and only if

$$\sum_{k=0}^{\infty} [\,|\Delta a_k| + t\Delta a_k\,] < \infty$$

This result is sharper than Theorem 3 of Garrett, Ress and Stanojević [2].

2.Lemmas.

For the proofs of our theorems, we need the following lemmas :

Lemma 1[1]. If $|c_k| \le 1$, then

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k \left[\frac{\sin\left(k+\frac{1}{2}\right)}{2\sin\frac{x}{2}} \right] \right| dx \le C\,(n+1) ,$$

where C is a positive absolute constant.

Lemma 2[3]. Let $D_n(x)$, $\bar{D}_n(x)$ and $F_n(x)$ denote Dirichlet, conjugate Dirichlet and Féjer kernels respectively, then

$$F_n(x) = D_n(x) - (1/(n+1))\ \bar{D}'_n(x) .$$

3. Proofs of the Results

Proof of Theorem 1. Making use of Lemma 2 and summation by parts,
we have
(3.1) $f_n(x)$ =

$$\frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

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$$= S_n(x) - \left(\frac{a_{n+1}}{n+1}\right) \bar{D}'_n(x)$$

$$= S_n(x) - a_{n+1} D_n(x) + a_{n+1} F_n(x)$$

$$= \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+1} F_n(x)$$

Applying summation by parts again, we get

$$(3.2) f_n(x) = \sum_{k=0}^{n-1} (k+1)\Delta^2 a_k F_k(x) + (n+1)\Delta a_n F_n(x) + a_{n+1} F_n(x)$$

$$= \sum_{k=0}^{n-1} (k+1)\Delta^2 a_k F_k(x) + (n+1) a_n F_n(x) - n\, a_{n+1} F_n(x).$$

Since $F_k(x) = O(1/kx^2)$ for $x \neq 0$, we have

$$0 \le \sum_{k=0}^{n-1} \Delta^2 a_k (k+1) F_k(x) \le (C/x^2)(a_0 - \Delta a_n),$$

so, $\lim_{n\to\infty} \sum_{k=0}^{n-1} (k+1)\Delta^2 a_k F_k(x)$ always exists for $x \neq 0$ and $a_k \to 0$.

The last two terms of (3.2) tend to zero as $n\to\infty$, and hence $\lim_{n\to\infty} f_n(x)$ exists and

$$f(x) = \sum_{k=0}^{\infty} (k+1)\Delta^2 a_k F_k(x).$$

Integrating term by term, we get

$$\int_0^{\pi} f(x)\,dx = (\pi/2) \sum_{k=0}^{\infty} (k+1)\Delta^2 a_k$$

$$= (\pi/2) a_0 < \infty,$$

and the conclusion follows.

Corollary 1. Let $\{a_n\}$ be a null sequence. Then

$$(1/x) \sum_{k=1}^{\infty} \Delta a_k \sin(k + (1/2))x = h(x)/x$$ converges for $x \neq 0$ and $(h(x)/x) \in L^1[0, \pi]$.

Proof. From relation (3.1), we have

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x) = \sum_{k=0}^{\infty} \frac{\Delta a_k \sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

$$= \frac{h(x)}{2\sin\frac{x}{2}}.$$

Then Theorem 1 implies that f(x) exists for $x \neq 0$ and

$$f(x) = \frac{h(x)}{2\sin\frac{x}{2}} \in L^1[0, \pi].$$

Proof of Theorem 2. Using relation (3.1), the pointwise limit f(x) of $f_n(x)$ exists in $(o, \pi]$ since

$\Sigma |\Delta a_k| < \infty$ and $|D_k(x)| < (\pi/x)$ for $x \in (0, \pi]$.

Let $c_n = \max\{t\,\Delta a_n, 0\}$ and $d_n = c_n - t\,\Delta a_n$. Then $2c_n = |\Delta a_n| + t\,\Delta a_n| + t\Delta a_n$ and $2d_n = |\Delta a_n| - t\Delta a_n$.

Thus $$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \Delta a_k D_k(x)$$

$$= \lim_{n\to\infty} \left\{ \sum_{k=0}^{n-1} \Delta^2 a_k (k+1) F_k(x) + \Delta a_n (n+1) F_n(x) \right\}$$

Since $a_n \to 0$ and $F_k(x) = O(1/(kx^2))$, so,

$$f(x) = \sum_{k=0}^{\infty} \Delta^2 a_k (k+1) F_k(x)$$

$$= \sum_{k=0}^{\infty} (\Delta a_k - \Delta a_{k+1})(k+1) F_k(x)$$

$$= \sum_{k=0}^{\infty} \Delta a_k (k+1) F_k(x) -$$

$$\sum_{k=0}^{\infty} \Delta a_{k+1}(k+1)\, F_k(x)$$

$$= \sum_{k=0}^{\infty} t(c_k - d_k)\,(k+1)\, F_k(x) - \sum_{k=0}^{\infty} t(c_{k+1} - d_{k+1})\,(k+1)\, F_k(x)$$

$$= \sum_{k=0}^{\infty} t\,\Delta c_k(k+1)\, F_k(x) - \sum_{k=0}^{\infty} t\Delta d_k\,(k+1)\, F_k(x).$$

Now

$$\left| \int_0^{\pi} \sum_{k=0}^{\infty} t\Delta d_k (k+1) F_k(x) dx \right|$$

$$= \left| t \sum_{k=0}^{\infty} \Delta d_k (k+1) \int_0^{\pi} F_k(x) dx \right|$$

$$= (\pi/2) \sum_{k=0}^{\infty} \Delta d_k(k+1)$$

$$= (\pi/2) \sum_{k=0}^{\infty} d_k < \infty .$$

Thus, $f \in L^1\,[0, \pi]$ if and only if

$$\left| \int_0^{\pi} \sum_{k=0}^{\infty} t\Delta c_k (k+1) F_k(x) dx \right|$$

$$= (\pi/2) \sum_{k=0}^{\infty} \Delta c_k\,(k+1)$$

$$= (\pi/2) \sum_{k=0}^{\infty} c_k < \infty.$$

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