Abstract:

The concept of triple connected graphs with real life application was introduced in [9] by considering the existence of a path containing any three vertices of G. A graph G is said to be triple connected if any three vertices lie on a path in G. In [3], the concept of triple connected dominating set was introduced. A set \( S \subseteq V \) is a triple connected dominating set if \( S \) is a dominating set of G and the induced sub graph \( <S> \) is triple connected. The triple connected domination number \( \gamma_{tc}(G) \) is the minimum cardinality taken over all triple connected dominating sets in G. In this paper, we introduce a new domination parameter, called Complementary perfect triple connected domination number of a graph. A set \( S \subseteq V \) is a complementary perfect triple connected dominating set if \( S \) is a triple connected dominating set of G and the induced sub graph \( <V-S> \) has a perfect matching. The complementary perfect triple connected domination number \( \gamma_{cptc}(G) \) is the minimum cardinality taken over all complementary perfect triple connected dominating sets in G. We determine this number for some standard classes of graphs and obtain some bounds for general graph. Their relationships with other graph theoretical parameters are investigated.

Key words: Domination Number, Triple connected graph, Complementary perfect triple connected domination number.

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1 Introduction

By a graph we mean a finite, simple, connected and undirected graph \( G (V, E) \), where \( V \) denotes its vertex set and \( E \) its edge set. Unless otherwise stated, the graph \( G \) has \( p \) vertices and \( q \) edges. Degree of a vertex \( v \) is denoted by \( d(v) \), the maximum degree of a graph \( G \) is denoted by \( \Delta(G) \). We denote a cycle on \( p \) vertices by \( C_p \), a path on \( p \) vertices by \( P_p \), and a complete graph on \( p \) vertices by \( K_p \). A graph \( G \) is connected if any two vertices of \( G \) are connected by a path. A maximal connected subgraph of a graph \( G \) is called a component of \( G \).

The number of components of \( G \) is denoted by \( \omega(G) \). The complement \( \overline{G} \) of \( G \) is the graph with vertex set \( V \) in which two vertices are adjacent if and only if they are not adjacent in \( G \). A tree is a connected acyclic graph. A bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets \( U \) and \( V \) such that every edge connects a vertex in \( U \) to one in \( V \). A complete bipartite graph is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set. The complete bipartite graph with partitions of order \( |V_1|=m \) and \( |V_2|=n \), is denoted \( K_{m,n} \). A star, denoted by \( K_{1,p} \), is a tree with one root vertex and \( p-1 \) pendant vertices. A bistar, denoted by \( B(m,n) \), is the graph obtained by joining the root vertices of the stars \( K_{1,m} \) and \( K_{1,n} \).

The friendship graph, denoted by \( F_n \), can be constructed by identifying \( n \) copies of the cycle \( C_3 \) at a common vertex. A wheel graph, denoted by \( W_n \), is a graph with \( n \) vertices, formed by connecting a single vertex to all vertices of an \((p-1)\) cycle. A helm graph, denoted by \( H_n \), is a graph obtained from the wheel \( W_n \) by joining a pendant vertex to each vertex in the outer cycle of \( W_n \) by means of an edge. Corona of two graphs \( G_1 \) and \( G_2 \), denoted by \( G_1 \circ G_2 \) is the disjoint union of one copy of \( G_1 \) and \( |V_1| \) copies of \( G_2 \), \( |V_2| \) is the number of vertices in \( G_1 \) in which \( r^p \) vertex of \( G_1 \) is joined to every vertex in the \( r^p \) copy of \( G_2 \). For any real number \( x \), \( \lceil x \rceil \) denotes the largest integer less than or equal to \( x \). If \( S \) is a subset of \( V \), then \( <S> \) denotes the vertex induced subgraph of \( G \) induced by \( S \). The open neighbourhood of a set \( S \) of vertices of a graph \( G \), denoted by \( N(S) \) is the set of all vertices adjacent to some vertex in \( S \) and \( N(S) \cup S \) is called the closed neighbourhood of \( S \), denoted by \( N[S] \). The diameter of a connected graph is the maximum distance between two vertices in \( G \) and is denoted by \( diam(G) \). A cut – vertex (cut edge) of a graph \( G \) is a vertex (edge) whose removal increases the number of components. A vertex cut, or separating set of a connected graph \( G \) is a set of vertices whose removal renders \( G \) disconnected. The connectivity or vertex connectivity of a graph \( G \), denoted by \( \kappa(G) \) (where \( G \) is not complete) is the size of a smallest vertex cut. A connected subgraph \( H \) of a
A connected graph $G$ is called a **H-cut** if $\omega(G - H) \geq 2$. The **chromatic number** of a graph $G$, denoted by $\chi(G)$, is the smallest number of colors needed to color all the vertices of a graph $G$ in which adjacent vertices receive different color. Terms not defined here are used in the sense of [2].

A subset $S$ of $V$ is called a **dominating set** of $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The **domination number** $\gamma(G)$ of $G$ is the minimum cardinality taken over all dominating sets in $G$. A dominating set $S$ of a connected graph $G$ is said to be a **connected dominating set** if $G$ if the induced sub graph $<S>$ is connected. The minimum cardinality taken over all connected dominating set is the **connected domination number** and is denoted by $\gamma_c$. A subset $S$ of $V$ of a nontrivial graph $G$ is said to be **Complementary perfect dominating set**, if $S$ is a dominating set and the subgraph induced by $<V-S>$ has a perfect matching. The minimum cardinality taken over all Complementary perfect dominating sets is called the **Complementary perfect domination number** and is denoted by $\gamma_{cp}$.

One can get a comprehensive survey of results on various types of domination number of a graph in [11].

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [1,8]. Recently the concept of triple connected graphs was introduced by Paulraj Joseph J. et. al.[9] by considering the existence of a path containing any three vertices of $G$. They have studied the properties of triple connected graph and established many results on them. A graph $G$ is said to be **triple connected** if any three vertices lie on a path in $G$. All paths and cycles, complete graphs and wheels are some standard examples of triple connected graphs.

A set $S \subseteq V$ is a triple connected dominating set if $S$ is a dominating set of $G$ and the induced sub graph $<S>$ is triple connected. The **connected domination number** $\gamma_c(G)$ is the minimum cardinality taken over all triple connected dominating sets in $G$.

A set $S \subseteq V$ is a complementary perfect triple connected dominating set if $S$ is a triple connected dominating set of $G$ and the induced sub graph $<V - S>$ has a perfect matching. The complementary perfect triple connected domination number $\gamma_{cp}(G)$ is the minimum cardinality taken over all complementary perfect triple connected dominating sets in $G$.

In this paper we use this idea to develop the concept of complementary perfect triple connected dominating set and complementary perfect triple connected domination number of a graph.
**Example 2.4** For the graph $G_2$ in Figure 2.2, any minimum dominating set must contain the supports and any connected dominating set containing these supports is not Complementary perfect triple connected and hence $\gamma_{cptc}$ does not exists.

**Figure 2.2**

**Remark 2.5** Throughout this paper we consider only connected graphs for which complementary perfect triple connected dominating set exists.

**Observation 2.6** The complement of the complementary perfect triple connected dominating set need not be a complementary perfect triple connected dominating set.

**Example 2.7** For the graph $G_3$ in Figure 2.3, $S = \{v_1, v_2, v_3\}$ forms a Complementary perfect triple connected dominating set of $G_3$. But the complement $V - S = \{v_4, v_5, v_6, v_7\}$ is not a Complementary perfect triple connected dominating set.

**Figure 2.3**

**Observation 2.8** Every Complementary perfect triple connected dominating set is a dominating set but not the converse.

**Example 2.9** For the graph $G_4$ in Figure 2.4, $S = \{v_1\}$ is a dominating set, but not a complementary perfect triple connected dominating set.

**Figure 2.4**

**Observation 2.10** For any connected graph $G$, $\gamma(G) \leq \gamma_p(G) \leq \gamma_{cptc}(G)$ and the inequalities are strict.

**Example 2.11** For $K_6$ in Figure 2.5, $\gamma(K_6) = \{v_1\} = 1$, $\gamma_p(K_6) = \{v_1, v_2\} = 2$ and $\gamma_{cptc}(K_6) = \{v_1, v_2, v_3, v_4\} = 4$. Hence $\gamma(G) \leq \gamma_p(G) \leq \gamma_{cptc}(G)$.

**Figure 2.5**

**Theorem 2.12** If the induced subgraph of each connected dominating set of $G$ has more than two pendant vertices, then $G$ does not contain a Complementary perfect triple connected dominating set.

**Proof** This theorem follows from Theorem 1.2.

**Example 2.13** For the graph $G_6$ in Figure 2.6, $S = \{v_5, v_6, v_7\}$ is a minimum connected dominating set so that $\gamma(G_6) = 4$. Here we notice that the induced subgraph of $S$ has three pendant vertices and hence $G$ does not have a Complementary perfect triple connected dominating set.

**Figure 2.6**

**Observation 2.14** If a spanning sub graph $H$ of a graph $G$ has a Complementary perfect triple
Theorem 2.19 For any connected graph \( G \) with \( p \geq 5 \), we have \( 3 \leq \gamma_{\text{cptc}}(G) \leq p - 2 \) and the bounds are sharp.

Proof The lower and upper bounds trivially follows from Definition 2.1. For \( C_5 \), the lower bound is attained and for \( C_5 \) the upper bound is attained.

**Theorem 2.20** For a connected graph \( G \) with 5 vertices, \( \gamma_{\text{cptc}}(G) = p - 2 \) if and only if \( G \) is isomorphic to \( C_5, W_5, K_{2,5}, F_5, K_3 - \{e\}, K_d(P_2), C_d(P_2), C_d(P_3), C_d(2P_2) \) or any one of the graphs shown in Figure 2.9.

Proof Suppose \( G \) is isomorphic to \( C_5, W_5, K_{2,5}, F_5, K_3 - \{e\}, K_d(P_2), C_d(P_2), C_d(P_3), C_d(2P_2) \) or any one of the graphs \( H_1 \) to \( H_7 \) given in Figure 2.9, then clearly \( \gamma_{\text{cptc}}(G) = p - 2 \). Conversely, let \( G \) be a connected graph with 5 vertices and \( \gamma_{\text{cptc}}(G) = 3 \). Let \( S = \{x, y, z\} \) be a \( \gamma_{\text{cptc}} \)-set, then clearly \( \langle S \rangle = P_3 \) or \( C_5 \). Let \( V - S = V(G) - V(S) = \{u, v\} \), then \( \langle V - S \rangle = K_2 = uv \). Case (i) \( \langle S \rangle = P_3 = xyz \).

Since \( G \) is connected, there exists a vertex say \( x \) or \( y \), \( z \) in \( P_3 \) is adjacent to \( u \) or \( v \) in \( K_2 \), then \( \gamma_{\text{cptc}} \)-set of \( G \) does not exists. But on increasing the degrees of the vertices of \( S \), let \( x \) be adjacent to \( u \) and \( z \) be adjacent to \( v \). If \( d(x) = d(y) = d(z) = 2 \), then \( G \cong C_5 \). Now by increasing the degrees of the vertices, by the above argument, we have \( G \cong K_5, K_3 - \{e\}, K_d(P_2), C_d(P_2), C_d(P_3), C_d(2P_2) \) or any one of the graphs \( H_1 \) to \( H_7 \) given in Figure 2.9. Since \( G \) is connected, there exists a vertex say \( y \) in \( P_3 \) is adjacent to \( u \) or \( v \) in \( K_2 \), then \( \gamma_{\text{cptc}} \)-set of \( G \) does not exists. But on increasing the degrees of the vertices of \( S \), let \( y \) be adjacent to \( v \), \( x \) be adjacent to \( u \) and \( z \) be adjacent to \( u \) and \( v \). If \( d(x) = 3 \),
3. Complementary Perfect Triple Connected Domination Number and Other Graph Theoretical Parameters

Theorem 3.1 For any connected graph $G$ with $p \geq 5$ vertices, $\gamma_{cptc}(G) + \kappa(G) \leq 2p - 3$ and the bound is sharp if and only if $G \cong K_5$.

Proof Let $G$ be a connected graph with $p \geq 5$ vertices. We know that $\kappa(G) \leq p - 1$ and by Theorem 2.19, $\gamma_{cptc}(G) \geq p - 2$. Hence $\gamma_{cptc}(G) + \kappa(G) \leq 2p - 3$. Suppose $G$ is isomorphic to $K_5$. Then clearly $\gamma_{cptc}(G) + \kappa(G) = 2p - 3$. Conversely, let $\gamma_{cptc}(G) + \kappa(G) = 2p - 3$. This is possible only if $\gamma_{cptc}(G) = p - 2$ and $\kappa(G) = p - 1$. But $\kappa(G) = p - 1$, and so $G \cong K_5$ for which $\gamma_{cptc}(G) = 3 = p - 2$ so that $p = 5$. Hence $G \cong K_5$.

Theorem 3.2 For any connected graph $G$ with $p \geq 5$ vertices, $\gamma_{cptc}(G) + \chi(G) \leq 2p - 2$ and the bound is sharp if and only if $G \cong K_5$.

Proof Let $G$ be a connected graph with $p \geq 5$ vertices. We know that $\chi(G) \leq p - 1$ and by Theorem 2.19, $\gamma_{cptc}(G) \geq p - 2$. Hence $\gamma_{cptc}(G) + \chi(G) \leq 2p - 2$. Suppose $G$ is isomorphic to $K_5$. Then clearly $\gamma_{cptc}(G) + \chi(G) = 2p - 3$. This is possible only if $\gamma_{cptc}(G) = p - 2$ and $\chi(G) = p$. But $\chi(G) = p$, and so $G$ is isomorphic to $K_5$ for which $\gamma_{cptc}(G) = 3 = p - 2$ so that $p = 5$. Hence $G \cong K_5$.

Theorem 3.3 For any connected graph $G$ with $p \geq 5$ vertices, $\gamma_{cptc}(G) + \Delta(G) \leq 2p - 3$ and the bound is sharp if and only if $G$ is isomorphic to $W_5$, $K_5$, $C_5(2)$, $K_5 - \{e\}$, $K_5(2P_2)$, $C_5(2P_2)$ or any one of the graphs shown in Figure 3.1.

![Figure 3.1](image_url)

**Proof** Let $G$ be a connected graph with $p \geq 5$ vertices. We know that $\Delta(G) \leq p - 1$ and by Theorem 2.19, $\gamma_{cptc}(G) \geq p - 2$. Hence $\gamma_{cptc}(G) + \Delta(G) \leq 2p - 3$. Let $G$ be isomorphic to $W_5$, $K_5$, $C_5(2)$, $K_5 - \{e\}$, $K_5(2P_2)$, $C_5(2P_2)$ or any one of the graphs $G_1$ to $G_4$ given in Figure 3.1, then clearly $\gamma_{cptc}(G) + \Delta(G) = 2p - 3$. Conversely, let $\gamma_{cptc}(G) + \Delta(G) = 2p - 3$. This is possible only if $\gamma_{cptc}(G) = p - 2$ and $\Delta(G) = p - 1$. But $\gamma_{cptc}(G) = p - 2$ and $\Delta(G) = p - 1$, by Theorem 2.20, we have $G \cong W_5$, $K_5$, $C_5(2)$, $K_5 - \{e\}$, $K_5(2P_2)$, $C_5(2P_2)$ and the graphs $G_1$ to $G_4$ given in Figure 3.1.
REFERENCES


