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Spinors as a tool to compute orbital ephemeris

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ABSTRACT

In this paper, the Spinors of the physical 3-D Euclidean spaces of Geometric Algebras over a finite dimensional vector space are defined. Action of Spinors on Euclidean spaces is discussed. The technique for the calculation of ephemeris using Spinors is illustrated. In this process sequences of Spinors in their half angle form as well as their matrix form are used and compared. It is shown that when large number of rotations is involved, Spinor matrix methods are advantageous over the Spinor half angle method.

Keywords - **Ephemeris, Euclidean space, Euler angles, Rotations, Spinors.**

I. INTRODUCTION

Hestenes [1] defined Spinors as a product of two vectors of i-*plane* and also proved that they can be treated as rotation operators on i-*plane*. Spinors are also defined as elements of a minimal left ideal [2], [3]. Rotation operators play a key role in the determination of orbits of celestial bodies such as Satellites and Spacecrafts. Geometric Algebra develops a unique and coordinate free method in this regard. There are various techniques using Quaternions, Matrices and Euler angles. Geometric Algebra unifies all these systems into a unique system called Spinors. Geometric Algebra not only integrates the above mentioned systems but also establishes the relation among these systems and thus providing a clear passage to move from one system to the other.

In this paper we study different parameterizations used for representing Spinors and rotations. We apply the Euler angle parameterization of Spinors to solve the problem of finding the orbital ephemeris of celestial bodies [4].

II. GEOMETRIC ALGEBRA

Let E be an n - dimensional vector space over R, the field of real numbers and g be a symmetric, positive definite, bilinear form $g : E \times E \longrightarrow$ R denoted by $g(\vec{x},\vec{y}) = \vec{x}.\vec{y} \quad \forall \vec{x},\vec{y} \in E$ .There exists a unique Clifford Algebra $(C(E),\rho)$ which is a universal algebra in which E is embedded. Clifford algebra is also called Geometric Algebra as all elements and operation used in it can be interpreted geometrically. We shall identify E with $\rho(E)$.

We choose and fix an orthonormal basis $B_n = \{e_1, e_2, ..., e_n\}$ for E .

Let $N = \{1,2,...., n\}$, $n = \dim E$ and $S \subseteq N$

Let $i_1, i_2, ..., i_m$ be the elements of S in the ascending order. We define

$e_S = e_{i_1} e_{i_2} .... e_{i_m}$ and $e_\phi = 1_L$.

We shall identify $e_{\{i\}}$ with e_i .

Note that if σ is a permutation of {1, 2, ….., m}, then

$e_{i_{\sigma(1)}} e_{i_{\sigma(2)}} .... e_{i_{\sigma(m)}} = (-1)^m e_{i_1} e_{i_2} .... e_{i_m}$

m stands for order of permutation.

$\|e_i\|^2 = 1_L$ … and $e_i e_j = -e_j e_i \quad$ if $i \neq j$

$C(E) = \oplus \mathrm{A}_k$ where $\mathrm{A}_k = \{ \sum_{\substack{s \\ |s|=k}} a_s e_s \}$

$\dim \mathrm{A}_k = nC_k$ and $\dim C(E) = 2^n$.

The operation 'Geometric Product' of vectors denoted by $\vec{a}\vec{b}$, is defined as

$$\vec{a}\vec{b} = \vec{a} \ . \ \vec{b} + \vec{a} \wedge \vec{b} \qquad \text{…………..} (1)$$

$$\vec{b}\vec{a} = \vec{b} \ . \ \vec{a} + \vec{b} \wedge \vec{a} = \vec{a} \ . \ \vec{b} \ - \ \vec{a} \wedge \vec{b} \qquad \text{……….} (2)$$

Here $\vec{a}.\vec{b} = \langle \vec{a}\vec{b} \rangle_0$ is the scalar part and $\vec{a} \wedge \vec{b} = \langle \vec{a}\vec{b} \rangle_2$ is the bivector part. Elements of Geometric Algebra are called multivectors as they are in the form A = $\langle A \rangle_0 + \langle A \rangle_1 + ….+ \langle A \rangle_n$. A multivector is said to be even (odd) if $\langle A \rangle_r = 0$ whenever r is odd (even) [5].

$\langle A \rangle_k \in \mathrm{A}_k$, denotes the k - vector part of the multivector A.

There exists a unique linear map $\dagger : \mathrm{C}(\mathrm{E}) \rightarrow C(E)$ that takes A to $\mathrm{A}^\dagger$ satisfying

(i) $e_S = e_{i_1} e_{i_2} .... e_{i_m}$ then

$e_S^\dagger = e_{i_m} e_{i_{m-1}} .... e_{i_2} e_{i_1} = (-1)^{\frac{m(m-1)}{2}} e_S$

(ii) $(\mathrm{AB})^\dagger = \mathrm{B}^\dagger \mathrm{A}^\dagger$

'$\dagger$' is called the reversion operator.

2.1 Euclidean nature of Geometric Algebra

2.1.1 Definition *Norm of a multivector* To every multivector, $\mathrm{A} \in C(E)$ the magnitude or modulus of A is defined as $|\mathrm{A}| = \langle \mathrm{A}^\dagger \mathrm{A} \rangle_0^{\frac{1}{2}}$. With this definition of norm,

$C(E)$ becomes a Euclidean algebra. The inverse of a non-zero element of A of $C(E)$, is also a multivector, defined by $A^{-1} = \frac{A^\dagger}{|A|^2}$.

2.1.2 Definition $k-space$ Every k- vector A_k determines a $k-space$.

2.2 Euclidean space E_3

Denote the Clifford Algebra constructed over a three dimensional vector space E with $C_3(E)$.

For a trivector A_3, designate a unit trivector '$\mathbf{i}$' proportional to A_3. That is $A_3 = |A_3|\,\mathbf{i}$.

'$\mathbf{i}$' represents the direction of the space represented by A_3.

The set of all vectors $\vec{x}$ which satisfy the equation $\vec{x} \wedge \mathbf{i} = 0$, is called the Euclidean 3- dimensional vector space corresponding to '$\mathbf{i}$' and is denoted by 'E_3'. E_3 is also be called an $\mathbf{i}-space$, the trivector $\mathbf{i}$ is called the pseudoscalar of the space as every other pseudoscalar is a scalar multiple of it.

Factorize $\mathbf{i} = \sigma_1\sigma_2\sigma_3$ where σ_1, σ_2 and σ_3 are orthonormal vectors. They represent a coordinate frame in the $\mathbf{i}-space$. $\vec{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$ is a parametric equation of the $\mathbf{i}-space$. x_1, x_2 and x_3 are called the rectangular components of vector $\vec{x}$ with respect to the basis $\{\sigma_1, \sigma_2, \sigma_3\}$. $\mathbf{i}-space$ of vectors is a 3 – dimensional vector space with the above basis.

2.2.1 Definition *Dual of vector* $\mathbf{i}_1 = \sigma_1\mathbf{i} = \sigma_2\sigma_3$; $\mathbf{i}_2 = \sigma_2\mathbf{i} = \sigma_3\sigma_1$; $\mathbf{i}_3 = \sigma_3\mathbf{i} = \sigma_1\sigma_2$

$\mathbf{i}_1\,\mathbf{i}_2 = -\,\mathbf{i}_3 = -\,\mathbf{i}_1\,\mathbf{i}_2$, $\mathbf{i}_2\,\mathbf{i}_3 = -\,\mathbf{i}_1 = -\,\mathbf{i}_3\,\mathbf{i}_2$ and $\mathbf{i}_3\,\mathbf{i}_1 = -\,\mathbf{i}_2 = -\,\mathbf{i}_1\,\mathbf{i}_3$

The set of bivectors in $C_3(E)$ (also denoted by $C_3(\mathbf{i})$)is a 3-dimensional vector space with basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$. Every bivector B is a dual of a vector $\mathbf{b}$ that is $B = \mathbf{b}\ \mathbf{i} = \mathbf{i}\ \mathbf{b}$.

2.3 Spinors of Euclidean space $C_3(\mathbf{i})$

2.3.1 Definition *Spinor* The Geometric product of two vectors in the $\mathbf{i}$ - *space* is called a Spinor and is denoted by 'R'. Thus

$R = \vec{x}\ \vec{y} = \vec{x}\ .\ \vec{y} + \vec{x} \wedge \vec{y}\ \forall\ \vec{x},\ \vec{y}\ \in E$.

'R' is a multivector, has a scalar part '$\alpha = \vec{x}\ .\ \vec{y}$' and a bivector part '$\beta\,\mathbf{i}_1 + \gamma\,\mathbf{i}_2 + \delta\,\mathbf{i}_3 = \vec{x} \wedge \vec{y}$'.

2.3.2 Definition *Spinor* $\mathbf{i}$ - *space* The Spinor $\mathbf{i}$ - *space* 'S_3' is defined as

$\{\ S_3 = \ R / R = \vec{x}\ \vec{y},\quad \vec{x}, \vec{y} \in \mathbf{i}\text{-}space\}$

S_3 can also be denoted by $C_3^+(E)$ or $C_3^+(\mathbf{i})$. 'R' is a multivector, has a scalar part 'α' and a bivector part '$\beta\,\mathbf{i}_1 + \gamma\,\mathbf{i}_2 + \delta\,\mathbf{i}_3$'. Spinors do not satisfy commutative property with respect to the operation 'Geometric product' in view of the above definition (2.2.1).

III. ACTION OF SPINORS ON EUCLIDEAN SPACE - ROTATIONS

Spinors of Euclidean space also can be treated as rotation operators on $\mathbf{i}$ - *space* of vectors. Rotation operators can be constructed by considering the group action of Spinors by conjugation (Hestenes 1986).

Consider a Spinor $R = \vec{v}\ \vec{u}$, where $\vec{u}$ and $\vec{v}$ are unit vectors of E. Define a rotation operator $\mathcal{R}$ on E as

$$\mathcal{R}\,(\vec{x}) = R^\dagger \vec{x} R = \vec{u}\ \vec{v}\ \vec{x}\ \vec{v}\ \vec{u}.$$

Unlike rotations in two dimensions, rotations in three dimensions are more complex as (i) the operation to be considered is the group action by conjugation, giving similarity transformations and (ii) the axis about which the rotation takes place is also to be specified. The resulting vector changes as the axis of rotation changes. This can be shown in the following examples.

Rotation of the vector $\vec{x}$ about the axis σ_1, the axis perpendicular to the plane $\sigma_2\sigma_3$ is represented by the bivector $\mathbf{i}_1 = \sigma_1\mathbf{i} = \sigma_2\sigma_3$.

Let $\vec{x} \in$ $\mathbf{i}-space$ of vectors and $\vec{x} = (x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$.

$$\mathbf{i}_1^\dagger \vec{x}\,\mathbf{i}_1 = \sigma_3\sigma_2\vec{x}\sigma_2\sigma_3 = \sigma_3\sigma_2(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)\sigma_2\sigma_3 = x_1\sigma_1 - (x_2\sigma_2 + x_3\sigma_3)$$

Rotation of the vector $\vec{x}$ about the axis σ_2, the axis perpendicular to the plane $\sigma_3\sigma_1$ is represented by the bivector $\mathbf{i}_2 = \sigma_2\mathbf{i} = \sigma_3\sigma_1$.

$$\mathbf{i}_2^\dagger \vec{x}\,\mathbf{i}_2 = \sigma_1\sigma_3\sigma_1\vec{x}\sigma_3\sigma_1 = \sigma_1\sigma_3(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)\sigma_3\sigma_1 = x_2\sigma_2 - (x_1\sigma_1 + x_3\sigma_3)$$

3.1 Composition of Rotations

Two rotations $\mathcal{R}_1 x = R_1^\dagger \vec{x} R_1$ and $\mathcal{R}_2 \vec{x} = R_2^\dagger \vec{x} R_2$ can be combined to give a new rotation $\mathcal{R}_3 \vec{x} = \mathcal{R}_2 \mathcal{R}_1 \vec{x} = R_2^\dagger R_1^\dagger \vec{x} R_1 R_2 = R_3^\dagger \vec{x} R_3$

Note that the order of rotations $\mathcal{R}_2\mathcal{R}_1$ is opposite to that of the corresponding Spinor $R_1 R_2$.

3.2 *Proposition* Product of two rotations is also a rotation.

Proof: As Geometric product of Spinors is a binary operation, R_3 is also a Spinor.

$$R_3^\dagger R_3 = R_2^\dagger R_1^\dagger R_1 R_2 = 1 \text{ as } R_1^\dagger R_1 = 1 = R_2^\dagger R_2$$

Hence R_3 is also a unitary Spinor and it can be proved that determinant of $\mathcal{R}_3$ is one (Hestenes 1986). As a consequence of this $\mathcal{R}_3$ is a rotation.

Product of two rotations in three dimensional spaces is not commutative in general.

The rotations determined by $\mathcal{R}_1\mathcal{R}_2$ and $\mathcal{R}_2\mathcal{R}_1$ are considered as two different rotations as their sequential order is not the same.

3.2.1 Spinors of the $\mathbf{i}$ - *space* in half angle form

As a unit vector is treated as a representation of the direction of a vector, a unit bivector can be treated as a representation of an angle, which is a relation between two directions.

Hence for $\vec{x}, \vec{y} \in C_3(\mathbf{i})$, let $\hat{x}, \hat{y}$ be their directions which are elements of the $\mathbf{i}$ - *space* .

From the definition of a Spinor of the $\mathbf{i}$ - *space* , the Spinor

$$R = \hat{x}\ \hat{y} = \hat{x}\ .\ \hat{y} + \hat{x} \wedge \hat{y}\,.$$

$= \cos(1/2)|\mathbf{A}| + \hat{\mathbf{A}}\sin(1/2)|\mathbf{A}|$ is the half angle form of the Spinor R.

$S = C^+(E)$ if dim $E \leq 3$. 'S' can be related to complex numbers for dim $E = 2$ and it can be related to Quaternion Algebra for $\dim E = 3$.

$R = \cos(1/2)|\mathbf{A}| + \hat{\mathbf{A}}\sin(1/2)|\mathbf{A}|$ can also be written as $e^{(1/2)\mathbf{A}}$.

Here $\hat{\mathbf{A}} = \hat{x} \wedge \hat{y}$, the bivector representing the plane of rotation and $|\mathbf{A}|$ gives the magnitude of the angle through which the rotation takes place.

3.2.2 Matrix form of a Spinor

There are different matrices to represent rotations. Instead, the use of Spinors to represent a rotation gives the matrix elements directly by the formula

$$e_{jk} = \sigma_j\ .\ e_k = \sigma_j\ .\ (\mathcal{R}\sigma_k).$$

There are also other forms such as Exponential form, Quaternion form etc. The advantages in using Spinor Algebra as a substitute for all the above algebras

(i) The coordinate free nature of Spinors.

(ii) The existence of Spinors in every dimension facilitating to perform rotations in higher dimensional spaces.

(iii) Representation of rotations using Spinors enables us to find the magnitude of the angle of rotation as well as the orientation of the rotation which is not the case with the matrix representation.

(iv) Spinors can be converted into other forms easily as and when required.

3.3 Representing a rotation using Euler angles

Rotations are orthogonal transformations. They transform one coordinate frame XYZ into another coordinate frame xyz preserving the angle between them. Euler stated that every rotation can be expressed as a product of two or three rotations about fixed axes of a standard basis in such a way that no two successive rotations have the same axis of rotation. This theorem is known as 'Euler's theorem'. Thus every Spinor can be divided further into a product of two or three Spinors that represent rotations about base vectors. Euler angles are widely used to represent rotations.

To represent the rotation using Euler angles, we select an axis for the first rotation among the three axes in the sequence. Then according to the rule that no two successive rotations have the same axis of rotation, we will have two axes to choose for the second rotation , and for the third, again two options are there to choose a different one from the previous. Thus we get totally 3 x 2 x 2 = 12 sets of Euler angles. Hence, one can represent the same rotation using different sets of Euler angles. If the axes chosen for the first rotation is same as that of third, such a sequence is called a Symmetric set or a Classical set of Euler angles.

3.4 Finding the Matrix form of the Euler angle sequence of Spinors

We consider the 3-2-1 symmetric sequence of rotations; the Spinor that represents the required rotation is given as a sequence of three rotations about the base vectors $\{\sigma_1, \sigma_2, \sigma_3\}$ is defined by

$$R = R_\phi Q_\theta R_\psi\ ,$$

Where $R_\psi = e^{(1/2)\,\mathbf{i}\,\sigma_3\psi} = \cos\frac{\psi}{2} + \sigma_1\sigma_2 \sin\frac{\psi}{2}$,

$$Q_\theta = e^{(1/2)\,\mathbf{i}\,\sigma_2\theta} = \cos\frac{\theta}{2} + \sigma_3\sigma_1 \sin\frac{\theta}{2},$$

$$R_\phi = e^{(1/2)\,\mathbf{i}\,\sigma_1\phi} = \cos\frac{\phi}{2} + \sigma_2\sigma_3 \sin\frac{\phi}{2}$$

The new set of axes after rotation are given by

$$e_k = \mathcal{R}\sigma_k = R^\dagger \sigma_k R$$

$$= R_\phi^\dagger Q_\theta^\dagger R_\psi^\dagger \sigma_k R_\psi Q_\theta R_\phi$$

This can be converted into the matrix form by calculating the elements of the matrix $[e_{jk}]$ given as $e_{jk} = \sigma_j . e_k = \mathcal{R}\sigma_k$

$$e_1 = \mathcal{R}\sigma_1 = R_\phi^\dagger Q_\theta^\dagger R_\psi^\dagger \sigma_1 R_\psi Q_\theta R_\phi$$

$$= R_\phi^\dagger Q_\theta^\dagger (R_\psi^\dagger \sigma_1 R_\psi) Q_\theta R_\phi$$

$$= R_\phi^\dagger Q_\theta^\dagger (\sigma_1(\cos\psi + \sigma_1\sigma_2 \sin\psi)) Q_\theta R_\phi$$

$$= R_\phi^\dagger Q_\theta^\dagger (\sigma_1 \cos\psi + \sigma_2 \sin\psi) Q_\theta R_\phi$$

$$= R_\phi^\dagger [(Q_\theta^\dagger \sigma_1 Q_\theta)\cos\psi + (Q_\theta^\dagger \sigma_2 Q_\theta)\sin\psi] R_\phi$$

$$= R_\phi^\dagger [\sigma_1(\cos\theta + \sigma_3\sigma_1 \sin\theta)\cos\psi + \sigma_2 \sin\psi] R_\phi$$

$$= R_\phi^\dagger (\sigma_1 \cos\theta\cos\psi - \sigma_3 \sin\theta\cos\psi + \sigma_2 \sin\psi) R_\phi$$

$$= (R_\phi^\dagger \sigma_1 R_\phi)\cos\theta\cos\psi - (R_\phi^\dagger \sigma_3 R_\phi)\sin\theta\cos\psi + (R_\phi^\dagger \sigma_2 R_\phi)\sin\psi$$

$$= \sigma_1 \cos\theta\cos\psi - \sigma_3(\cos\phi + \sigma_2\sigma_3 \sin\phi)\sin\theta\cos\psi + \sigma_2(\cos\phi + \sigma_2\sigma_3 \sin\phi)\sin\psi$$

$$= \sigma_1 \cos\theta\cos\psi - \sigma_3 \cos\phi\sin\theta\cos\psi + \sigma_2 \sin\phi\sin\theta\cos\psi + \sigma_2 \cos\phi\sin\psi + \sigma_3 \sin\phi\sin\psi$$

$$= \sigma_1(\cos\theta\cos\psi) + \sigma_2(\sin\phi\sin\theta\cos\psi + \cos\phi\sin\psi) + \sigma_3(\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi)$$

$$e_2 = \mathcal{R}\sigma_2 = R_\phi^\dagger Q_\theta^\dagger R_\psi^\dagger \sigma_2 R_\psi Q_\theta R_\phi$$

$$= R_\phi^\dagger Q_\theta^\dagger (R_\psi^\dagger \sigma_2 R_\psi) Q_\theta R_\phi$$

$$= R_\phi^\dagger Q_\theta^\dagger [\sigma_2(\cos\psi + \sigma_1\sigma_2 \sin\psi)] Q_\theta R_\phi$$

$$= R_\phi^\dagger Q_\theta^\dagger (\sigma_2 \cos\psi - \sigma_1 \sin\psi) Q_\theta R_\phi$$

$$= R_\phi^\dagger ((Q_\theta^\dagger \sigma_2 Q_\theta)\cos\psi - (Q_\theta^\dagger \sigma_1 Q_\theta)\sin\psi) R_\phi$$

$$= R_\phi^\dagger (\sigma_2 \cos\psi - \sigma_1(\cos\theta + \sigma_3\sigma_1 \sin\theta)\sin\psi) R_\phi$$

$$= R_\phi^\dagger (\sigma_2 \cos\psi - \sigma_1 \cos\theta\sin\psi + \sigma_3 \sin\theta\sin\psi) R_\phi$$

$$= (R_\phi^\dagger \sigma_2 R_\phi)\cos\psi - (R_\phi^\dagger \sigma_1 R_\phi)\cos\theta\sin\psi + (R_\phi^\dagger \sigma_3 R_\phi)\sin\theta\sin\psi$$

$$= \sigma_2(\cos\phi + \sigma_2\sigma_3 \sin\phi)\cos\psi - \sigma_1 \cos\theta\sin\psi + \sigma_3(\cos\phi + \sigma_2\sigma_3 \sin\phi)\sin\theta\sin\psi$$

$$= \sigma_2 \cos\phi\cos\psi + \sigma_3 \sin\phi\cos\psi - \sigma_1 \cos\theta\sin\psi + \sigma_3 \cos\phi\sin\theta\sin\psi - \sigma_2 \sin\phi\sin\theta\sin\psi$$

$$= \sigma_1(-\cos\theta\sin\psi) + \sigma_2(\cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi) + \sigma_3(\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)$$

$$e_3 = \mathcal{R}\sigma_3 = R_\phi^\dagger Q_\theta^\dagger R_\psi^\dagger \sigma_3 R_\psi Q_\theta R_\phi$$

$$= R_\phi^\dagger Q_\theta^\dagger (R_\psi^\dagger \sigma_3 R_\psi) Q_\theta R_\phi$$

$$= R_\phi^\dagger Q_\theta^\dagger (\sigma_3) Q_\theta R_\phi$$

$$= R_\phi^\dagger (Q_\theta^\dagger \sigma_3 Q_\theta) R_\phi$$

$$= R_\phi^\dagger [\sigma_3(\cos\theta + \sigma_3\sigma_1 \sin\theta)] R_\phi$$

$$= R_\phi^\dagger (\sigma_3 \cos\theta + \sigma_1 \sin\theta) R_\phi$$

$$= (R_\phi^\dagger \sigma_3 R_\phi)\cos\theta + (R_\phi^\dagger \sigma_1 R_\phi)\sin\theta$$

$$= \sigma_3(\cos\phi + \sigma_2\sigma_3 \sin\phi)\cos\theta + \sigma_1 \sin\theta$$

$$= \sigma_3 \cos\phi\cos\theta - \sigma_2 \sin\phi\cos\theta + \sigma_1 \sin\theta$$

$$= \sigma_1 \sin\theta - \sigma_2 \sin\phi\cos\theta + \sigma_3 \cos\phi\cos\theta$$

The matrix obtained is as follows

$$\begin{vmatrix} \cos\theta\cos\psi & -\cos\theta\sin\psi & \sin\theta \\ \sin\phi\sin\theta\cos\psi + \cos\phi\sin\psi & \cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\theta \\ \sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi & \sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{vmatrix}$$

This sequence is used in aerospace applications and to find orbital ephemeris.

IV EULER ANGLES AND EQUIVALENT ROTATIONS

The set of Euler angles that represent a particular rotation are not unique. Representing the same rotation by two different Euler angle sequences gives Equivalent rotations. In this paper we use sequences of Spinors in their matrix form and also in their half angle form in place of rotation sequences to obtain the relationship between the two sets of angles (i) orbital ephemeris set (L, σ and α) and (ii) orbital parameters (Ω, ι and ν) set. Thus established the equivalence between Spinor methods and Quaternion methods.

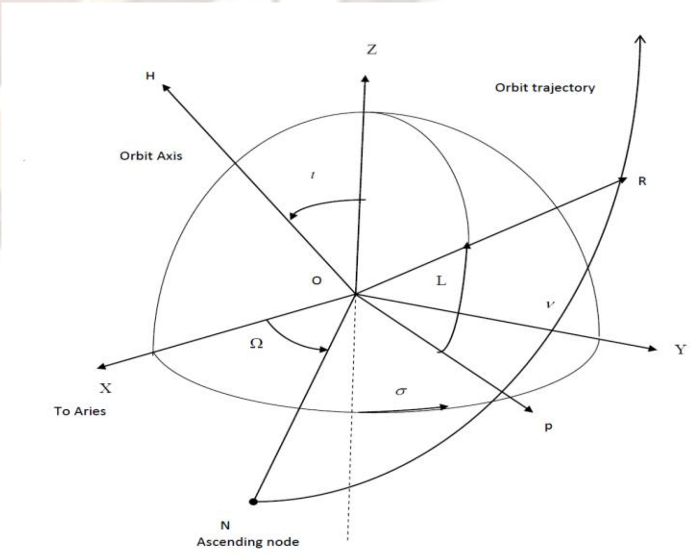

Figure 1: ephemeris sequence

L = Earth-Latitude of orbiting body

$\sigma = \lambda + \lambda_0$

α = Ephemeris path direction angle

λ = Earth-Longitude of orbiting body

λ_0 = Greenwich with respect to X axis

Ω = Angle to the orbit ascending node, also called 'Right Ascension of the ascending node'. It is the angle subtended at the center of the Earth from the Vernal Equinox (positive X axis) to the ascending node 'N'.

ι = The angle of inclination. It is the angle between orbital plane and equatorial plane.

ν = The argument of the latitude to orbiting body

The position vector of the object is given by the vector $\overrightarrow{OR}$. 'P' is the point on the surface of the Earth in the radial direction of the object. The reference frame XYZ is the Equatorial frame of reference that is the plane that contains the Earth's equator. X, Y axes are contained in the equatorial plane of the Earth. The Z axis is normal to this XY plane such that XYZ frame forms a right handed frame of reference. 'NOR' is the orbital plane. The trajectory of the object is as indicated in the figure 1. ON is the line segment in which the orbit trajectory and the reference frame intersect each other.

4.1 Matrix method:

4.1.1 Orbit ephemeris sequence

A tabulation of data of the Earth longitude and Latitude of a body, as a function of time is called the orbit ephemeris of the body.

In this sequence the body frame (xyz) of the object will be related to an inertial frame of reference (XYZ) through the orbital Ephemeris (L, $\sigma = \lambda + \lambda_0$, α). This requires the 3-2-1 symmetric sequence of rotations; the Spinor that represents the required rotation is given as a sequence of three rotations about the base vectors is defined by

$$\mathcal{R} = \mathcal{R}_\alpha Q_{-L} \mathcal{R}_\sigma = R_\phi^\dagger Q_\theta^\dagger R_\sigma^\dagger \sigma_k R_\sigma Q_{-L} R_\alpha ,$$

Where $R_\sigma = e^{(1/2)\, i\, \sigma_3 \sigma} = \cos\frac{\sigma}{2} + \sigma_1\sigma_2 \sin\frac{\sigma}{2}$,

$$Q(-L) = e^{(1/2)\, i\, \sigma_2 (-L)}$$

$$= \cos\frac{(-L)}{2} + \sigma_3\sigma_1 \sin\frac{(-L)}{2} = \cos\frac{L}{2} - \sigma_3\sigma_1 \sin\frac{L}{2},$$

$$R_\alpha = e^{(1/2)\, i\, \sigma_1 \alpha} = \cos\frac{\alpha}{2} + \sigma_2\sigma_3 \sin\frac{\alpha}{2}$$

The matrix is obtained is as follows

$$= \begin{vmatrix} (\cos L \cos\sigma) & (-\cos L \sin\sigma) & -\sin L \\ (-\sin\alpha \sin L \cos\sigma + \cos\alpha \sin\sigma) & (\cos\alpha\cos\sigma + \sin\alpha \sin L \sin\sigma) & -\sin\alpha \cos L \\ (\sin\alpha \sin\sigma + \cos\alpha \sin L \cos\sigma) & \sin\alpha\cos\sigma - \cos\alpha \sin L \sin\sigma & \cos\alpha\cos L \end{vmatrix}$$

In this application we relate the body frame (xyz) of a spacecraft or a near earth orbiting satellite to an inertial frame of reference (XYZ, NED frame of reference) through the orbital parameters (Ω, ι and ν). This can be done by using 3-1-3 sequence of Euler angles.

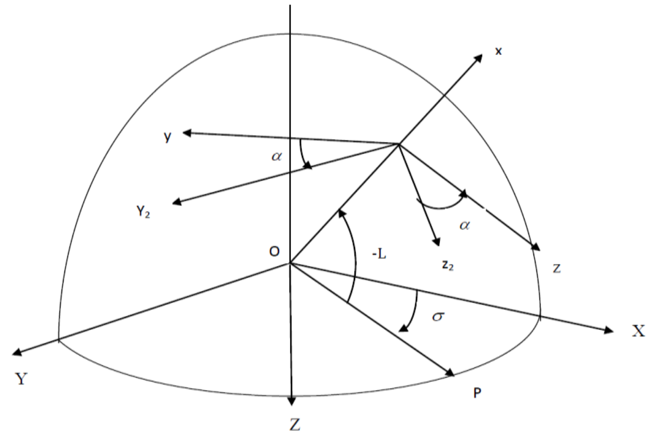

Figure 2: Orbital sequence

4.1.2 Orbital parameter sequence

In this application, the aircraft's body frame xyz is related to the NED reference coordinate frame XYZ (refer fig 3) defined as XY plane is the Tangent plane to the Earth pointing towards North and East directions respectively. Z axis points towards the centre of the Earth (NED frame of reference). The positive x axis of the body frame is directed along the longitudinal axis. The positive y axis is directed along its right wing and the positive z axis is perpendicular to the xy plane such that xyz forms a right handed system. These two frames are related by the heading and attitude sequence of rotations followed by a third and final rotation about the newest x axis which is the position vector of the spacecraft through an angle α as depicted by the figure 2.

This requires the 3-1-3 symmetric sequence of rotations; the Spinor that represents the required rotation is given as a sequence of three rotations about the base vectors is defined by

$$\mathcal{R} = \mathcal{R}_\nu Q_\iota \mathcal{R}_\Omega = R_\nu^\dagger Q_\iota^\dagger R_\Omega^\dagger \sigma_k R_\Omega Q_\iota R_\nu ,$$

Where $R_\Omega = e^{(1/2)\, i\, \sigma_3 \Omega} = \cos\frac{\Omega}{2} + \sigma_1\sigma_2 \sin\frac{\Omega}{2}$,

$$Q_\iota = e^{(1/2)\, i\, \sigma_1 \iota} = \cos\frac{\iota}{2} + \sigma_2\sigma_3 \sin\frac{\iota}{2},$$

$$R_\nu = e^{(1/2)\, i\, \sigma_3 \nu} = \cos\frac{\nu}{2} + \sigma_1\sigma_2 \sin\frac{\nu}{2}$$

The matrix obtained is

$$\begin{vmatrix} \cos\Omega\cos\nu - \sin\nu\cos\iota\sin\Omega & -\sin\nu\cos\iota\cos\Omega - \sin\Omega\cos\nu & \sin\nu\sin\iota \\ \cos\Omega\sin\nu + \cos\nu\cos\iota\sin\Omega & \cos\nu\cos\iota\cos\Omega - \sin\Omega\sin\nu & -\cos\nu\sin\iota \\ \sin\iota\sin\Omega & \sin\iota\cos\Omega & \cos\iota \end{vmatrix}$$

As the orbital sequence and ephemeris sequence represent the same body frame of reference, they can be equated to get the relations between the orbital ephemeris in terms of the orbital elements. Thus we obtain

$$\sin L = -\sin\nu \sin\iota$$

$$\tan\alpha = \frac{\cos\nu \sin\iota}{\cos\iota} = \cos\nu \tan\iota$$

$$\tan\sigma = \frac{\cos L \sin\sigma}{\cos L \cos\sigma} = \frac{(\tan\Omega + \tan\nu\cos\iota)}{1 - \tan\nu\cos\iota\tan\Omega}$$

4.2 Spinor half angle method

Equating the orbit sequence with Ephemeris sequence

$$R_\alpha Q_{-L} R_\sigma = R_\nu Q_\iota R_\Omega$$

,

$$R_\sigma = e^{(1/2)\ i\ \sigma_3\sigma} = \cos\frac{\sigma}{2} + \sigma_1\sigma_2 \sin\frac{\sigma}{2};$$

$$Q_{-L} = e^{(1/2)\ i\ \sigma_2(-L)} = \cos\frac{L}{2} - \sigma_3\sigma_1 \sin\frac{L}{2}$$

$$R_\alpha = e^{(1/2)\ i\ \sigma_1\alpha} = \cos\frac{\alpha}{2} + \sigma_2\sigma_3 \sin\frac{\alpha}{2};$$

$$R_\nu = e^{(1/2)\ i\ \sigma_1\nu} = \cos\frac{\nu}{2} + \sigma_1\sigma_2 \sin\frac{\nu}{2};$$

$$Q_\iota = e^{(1/2)\ i\ \sigma_2\iota} = \cos\frac{\iota}{2} + \sigma_1\sigma_2 \sin\frac{\iota}{2} \quad \text{and}$$

$$R_\Omega = e^{(1/2)\ i\ \sigma_3\Omega} = \cos\frac{\Omega}{2} + \sigma_1\sigma_2 \sin\frac{\Omega}{2}$$

$$R_\alpha Q_L R_\sigma = R_\nu Q_\iota R_\Omega \Rightarrow$$

$$R_\alpha Q_{-L} = R_\nu Q_\iota R_\Omega R_\sigma^{-1} = R_\nu Q_\iota R_\Omega R_{-\sigma} = R_\nu Q_\iota R_{\Omega-\sigma} = R_\nu Q_\iota R_\psi$$

$$\Rightarrow R_\alpha Q_{-L} = R_\nu Q_\iota R_\psi$$

To avoid half angles

$$\alpha = 2\rho, L = 2\varepsilon, \Omega = 2\omega, \iota = 2\beta, \Omega\ \ \sigma = \psi = 2\tau, \nu = 2\gamma$$

$$R_\alpha Q_{-L} = R_\nu Q_\iota R_\psi \Rightarrow R_{2\rho} Q_{-2\varepsilon} = R_{2\gamma} Q_{2\beta} R_{2\tau}$$

Gives

$$(\cos\rho + \sigma_2\sigma_3\sin\rho)(\cos\varepsilon + \sigma_3\sigma_1\sin\varepsilon)$$
$$= (\cos\gamma + \sigma_1\sigma_2\sin\gamma)(\cos\beta + \sigma_2\sigma_3\sin\beta)(\cos\tau + \sigma_1\sigma_2\sin\tau)$$

$$\Rightarrow \cos\rho\cos\varepsilon + \sigma_1\sigma_2\sin\rho\sin\varepsilon + \sigma_2\sigma_3\sin\rho\cos\varepsilon - \sigma_3\sigma_1\sin\varepsilon\cos\rho$$

$$= (\cos\gamma + \sigma_1\sigma_2\sin\gamma)[\cos\beta\cos\tau + \sigma_1\sigma_2\cos\beta\sin\tau + \sigma_2\sigma_3\sin\beta\cos\tau + \sigma_3\sigma_1\sin\beta\sin\tau]$$

$$= \cos\gamma\cos\beta\cos\tau + \sigma_1\sigma_2\cos\gamma\cos\beta\sin\tau + \sigma_2\sigma_3\cos\gamma\sin\beta\cos\tau + \sigma_3\sigma_1\cos\gamma\sin\beta\sin\tau +$$
$$\sigma_1\sigma_2\sin\gamma\cos\beta\cos\tau - \sin\gamma\cos\beta\sin\tau + \sigma_1\sigma_3\sin\gamma\sin\beta\cos\tau + \sigma_2\sigma_3\sin\gamma\sin\beta\sin\tau$$

$$= \cos\gamma\cos\beta\cos\tau - \sin\gamma\cos\beta\sin\tau + \sigma_1\sigma_2(\cos\gamma\cos\beta\sin\tau + \sin\gamma\cos\beta\cos\tau)$$
$$+ \sigma_2\sigma_3(\cos\gamma\sin\beta\cos\tau + \sin\gamma\sin\beta\sin\tau) + \sigma_3\sigma_1(\cos\gamma\sin\beta\sin\tau - \ \sin\gamma\sin\beta\cos\tau)$$

$$\Rightarrow \cos\rho\cos\varepsilon = \cos\gamma\cos\beta\cos\tau \ \ - \sin\gamma\cos\beta\sin\tau = \cos\beta\cos(\tau+\gamma) \ldots.(1)$$

$$-\sin\rho\sin\varepsilon = \cos\gamma\cos\beta\sin\tau + \sin\gamma\cos\beta\cos\tau = \cos\beta\sin(\tau+\gamma) \ldots.(2)$$

$$\sin\rho\cos\varepsilon = \cos\gamma\sin\beta\cos\tau + \sin\gamma\sin\beta\sin\tau = \sin\beta\cos(\tau-\gamma) \ldots(3)$$

$$\sin\varepsilon\cos\rho = \cos\gamma\sin\beta\sin\tau - \ \ \sin\gamma\sin\beta\cos\tau = \sin\beta\sin(\tau-\gamma) \ldots(4)$$

Dividing (3) by (1) and (2) by (4) gives

$$\tan\rho = \tan\beta\frac{\cos(\tau-\gamma)}{\cos(\tau+\gamma)}$$

$$\Rightarrow \frac{\tan\rho}{\tan\beta} = \frac{\cos(\tau-\gamma)}{\cos(\tau+\gamma)} \ldots.(5)$$

$$-\tan\rho = \frac{1}{\tan\beta}\frac{\sin(\tau+\gamma)}{\sin(\tau-\gamma)}$$

$$\Rightarrow -\tan\rho\tan\beta = \frac{\sin(\tau+\gamma)}{\sin(\tau-\gamma)} = \frac{\sin(\tau+\gamma)}{\sin(\gamma-\tau)} \ldots(6)$$

Multiplying (5) and (6) gives

$$-\tan^2\rho = \frac{\cos(\tau-\gamma)\sin(\tau+\gamma)}{\sin(\tau-\gamma)\cos(\tau+\gamma)} = \frac{\tan(\tau+\gamma)}{\tan(\tau-\gamma)} \ldots..(7)$$

dividing (6) by (5) gives

$$-\tan^2\beta = \frac{\sin(\tau+\gamma)\cos(\tau+\gamma)}{\sin(\tau-\gamma)\cos(\tau-\gamma)} = \frac{\sin 2(\tau+\gamma)}{\sin 2(\tau-\gamma)}$$

$$\text{but} \quad \cos 2\beta = \frac{1-\tan^2\beta}{1+\tan^2\beta}$$

$$= \frac{\sin 2(\tau-\gamma) + \sin 2(\tau+\gamma)}{\sin 2(\tau-\gamma) - \sin 2(\tau+\gamma)} = -\frac{\tan 2\tau}{\tan 2\gamma}$$

$$\tan 2\tau = -\cos 2\beta\tan 2\gamma$$

substituting $\iota = 2\beta, \Omega - \sigma = \psi = 2\tau, \nu = 2\gamma$ gives

$$\tan\psi = -\cos\iota\tan\nu$$

$$-\cos\iota\tan\nu = \tan\psi = \tan(\Omega - \sigma) = \frac{\tan\Omega - \tan\sigma}{1 + \tan\Omega\tan\sigma}$$

After simplification we get

$$\tan\sigma = \frac{\cos\iota\tan\nu + \tan\Omega}{1 - \tan\Omega\cos\iota\tan\nu}$$

Similarly we get the other relations.

4.3 Singularities in Euler sequences

A singularity occurs in every sequence of Euler angles. For example in the 3-1-3 sequence,

$\tan\phi = \dfrac{e_{12}}{e_{11}} = \dfrac{\sin\phi\cos\theta}{\cos\phi\cos\theta} = \dfrac{\sin\phi}{\cos\phi}$. Singularity occurs at $\phi = \dfrac{\pi}{2}$.

But Euler angles work well for small angles or Infinitesimal rotations. Hence in the problems like finding the orientation of the spacecraft and tracking an aero plane employment of Euler angles is advantageous.

V. DISCUSSIONS

Even though there are many representations for rotations in 3-D namely Quaternions, Spinors, Euler axis-angle and sequences of Euler angles, each one can be transformed into the other. Each system has its own merits or demerits over the other systems. Singularities occur for every set of Euler angles but they work well in infinitesimal rotations. Comparatively, the other methods of using Quaternions and Euler axis and angle for representing rotations provide a better procedure as they need four parameters to be defined and they can be converted into any other convenient form depending upon the information available. In the case of Spinors, whose parametric form is given as $s = \alpha + \mathbf{i}\vec{\beta}$, the number of parameters reduce further as the parameter α is not independent of $\vec{\beta}$.

VI. CONCLUSIONS

The technique of using Spinors can replace the conventional methods and also provide a richer formalism. If large number of rotations is involved Spinor matrix methods are advantageous over the Spinor half angle method.

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REFERENCES

Books:

[1] Hestenes D., *New Foundations for Classical Mechanics*, D. Reidel Publishing, 1986.
[2] Hestenes D., *Space-Time Algebra*, Gordon and Breach (1966).
[3] Lounesto P., *Clifford Algebras and Spinors*, 2nd Edition, Cambridge University Press (2001).
[4] Kuipers J.B., *Quaternions and Rotation Sequences*, Princeton University Press, Princeton, New Jersey (1999).

Thesis:

Hasan S. N., *Use of Clifford Algebra in Physics: A Mathematical Model*, M.Phil Thesis, University of Hyderabad, India (1987).