He's Homotopy Perturbation Method for Nonlinear Ferdholm Integro-Differential Equations Of Fractional Order

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ABSTRACT

In this paper, homotopy perturbation method is applied to solve non-linear Fredholm integro-differential equations of fractional order. Examples are presented to illustrate the ability of the method. The results reveal that the proposed methods very effective and simple and show that this method can be applied to the non-fractional cases.

Keywords - Fractional calculus, Fredholm integro-differential equations, Homotopy-perturbation method

I. INTRODUCTION

The generalization of the concept of derivative $D^{\alpha}f(x)$ to non-integer values of a goes to the beginning of the theory of differential calculus. In fact, Leibniz, in his correspondence with Bernoulli, L'Hopital and Wallis (1695), had several notes about the calculation of $D^{1/2}f(x)$. Nevertheless, the development of the theory of fractional derivatives and integrals is due to Euler, Liouville and Abel (1823). However, during the last 10 years fractional calculus starts to attract much more attention of physicists and mathematicians. It was found that various; especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [3], and the fluid-dynamic traffic model with fractional derivatives caneliminate the deficiency arising from the assumption of continuum traffic flow [4]. In the fields of physics and chemistry, fractional derivatives and integrals are presently associated with the application of fractals in the modeling of electro-chemical reactions, irreversibility and electro magnetism [10], heat conduction in materials with memory and radiation problems. Many mathematical formulations of mentioned phenomena contain nonlinear integro-differential equations with fractional order. Nonlinear phenomena are also of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to be solved either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models. So the aim of this work is to present a numerical method (Homotopy-perturbation method) for approximating the solution of a nonlinear fractional integro-differential equation of the second kind:

$$D^{\alpha}f(x)-\lambda\int_0^1 k(x,t)\,F\big(f(t)\big)dt = g(x) \qquad (1)$$

Where $F(f(t)) = [f(t)]^q \qquad q > 1$

With these supplementary conditions:

$$f^{(i)}(0)=\delta_i, i=0,1,\dots,r-1, \quad r-1<\alpha\le r \quad , r\in N \qquad (2)$$

Where,$g\in L^2([0,1))$, $k\in L^2([0,1)^2)$ are known functions, f(x) is the unknown function, $D^{\alpha}$is the Caputo fractional differentiation operator and q is a positive integer. There are several techniques for solving such equations like Adomian decomposition method [7, 9], collocation method [8], CAS wavelet method [2] and differential transform method [1]. Most of the methods have been utilized in linear problems and a few numbers of works have considered nonlinear problems. In this paper, homotopy perturbation method is applied to solve non-linear Fredholm integro-differential equations of fractional order.

II. FRACTIONAL CALCULUS

There are several definitions of a fractional derivative of order $\alpha > 0$. The two most commonly used definitions are the Riemann–Liouville and Caputo.

Definition1.2.The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$ is defined as

$$J^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\ \alpha>0, x>0, \qquad (3)$$

$$J^0 f(x)=f(x)$$

IIt has the following properties:

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \quad \gamma > -1, \qquad (4)$$

Definition2.2.The Caputo definition of fractal derivative operator is given by

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \qquad (5)$$

Where, $m-1 \le \alpha \le m$, $m \in N$, $x > 0$. It has the following two basic properties:

$$D^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0 \qquad (6)$$

III. Homotopy perturbation method for solving Eq. (1)

To illustrate the basic concept of homotopy perturbation method, consider the following non-linear functional equation:

$$A(U) = f(r) \qquad (7)$$

With the following boundary conditions:

$$\left(U, \frac{\partial U}{\partial n}\right) = 0, \qquad r \in \Gamma$$

Where A is a general functional operator, B is a boundary operator, f(r) is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking the operator A can be decomposed into two parts L and N, where L is a linear and N is a non-linear operator. Eq. (7), therefore, can be rewritten as the following:

$$L(U) + N(U) - f(r) = 0$$

We construct a homotopy $V(r,p): \Omega \times [0,1] \to R$, which satisfies

$$H(V,p) = (1-p)[L(V) - L(U_0)] + p[A(V) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega \qquad (8)$$

Or

$$H(V,p) = L(V) - L(U_0) + pL(U_0) + p[N(V) - f(r)] = 0$$

Where, u_0 is an initial approximation for the solution of Eq. (7).In this method, we use the homotopy parameter p to expand

$$V = V_0 + pV_1 + p^2V_2 + \cdots$$

The solution, usually an approximation to the solution, will be obtained by taking the limit as p tends to 1,

$$U = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \cdots \qquad (9)$$

To illustrate the homotopy perturbation method, for nonlinear Ferdholm integro-differential equations of fractional order, we consider

$$(1-p)\left(D^\alpha f_i(x)\right) + p\left(D^\alpha f_i(x) - g_i(x) - \lambda \int_0^1 k(x,t)\,[f_i(t)]^q dt\right) = 0 \qquad (10)$$

Or

$$D^\alpha f_i(x) = p\left(g_i(x) + \lambda \int_0^1 k(x,t)\,[f_i(t)]^q dt\right) \qquad (11)$$

P is an embedding parameter which changes from zero to unity. Using the parameter p, we expand the solution of the Eq. (1) in the following form:

$$f_i(t) = f_0 + pf_1 + p^2 f_2 + \cdots. \qquad (12)$$

Substituting (12) into (11) and collecting the terms with the same powers of p, we obtain a series of equations of the form:

$$p^0: \quad D^\alpha f_0 = 0 \qquad (13)$$

$$p^1: \quad D^\alpha f_1 = g(x) + \lambda \int_0^1 k(x,t)\,[f_0(t)]^q dt \qquad (14)$$

$$p^2: \quad D^\alpha f_2 = \lambda \int_0^1 k(x,t)\,[f_1(t)]^q dt \qquad (15)$$

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It is obvious that these equations can be easily solved by applying the operator J^α, the inverse of the operatorD^α, which is defined by (3). Hence, the components $f_i (i = 0, 1, 2, ...)$ of the HPM solution can be determined. That is, by setting $p = 1$. In (11) we can entirely determine the HPM series solutions, $f(x) = \sum_{k=0}^{\infty} f_i(x)$.

IV. Applications

In this section, in order to illustrate the method, we solve two examples and then we will compare the obtained results with the exact solutions.

Example 4.1. Consider the following fractional nonlinear integro-differential equation:

$$D^\alpha f(x) = 2 - \frac{32}{15}x^2 + \int_0^1 x^2 t^2 [f(t)]^2\, dt, \quad 0 \le x < 1, \quad 0 < \alpha \le 1 \qquad (16)$$

With this supplementary condition $f(0) = 1$.

For $\alpha = 1$, we can get the exact solution

$$f(x) = 1 + 2x.$$

For $\alpha = \frac{1}{4}$, According to (11) we construct the following homotopy

$$D^{1/4}f(x) = p(g(x) + \int_0^1 x^2t^2[f(t)]^2dt) \qquad (17)$$

Substituting (12) into (17)

$$p^0 \ : \ D^{1/4}f_0(x) = 0$$

$$p^1 \ : \ D^{1/4}f_1(x) = g(x) + \int_0^1 x^2t^2[f_0(t)]^2dt$$

$$p^2: \ D^{1/4}f_2(x) = \int_0^1 x^2t^2[2f_0(t)f_1(t)]dt$$

$$p^3: \ D^{1/4}f_3(x) = \int_0^1 x^2t^2[2f_0(t)f_2(t) + f_1^2(t)]dt$$

$$p^4 \ : D^{1/4}f_4(x) = \int_0^1 x^2t^2[2f_0(t)f_3(t) + 2f_2(t)f_1(t)]dt$$

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Consequently, by applying the operators J^α to the above sets

$$f_0(x) = 1$$

$$f_1 = \frac{1154}{523}x^{1/4} - \frac{764}{541}x^{9/4}$$

$$f_2(x) = \frac{893}{1386}x^{9/4}$$

$$f_3(x) = \frac{718}{1169}x^{9/4}$$

$$f_4(x) = \frac{317}{1250}x^{9/4}$$

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Therefore the approximations to the solutions of Example 4.1. For $\alpha = \frac{1}{4}$ will be determined as

$$f(x) = \sum_{i=0}^{\infty} f_i(x) = f_0 + f_1 + f_2 + f_3 + f_4 + \cdots = 1 + \frac{1154}{523}\sqrt[4]{x}$$

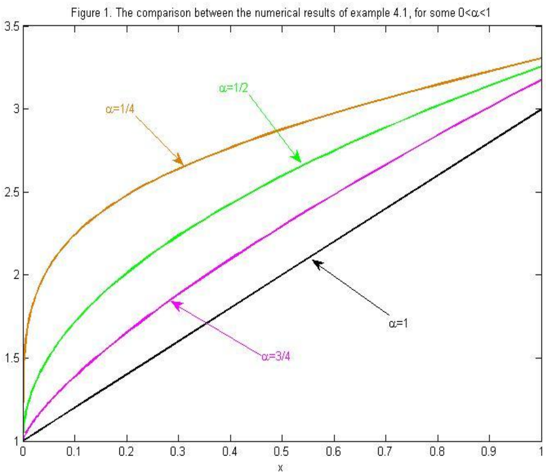
The approximations to the solutions for $\alpha = \frac{1}{2}$ is

$$f(x) = 1 + \frac{2821}{1250}\sqrt[2]{x}\,.$$

Hence for $\alpha = \frac{3}{4}$,

$$f(x) = 1 + \frac{346}{159}\sqrt[4]{x^3}.$$

Fig. 1 shows the numerical results for $\alpha = \frac{1}{4}. \frac{1}{2}. \frac{3}{4}.$ The comparisons show that as $\alpha \to 1$, the approximate solutions tend to $f(x) = 1 + 2x$, which is the exact solution of the equation in the case of $\alpha = 1$.



Example 4.2. Consider the following nonlinear Fredholm integro-differential equation, of order $\alpha = \frac{1}{2}$

$$D^{1/2}f(x) = g(x) + \int_0^1 xt[f(t)]^2dt \qquad 0 \le x < 1 \qquad (18)$$

Where, $g(x) = \frac{1}{\Gamma(\frac{3}{2})}\sqrt{x} - \frac{17}{12}x$ with these supplementary conditions $f(0) = 1$, with the exact solution, $f(x) = 1 + x$

According to (11) we construct the following homotopy:

$$D^{1/2}f(x) = p(g(x) + \int_0^1 xt[f(t)]^2dt) \qquad (19)$$

Substituting (12) into (18)

$$p^0 \ : \ D^{1/2}f_0(x) = 0$$

$$p^1 \ : \ D^{1/2}f_1(x) = g(x) + \int_0^1 xt[f_0(t)]^2dt$$

$$p^2 \ : \ D^{1/2}f_2(x) = \int_0^1 xt[2f_0(t)f_1(t)]dt$$

$$p^3 \ : \ D^{1/2}f_3(x) = \int_0^1 xt[2f_0(t)f_2(t) + f_1^2(t)]dt$$

$$p^4 \ : \ D^{1/2}f_4(x) = \int_0^1 xt[2f_0(t)f_3(t) + 2f_2(t)f_1(t)]dt$$

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Consequently, by applying the operators J^α to the above sets

$$f_0(x) = 1$$

$$f_1 = x - \frac{1379}{2000}x^{3/2}$$

$$f_2(x) = \frac{41}{200}x^{3/2}$$

$$f_3(x) = \frac{617}{5269} x^{3/2}$$
$$f_4(x) = \frac{471}{6173} x^{3/2}$$

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Therefore the approximations to the solutions of Example 4.2. Will be determined as

$$f(x) = \sum_{i=0}^{\infty} f_i(x) = f_0 + f_1 + f_2 + f_3 + f_4 + \cdots$$
$$= 1 + x$$

And hence, $f(x) = 1 + x$, which are the exact solution.

Example 4.3. Consider the following nonlinear Fredholm integro-differential equation, of order $\alpha = \frac{3}{4}$

$$D^{3/4} f(x) = \frac{8}{5\Gamma\left(\frac{5}{4}\right)} x^{5/4} - \frac{15}{8} x + \int_0^1 xt[f(t)]^3 dt \qquad 0 \le x < 1 \qquad (20)$$

With these supplementary Condition $f(0) = 1$, with the exact solution, $f(x) = 1 + x^2$

According to (11) we construct the following homotopy:

$$D^{3/4} f(x) = p\left(g(x) + \int_0^1 xt[f(t)]^3 dt\right) \qquad (21)$$

Substituting (12) into (21)

$$p^0 : \quad D^{3/4} f_0(x) = 0$$
$$p^1 : \quad D^{3/4} f_1(x) = g(x) + \int_0^1 xt[f_0(t)]^3 dt$$
$$p^2 : \quad D^{3/4} f_2(x) = \int_0^1 xt[3f_0(t)^2 f_1(t)] dt$$
$$p^3 : \quad D^{3/4} f_3(x) = \int_0^1 xt[3f_0(t)^2 f_2(t) + 3f_0(t) f_1(t)^2] dt$$
$$p^4 : \quad D^{3/4} f_4(x) = \int_0^1 xt[3f_0(t)^2 f_3(t) + 6f_0(t) + f_1(t) + f_2(t) + f_3^2(t)] dt$$

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Consequently, by applying the operators J^{α} to the above sets

$$f_0(x) = 1$$
$$f_1 = x^2 - \frac{2137}{2500} x^{7/4}$$
$$f_2(x) = \frac{203}{479} x^{7/4}$$
$$f_3(x) = \frac{232}{921} x^{7/4}$$
$$f_4(x) = \frac{779}{5000} x^{7/4}$$

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Therefore the approximations to the solutions of Example 4.3. Will be determined as

$$f(x) = \sum_{i=0}^{\infty} f_i(x) = f_0 + f_1 + f_2 + f_3 + f_4 + \cdots$$
$$= 1 + x^2$$

And hence $f(x) = 1 + x^2$, which are the exact solution.

V. Conclusion

This paper presents the use of the He's homotopy perturbation method, for non-linear ferdholm integro-differential equations of fractional order. We usually derive very good approximations to the solutions. It can be concluded that the He's homotopy perturbation method is a powerful and efficient technique infinding very good solutions for this kind of equations.

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