Restrained Triple Connected Domination Number of a Graph

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Abstract
The concept of triple connected graphs with real life application was introduced in [9] by considering the existence of a path containing any three vertices of a graph G. In [3], G. Mahadevan et. al., introduced the concept of triple connected domination number of a graph. In this paper, we introduce a new domination parameter, called restrained triple connected domination number of a graph. A subset \( S \) of \( V \) of a nontrivial graph \( G \) is called a dominating set of \( G \) if every vertex in \( V - S \) is adjacent to at least one vertex in \( S \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality taken over all dominating sets in \( G \). A subset \( S \) of \( V \) of a nontrivial graph \( G \) is called a restrained dominating set of \( G \) if every vertex in \( V - S \) is adjacent to at least one vertex in \( S \) as well as another vertex in \( V - S \). The restrained domination number \( \gamma(G) \) of \( G \) is the minimum cardinality taken over all restrained dominating sets in \( G \). A subset \( S \) of \( V \) of a nontrivial graph \( G \) is said to be triple connected dominating set, if \( S \) is a dominating set and the induced subgraph \( <S> \) is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number and is denoted by \( \gamma_{tc} \). A subset \( S \) of \( V \) of a nontrivial graph \( G \) is said to be restrained triple connected dominating set, if \( S \) is a restrained dominating set and the induced subgraph \( <S> \) is triple connected. The minimum cardinality taken over all restrained triple connected dominating sets is called the restrained triple connected domination number and is denoted by \( \gamma_{rtc} \). We determine this number for some standard graphs and obtain bounds for general graph. Its relationship with other graph theoretical parameters are also investigated.

Key words: Domination Number, Triple connected graph, Triple connected domination number, Restrained Triple connected domination number.

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1. Introduction
By a graph we mean a finite, simple, connected and undirected graph \( G(V, E) \), where \( V \) denotes its vertex set and \( E \) its edge set. Unless otherwise stated, the graph \( G \) has \( p \) vertices and \( q \) edges. Degree of a vertex \( v \) is denoted by \( d(v) \), the maximum degree of a graph \( G \) is denoted by \( \Delta(G) \). We denote a cycle on \( p \) vertices by \( C_p \), a path on \( p \) vertices by \( P_p \), and a complete graph on \( p \) vertices by \( K_p \). A graph \( G \) is connected if any two vertices of \( G \) are connected by a path. A maximal connected subgraph of a graph \( G \) is called a component of \( G \).

The number of components of \( G \) is denoted by \( \omega(G) \). The complement \( \bar{G} \) of \( G \) is the graph with vertex set \( V \) in which two vertices are adjacent if and only if they are not adjacent in \( G \). A tree is a connected cyclic graph. A bipartite graph (or bigraph) is a graph whose vertex set can be divided into two disjoint sets \( V_1 \) and \( V_2 \) such that every edge has one end in \( V_1 \) and another end in \( V_2 \). A complete bipartite graph is a bipartite graph where every vertex of \( V_1 \) is adjacent to every vertex in \( V_2 \). The complete bipartite graph with partitions of order \( |V_1|=m \) and \( |V_2|=n \), is denoted by \( K_{m,n} \). A star, denoted by \( K_{1,p} \), is a tree with one root vertex and \( p-1 \) pendant vertices. A bistar, denoted by \( B(m, n) \) is the graph obtained by joining the root vertices of the stars \( K_{m,1} \) and \( K_{1,n} \). The friendship graph, denoted by \( F_p \), can be constructed by identifying a copies of the cycle \( C_3 \) at a common vertex. A wheel graph, denoted by \( W_p \), is a graph with \( p \) vertices, formed by connecting a single vertex to all vertices of \( C_{p-1} \). If \( S \) is a subset of \( V \), then \( <S> \) denotes the vertex induced subgraph of \( G \) induced by \( S \). The open neighbourhood of a set \( S \) of vertices of a graph \( G \), denoted by \( N(S) \) is the set of all vertices adjacent to some vertex in \( S \) and \( N(S) \cup S \) is called the closed neighbourhood of \( S \), denoted by \( N[S] \). A cut – vertex (cut edge) of a graph \( G \) is a vertex (edge) whose removal increases the number of components. A vertex cut, or separating set of a connected graph \( G \) is a set of vertices whose removal results in a disconnected. The connectivity or vertex connectivity of a graph \( G \), denoted by \( \kappa(G) \) (where \( G \) is not complete) is the size of a smallest vertex cut. The chromatic number of a graph \( G \), denoted by \( \chi(G) \) is the smallest number of colors needed to colour all the vertices of a graph \( G \) in which adjacent vertices receive different colours. For any real number \( x \), \( [x] \) denotes the largest integer less than or equal to \( x \). A Nordhaus - Gaddum-type result is a (tight) lower or upper
bound on the sum or product of a parameter of a graph and its complement. Terms not defined here are used in the sense of [2].

A subset $S$ of $V$ is called a dominating set of $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality taken over all dominating sets in $G$. A dominating set $S$ of a connected graph $G$ is said to be a connected dominating set of $G$ if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality taken over all connected dominating sets is the connected domination number and is denoted by $\gamma_c$.

A dominating set is said to be restrained dominating set if every vertex in $V - S$ is adjacent to at least one vertex in $S$ as well as another vertex in $V - S$. The minimum cardinality taken over all restrained dominating sets is called the restrained domination number and is denoted by $\gamma_r$.

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [11, 12]. Recently, the concept of triple connected graphs has been introduced by J. Paulraj Joseph et. al. [9] by considering the existence of a path containing any three vertices of $G$. They have studied the properties of triple connected graphs and established many results on them. A graph $G$ is said to be triple connected if any three vertices lie on a path in $G$. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs.

In [3], G. Mahadevan et. al., introduced the concept of triple connected domination number of a graph. A subset $S$ of $V$ of a nontrivial graph $G$ is said to be a triple connected dominating set, if $S$ is a dominating set and the induced subgraph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number of $G$ and is denoted by $\gamma_{tc}(G)$. Any triple connected dominating set with $\gamma_{tc}$ vertices is called a $\gamma_{tc}$-set of $G$. In [4, 5, 6], G. Mahadevan et. al., was introduced complementary triple connected domination number, complementary perfect triple connected domination number and paired triple connected domination number of a graph and investigated new results on them.

In this paper, we use this idea to develop the concept of restrained triple connected dominating set and restrained triple connected domination number of a graph.

**Theorem 1.1** [9] A tree $T$ is triple connected if and only if $T \cong P_r$, $r \geq 3$.

**Notation 1.2** Let $G$ be a connected graph with $m$ vertices $v_1, v_2, \ldots, v_m$. The graph obtained from $G$ by attaching $n_i$ times a pendant vertex of $P_{l_i}$ on the vertex $v_i$, $n_2$ times a pendant vertex of $P_{l_2}$ on the vertex $v_2$ and so on, is denoted by $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \ldots, n_mP_{l_m})$ where $n_i, l_i \geq 0$ and $1 \leq i \leq m$.

**Example 1.3** Let $v_1, v_2, v_3, v_4$ be the vertices of $K_4$. The graph $K_4(2P_2, 2P_3, 2P_5, P_3)$ is obtained from $K_4$ by attaching 2 times a pendant vertex of $P_2$ on $v_1$, 2 times a pendant vertex of $P_3$ on $v_2$, 2 times a pendant vertex of $P_3$ on $v_3$ and 1 time a pendant vertex of $P_3$ on $v_4$ and is shown in Figure 1.1.

![Figure 1.1](image-url)

2. Restrained Triple connected domination number

**Definition 2.1** A subset $S$ of $V$ of a nontrivial graph $G$ is said to be a restrained triple connected dominating set, if $S$ is a restrained dominating set and the induced subgraph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all restrained triple connected dominating sets is called the restrained triple connected domination number of $G$ and is denoted by $\gamma_{rtc}(G)$. Any triple connected two dominating set with $\gamma_{tc}$ vertices is called a $\gamma_{rtc}$-set of $G$.

**Example 2.2** For the graphs $G_1$, $G_2$, $G_3$ and $G_6$, in Figure 2.1, the heavy dotted vertices forms the restrained triple connected dominating sets.

![Figure 2.1](image-url)

**Figure 2.1 :** Graph with $\gamma_{rtc} = 3$.

**Observation 2.3** Restrained Triple connected dominating set (rtcd set) does not exists for all graphs and if exists, then $\gamma_{rtc}(G) \geq 3$. 

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Example 2.4 For the graph $G_5, G_6$ in Figure 2.2, we cannot find any restrained triple connected dominating set.

Example 2.11 Consider the graph $G_8$ and its spanning subgraph $H_8$ and $G_9$ and its spanning subgraph $H_9$ shown in Figure 2.4.

Figure 2.2 : Graphs with no rtd set
Throughout this paper we consider only connected graphs for which triple connected two dominating set exists.

Observation 2.5 The complement of the restrained triple connected dominating set need not be a restrained triple connected dominating set.

Example 2.6 For the graph $G_1$ in Figure 2.3, $S = \{v_1, v_5, v_3\}$ forms a restrained triple connected dominating set of $G_5$. But the complement $V - S = \{v_2, v_4\}$ is not a restrained triple connected dominating set.

Figure 2.3 : Graph in which $V - S$ is not a rtd set
Observation 2.7 Every restrained triple connected dominating set is a dominating set but not conversely.

Observation 2.8 Every restrained triple connected dominating set is a connected dominating set but not conversely.

Exact value for some standard graphs:
1) For any cycle of order $p \geq 5$, $\gamma_{r}(C_p) = p - 2$.
2) For any complete graph of order $p \geq 5$, $\gamma_{r}(K_p) = 3$.
3) For any complete bipartite graph of order $p \geq 5$, $\gamma_{r}(K_{m,n}) = 3$.

(Where $m, n \geq 2$ and $m + n = p$).

Observation 2.9 If a spanning sub graph $H$ of a graph $G$ has a restrained triple connected dominating set, then $G$ also has a restrained triple connected dominating set.

Observation 2.10 Let $G$ be a connected graph and $H$ be a spanning sub graph of $G$. If $H$ has a restrained triple connected dominating set, then $\gamma_{r}(G) \leq \gamma_{r}(H)$ and the bound is sharp.
The Nordhaus by increasing the degrees of the vertices, we have \( G \) dominating set, \( v \) is adjacent to \( x \), then \( G \) and \( z \), \( x \) is adjacent to \( z \), then \( G \) adjacent to \( y \), then \( G \) dominating set, \( v \) is adjacent to \( x \) (or) \( y \) (or) \( z \). If \( v \) is adjacent to \( z \), then

\[
\gamma_r(G) = \text{set of } S \text{ is adjacent to } v \text{ (or) } u \text{ and hence } <V - S> = G = G \text{ is isomorphic to } K_2 = uv.
\]

**Case (i)** \(<S> = P_1 = xyz>.

Since \( G \) is connected, \( x \) (or equivalently \( z \)) is adjacent to \( u \) (or equivalently \( v \)) (or) \( y \) is adjacent to \( u \) (or equivalently \( v \)). If \( x \) is adjacent to \( u \). Since \( S \) is a restrained triple connected dominating set, \( v \) is adjacent to \( x \) (or) \( y \) (or) \( z \). If \( v \) is adjacent to \( z \), then \( G \cong C_5 \). If \( v \) is adjacent to \( y \), then \( G \cong C_4(P_3) \). Now by increasing the degrees of the vertices of \( K_2 = uv \), we have \( G \cong G_1 \) to \( G_3 \), \( K_5 - e, C_3(P_3) \). Now let \( y \) be adjacent to \( u \). Since \( S \) is a restrained triple connected dominating set, \( v \) is adjacent to \( x \) (or) \( y \) (or) \( z \). If \( v \) is adjacent to \( y \), then \( G \cong C_5(2P_2) \). If \( v \) is adjacent to \( y \) and \( z \), \( x \) is adjacent to \( z \), then \( G \cong K_5(P_2) \). Now by increasing the degrees of the vertices, we have \( G \cong G_6 \) to \( G_5, C_5(2P_2) \).

**Case (ii)** \(<S> = C_3 = xyz>.

Since \( G \) is connected, there exists a vertex in \( C_3 \), say \( x \) is adjacent to \( u \) (or) \( v \). Let \( x \) be adjacent to \( u \). Since \( S \) is a restrained triple connected dominating set, \( v \) is adjacent to \( x \), then \( G \cong F_2 \). Now by increasing the degrees of the vertices, we have \( G \cong G_6, K_5 \). In all the other cases, no new graph exists.

The Nordhaus – Gaddum type result is given below:

**Theorem 2.16** Let \( G \) be a graph such that \( G \) and \( \bar{G} \) have no isolates of order \( p \geq 5 \). Then

(i) \( \gamma_{rn}(G) + \gamma_{rn}(\bar{G}) \leq 2p - 4 \)

(ii) \( \gamma_{rn}(G), \gamma_{rn}(\bar{G}) \leq (p - 2)^2 \) and the bound is sharp.

**Proof** The bound directly follows from Theorem 2.12. For cycle \( C_5 \), both the bounds are attained.

3 Relation with Other Graph Theoretical Parameters

**Theorem 3.1** For any connected graph \( G \) with \( p \geq 5 \) vertices, \( \gamma_{rn}(G) + \kappa(G) \leq 2p - 3 \) and the bound is sharp if and only if \( G \cong K_5 \).

**Proof** Let \( G \) be a connected graph with \( p \geq 5 \) vertices. We know that \( \kappa(G) \leq p - 1 \) and by Theorem 2.12, \( \gamma_{rn}(G) \leq p - 2 \). Hence \( \gamma_{rn}(G) + \kappa(G) \leq 2p - 3 \). Suppose \( G \) is isomorphic to \( K_5 \). Then clearly \( \gamma_{rn}(G) + \kappa(G) = 2p - 3 \). Conversely, \( G \cong K_5 \).

**Theorem 3.2** For any connected graph \( G \) with \( p \geq 5 \) vertices, \( \gamma_{rn}(G) + \chi(G) \leq 2p - 2 \) and the bound is sharp if and only if \( G \cong K_5 \).

**Proof** Let \( G \) be a connected graph with \( p \geq 5 \) vertices. We know that \( \chi(G) \leq p \) and by Theorem 2.12, \( \gamma_{rn}(G) \leq p - 2 \). Hence \( \gamma_{rn}(G) + \chi(G) \leq 2p - 2 \). Suppose \( G \) is isomorphic to \( K_5 \). Then clearly \( \gamma_{rn}(G) + \chi(G) = 2p - 2 \). Conversely, let \( \gamma_{rn}(G) + \chi(G) =
2p – 2. This is possible only if $\gamma_{rtc}(G) = p – 2$ and $\chi(G) = p$. Since $\chi(G) = p$, $G$ is isomorphic to $K_p$ for which $\gamma_{rtc}(G) = 3 = p – 2$. Hence $G \cong K_p$.

**Theorem 3.3** For any connected graph $G$ with $p \geq 5$ vertices, $\gamma_{rtc}(G) + \Delta(G) \leq 2p – 3$ and the bound is sharp.

**Proof** Let $G$ be a connected graph with $p \geq 5$ vertices. We know that $\Delta(G) \leq p – 1$ and by Theorem 2.12, $\gamma_{rtc}(G) \leq p$. Hence $\gamma_{rtc}(G) + \Delta(G) \leq 2p – 3$. For $K_p$, the bound is sharp.

**REFERENCES**