On Comparative Growth Rates of Differential Polynomials Generated by Meromorphic Functions

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Abstract
In this paper we discuss some growth rates of composite entire and meromorphic functions and differential polynomials generated by one of the factors improving some earlier results.
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1. Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$.Also let $n_{0j}, n_{1j},...n_{kj} (k \geq 1)$ be non-negative integers such that for each $j, \sum_{i=0}^{k} n_{ij} \geq 1$.We call $M_j[f] = A_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}}...(f^{(k)})^{n_{kj}}$ where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The $\gamma_{M_i} = \sum_{i=0}^{k} n_{ij}$ and $\Gamma_{M_i} = \sum_{i=0}^{k}(i+1)n_{ij}$ are called respectively the degree and weight of $M_j[f]$ {[3],[9]}. The expression $P[f] = \sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f.The numbers $\gamma_{P[f]} = \max_{1<j<s} \gamma_{M_j}$ and $\Gamma_{P[f]} = \max_{1<j<s} \Gamma_{M_j}$ are called respectively the degree and weight of $P[f]$ {[3],[9]}. Also we call the numbers $\underset{-}{\gamma}_{P[f]} = \min_{1<j<s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underset{-}{\gamma}_{P[f]} = \gamma_{P[f]}$, $P[f]$ is called a homogeneous differential polynomial.

In the paper we establish some newly developed results based on the comparative growth of composite entire and meromorphic functions and differential polynomials generated by their factors on the basis of (p,q) th order and (p,q) th lower order where p,q are positive integers and $p > q$. Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e. for which $n_{0j} = 0$ for $j = 1,2,......s$. We consider only those $P[f], P_0[f]$ singularities of whose individual terms do not cancel each other.

We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [12] and [4]. In the sequel we use the following notations:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1,2,3,... \text{ and } \log^{[0]} x = x;$$

and

$$\exp^{[k]} x = \exp(\exp^{[k-1]} x) \text{ for } k = 1,2,3,... \text{ and } \exp^{[0]} x = x.$$

The following definitions are well known.

Definition 1 The quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r,a;f)}{T(r,f)}.$$

Definition 2 [7] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r,a;f)$ denote the number of zeros of $f - a$ in $|z| \le r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\ge p$ is counted exactly p times; and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way .We define

$$\delta_p(a; f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{T(r,f)}.$$

Definition 3 [2] $P[f]$ is said to be admissible if

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m(r,f) = S(r,f)$.

Definition 4 The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

If f is meromorphic then

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad and \quad \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Extending this notion, Sato [8] defined the generalized order and generalized lower order of an entire function as follows :

Definition 5 [8] The generalized order $\rho_f^{[l]}$ and the generalized lower order $\lambda_f^{[l]}$ of an entire function f are defined as :

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M(r,f)}{\log r} \quad and \quad \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M(r,f)}{\log r}.$$

If f is meromorphic then

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} T(r,f)}{\log r} \quad and \quad \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} T(r,f)}{\log r}.$$

Juneja, Kapoor and Bajpai[5] defined the (p,q) th order and (p,q) th lower order of an entire function f respectively as follows :

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \quad and \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p, q are positive integers with $p > q.$

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \quad and \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q.$

If $p = 2$ and $q = 1$ then we write $\rho_f(p,q) = \rho_f$ and $\lambda_f(p,q) = \lambda_f$.

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [1] If f be a meromorphic function and g be an entire function then for all sufficiently large values of r,

$$T(r, f\circ g) \le \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2 [6] Let f be an entire function of finite lower order. If there exists entire functions $d_i(i=1,2,\ldots n; n\le\infty)$ satisfying $T(r,d_i)=o\{T(r,f)\}$ and $\sum_i^n \delta(d_i,f)=1$, then

$$\lim_{r\to\infty}\frac{T(r,f)}{\log M(r,f)}=\frac{1}{\pi}.$$

Lemma 3 Let g be an entire function with $\lambda_g<\infty$ and assume that $d_i(i=1,2,\ldots n; n\le\infty)$ satisfying $T(r,d_i)=o\{T(r,g)\}$. If $\sum_i^n \delta(d_i,g)=1$, then for any integer $q\ge 2$,

$$\lim_{r\to\infty}\frac{\log^{[q]}T(r,f)}{\log^{[q+1]}M(r,f)}=1.$$

Proof. In view of Lemma 2 we get for all sufficiently large values of r that

Now in view of Lemma 2 we obtain for a sequence of values of r tending to infinity that

$$\frac{1}{\pi}-\varepsilon\le\frac{T(r,f)}{\log M(r,f)}\le\frac{1}{\pi}+\varepsilon$$

$$i.e.,\ (\frac{1}{\pi}-\varepsilon)\log M(r,f)\le T(r,f)\le(\frac{1}{\pi}+\varepsilon)\log M(r,f)$$

$$i.e.,\ \log^{[q+1]}M(r,f)+O(1)\le\log^{[q]}T(r,f)\le\log^{[q+1]}M(r,f)+O(1)$$

$$i.e.,\ 1+\frac{O(1)}{\log^{[q+1]}M(r,f)}\le\frac{\log^{[q]}T(r,f)}{\log^{[q+1]}M(r,f)}\le 1+\frac{O(1)}{\log^{[q+1]}M(r,f)}$$

Therefore we get from above that

$$1\le\liminf_{r\to\infty}\frac{\log^{[q]}T(r,f)}{\log^{[q+1]}M(r,f)}\le\limsup_{r\to\infty}\frac{\log^{[q]}T(r,f)}{\log^{[q+1]}M(r,f)}\le 1$$

$$i.e.,\ \lim_{r\to\infty}\frac{\log^{[q]}T(r,f)}{\log^{[q+1]}M(r,f)}=1.$$

This proves the lemma.

Lemma 4 [2] Let $P_0[f]$ be admissible. If f is of finite order or of non zero lower order and $\sum_{a\neq\infty}\Theta(a;f)=2$ then

$$\lim_{r\to\infty}\frac{T(r,P_0[f])}{T(r,f)}=\Gamma_{P_0[f]}.$$

Lemma 5 [2Let f be either of finite order or of non-zero lower order such that $\Theta(a;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$or $\delta(a;f)=\sum_{a\neq\infty}\delta(a;f)=1$. Then for homogeneous $P_0[f]$,

$$\lim_{r\to\infty}\frac{T(r,P_0[f])}{T(r,f)}=\gamma_{P_0[f]}.$$

Lemma 6 Let f be a meromorphic function of finite order or of non zero lower order. If $\sum_{a\neq\infty}\Theta(a;f)=2$, then the order (lower order) of homogeneous $P_0[f]$ is same as that of f.

Proof. By Lemma 4, $\lim_{r\to\infty}\frac{\log^{[p]}T(r,P_0[f])}{\log^{[p]}T(r,f)}$ exists and is equal to 1.

$$\rho_{P_0[f]}=\limsup_{r\to\infty}\frac{\log T(r,P_0[f])}{\log r}$$

$$= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} . \lim_{r\to\infty} \frac{\log T(r,P_0[f])}{\log T(r,f)}$$
$$= \rho_f .1$$
$$= \rho_f .$$

In a similar manner, $\lambda_{P_0[f]} = \lambda_f$.

This proves the lemma.

Lemma 7 Let f be a meromorphic function of finite order or of non zero lower order such that $\Theta(a;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$. Then the order (lower order) of homogeneous $P_0[f]$ and f are same.

We omit the proof of Lemma 7 because it can be carried out in the line of Lemma 6 and with the help of Lemma 5.

In a similar manner we can state the following lemma without proof.

Lemma 8 Let f be a meromorphic function of finite order or of non-zero lower order such that $\delta(a;f) = \sum_{a\neq\infty} \delta(a;f) = 1$.Then for every homogeneous $P_0[f]$ the order (lower order) of $P_0[f]$ is same as that of f .

In order to state our next lemma we need the following notion of sharing of values of meromorphic functions.

Let $S(f)$ be the set of all meromorphic functions $a \equiv a(z)$ in $|z| < \infty$ which satisfy $T(r,a) = o\{T(r,f)\}$ as $r \to \infty$.We consider $\bar{C} = \mathbb{C} \cup \{\infty\}$ as a subset of $S(f)$ .We shall call any $a \in S(f)$ as a small function (with respect to f).

Let $E(f = a) = \{z : f(z) - a(z) = 0\}$..

The meromorphic functions f and h are said to share "a" if and only if $E(f = a) = E(h = a)$.We say that two meromorphic functions f and h share a value "a" IM (CM) if $f - a$ and $h - a$ have the same zeros ignoring multiplicities (with the same multiplicity). In addition, we introduce the following notations:

$S(m,n)(b) = \{z | z$ is a common zero of $f - b$ and $f^{/} - b$ with multiplicities m and n respectively$\}$.

$\bar{N}(m,n)(r, \frac{1}{(f-b)})$ denotes the counting function of f with respect to the set $S(m,n)(b)$.

Lemma 9 [11] If f and h are any two meromorphic functions that share six small functions $a_i \in S(f) \cap S(h)$ for $i = 1,2,...6$ ignoring multiplicities then outside a set of r of finite linear measure,

$$\lim_{r\to\infty} \frac{T(r,h)}{T(r,f)} = 1.$$

Lemma 10 If f and h are any two meromorphic functions with finite orders or of non zero lower orders and $\sum_{a\neq\infty} \Theta(a;f) = \sum_{a\neq\infty} \Theta(a;h) = 2$. Also f and h share six small functions $a_i \in S(f) \cap S(h)$ for $i = 1,2,...6$ ignoring multiplicities then outside a set of r of finite linear measure and for any positive integer p ,

$$\lim_{r\to\infty} \frac{\log^{[p]} T(r,P_0[h])}{\log^{[p]} T(r,P_0[f])} = 1.$$

Proof. In view of Lemma 4 and Lemma 9 we get that

$$\lim_{r\to\infty} \frac{T(r,P_0[h])}{T(r,P_0[f])} = \lim_{r\to\infty} (\frac{T(r,P_0[h])}{T(r,h)} . \frac{T(r,f)}{T(r,P_0[f])} . \frac{T(r,h)}{T(r,f)})$$

$$i.e., \lim_{r\to\infty} \frac{T(r,P_0[h])}{T(r,P_0[f])} = \lim_{r\to\infty} \frac{T(r,P_0[h])}{T(r,h)} . \lim_{r\to\infty} \frac{T(r,f)}{T(r,P_0[f])} . \lim_{r\to\infty} \frac{T(r,h)}{T(r,f)}$$

$$i.e., \lim_{r\to\infty}\frac{T(r,P_0[h])}{T(r,P_0[f])}=\frac{\Gamma_{P_0[h]}}{\Gamma_{P_0[f]}}.$$

Therefore from above for any positive integer p,

$$\lim_{r\to\infty}\frac{\log^{[p]}T(r,P_0[h])}{\log^{[p]}T(r,P_0[f])}$$

exists and is equal to 1.
Thus the lemma follows.

Lemma 11 Let f and h be any two meromorphic functions with finite orders or of non zero lower orders such $\Theta(a;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ and $\Theta(a;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$. Also let f and h share six small functions $a_i\in S(f)\cap S(h)$ for $i=1,2,...6$ ignoring multiplicities. Then outside a set of r of finite linear measure and for any positive integer p

$$\lim_{r\to\infty}\frac{\log^{[p]}T(r,P_0[h])}{\log^{[p]}T(r,P_0[f])}=1.$$

We omit the proof of Lemma 11 because it can be carried out in the line of Lemma 10 and with the help of Lemma 5.

In a similar manner we can state the following lemma without proof.

Lemma 12 Let f and h be any two meromorphic functions with finite orders or of non zero lower orders such $\delta(a;f)=\sum_{a\neq\infty}\delta(a;f)=1$ and $\delta(a;h)=\sum_{a\neq\infty}\delta(a;h)=1$. If f and h share six small functions $a_i\in S(f)\cap S(h)$ for $i=1,2,...6$ ignoring multiplicities then outside a set of r of finite linear measure and for any positive integer p

$$\lim_{r\to\infty}\frac{\log^{[p]}T(r,P_0[h])}{\log^{[p]}T(r,P_0[f])}=1.$$

Lemma 13 [10] If f be a non constant entire function, a and b two distinct small functions of f with $a\neq\infty$ and $b\neq\infty$. If f and f' share a and b IM then $f\equiv f'$.

3. Theorems.

In this section we present the main results of the paper.

Theorem 1 Let f and h be any two meromorphic functions of finite order or of non zero lower order such that $\sum_{a\neq\infty}\Theta(a;f)=\sum_{a\neq\infty}\Theta(a;h)=2$. Again g be an entire function such that $\rho_g(m,n)<\lambda_h\leq\rho_h<\infty$ where m,n are positive integers with $m>n.$ Also if f and h share six small functions $a_i\in S(f)\cap S(h)$ for $i=1,2,...6$ ignoring multiplicities then outside a set of r of finite linear measure,

$$\lim_{r\to\infty}\frac{\log^{[m-1]}T(\exp^{[n-1]}r,f\circ g)}{T(r,P_0[f])}=0.$$

Proof. Since $\rho_g(m,n)<\lambda_h$ we can choose $\varepsilon(>0)$ in such a way that

$$\rho_g(m,n)+\varepsilon<\lambda_h-\varepsilon \tag{1}$$

As $T(r,g)\leq\log^+M(r,g)$, we have from Lemma 1 for all sufficiently large values of r,

$$\log T(\exp^{[n-1]}r,f\circ g)\leq\log T(M(\exp^{[n-1]}r,g),f)+O(1)$$
$$i.e., \log T(\exp^{[n-1]}r,f\circ g)\leq(\rho_f+\varepsilon)\log M(\exp^{[n-1]}r,g)+O(1). \tag{2}$$

Then for all sufficiently large values of r we get from (2) that

$$\log T(\exp^{[n-1]}r,f\circ g)\leq(\rho_f+\varepsilon)\exp^{[m-1]}\log^{[m]}M(\exp^{[n-1]}r,g)+O(1). \tag{3}$$

Again for all sufficiently large values of r,

$$\log^{[m]} M(\exp^{[n-1]} r, g) \leq (\rho_g(m,n)+\varepsilon)\log^{[n]}\exp^{[n-1]} r$$

$$i.e., \log^{[m]} M(\exp^{[n-1]} r, g) \leq (\rho_g(m,n)+\varepsilon)\log r$$

$$i.e., \log^{[m]} M(\exp^{[n-1]} r, g) \leq \log r^{(\rho_g(m,n)+\varepsilon)}$$

$$i.e., \exp^{[m-1]}\log^{[m]} M(\exp^{[n-1]} r, g) \leq \exp^{[m-1]}\log r^{(\rho_g(m,n)+\varepsilon)}$$

$$i.e., \exp^{[m-1]}\log^{[m]} M(\exp^{[n-1]} r, g) \leq \exp^{[m-2]} r^{(\rho_g(m,n)+\varepsilon)}. \tag{4}$$

Now from (3) and (4) we have for all sufficiently large values of r,

$$\log T(\exp^{[n-1]} r, f\circ g) \leq (\rho_f+\varepsilon)\exp^{[m-1]}\exp^{[m-2]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$i.e., \log^{[2]} T(\exp^{[n-1]} r, f\circ g) \leq \exp^{[m-3]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$i.e., \log^{[m-1]} T(\exp^{[n-1]} r, f\circ g) \leq \log^{[m-3]}\exp^{[m-3]} r^{(\rho_g(m,n)+\varepsilon)} + O(1)$$

$$i.e., \log^{[m-1]} T(\exp^{[n-1]} r, f\circ g) \leq r^{(\rho_g(m,n)+\varepsilon)} + O(1). \tag{5}$$

For all sufficiently large values of r we obtain in view of Lemma 6 that

$$\log T(r, P_0[h]) \geq (\lambda_{P_0[h]}-\varepsilon)\log r$$

$$i.e., \log T(r, P_0[h]) \geq (\lambda_h-\varepsilon)\log r$$

$$i.e., \log T(r, P_0[h]) \geq \log r^{(\lambda_h-\varepsilon)}$$

$$i.e., \log T(r, P_0[h]) \geq r^{(\lambda_h-\varepsilon)}. \tag{6}$$

Again combining (5) and (6) we obtain for all sufficiently large values of r that

$$\frac{\log^{[m-1]} T(\exp^{[n-1]} r, f\circ g)}{\log T(r, P_0[h])} \leq \frac{r^{(\rho_g(m,n)+\varepsilon)} + O(1)}{r^{(\lambda_h-\varepsilon)}}. \tag{7}$$

Now in view of (1) it follows from (7) that

$$\limsup_{r\to\infty}\frac{\log^{[m-1]} T(\exp^{[n-1]} r, f\circ g)}{T(r, P_0[h])} = 0$$

$$i.e., \lim_{r\to\infty}\frac{\log^{[m-1]} T(\exp^{[n-1]} r, f\circ g)}{T(r, P_0[h])} = 0. \tag{8}$$

Since f and h share six small functions $a_i \in S(f)\cap S(h)$ *for* $i = 1,2,\ldots 6$ ignoring multiplicities then in view of Lemma 10 we may write outside a set of r of finite linear measure,

$$\lim_{r\to\infty}\frac{T(r, P_0[h])}{T(r, P_0[f])} = \frac{\Gamma_{P_0[h]}}{\Gamma_{P_0[f]}}. \tag{9}$$

Thus the theorem follows from (8) and (9).

Remark 1 The conclusion of Theorem 1 can also be drawn under the hypothesis $\Theta(a;f) = \sum_{a\neq\infty}\delta_p(a;f) = 1$ and $\Theta(a;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ or $\delta(a;f) = \sum_{a\neq\infty}\delta(a;f) = 1$ and $\delta(a;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ instead of $\sum_{a\neq\infty}\Theta(a;f) = 2$ and $\sum_{a\neq\infty}\Theta(a;h) = 2$.

Theorem 2 Let f and h be any two meromorphic functions of finite order or non zero lower order such that $\sum_{a\neq\infty}\Theta(a;f) = 2$ and $\sum_{a\neq\infty}\Theta(a;h) = 2$. Again let g be an entire function such that $\lambda_g(m,n) < \lambda_h \leq \rho_h < \infty$ where m, n are positive integers with $m > n$. Also if f and h share six small functions $a_i \in S(f)\cap S(h)$ *for* $i = 1,2,\ldots 6$ ignoring multiplicities then outside a set of r of finite linear measure,

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(\exp^{[n-1]}r, f\circ g)}{T(r,P_0[f])}=0.$$

Proof. For a sequence of values of r tending to infinity that

$$\log^{[m]}M(\exp^{[n-1]}r,g)\le(\lambda_g(m,n)+\varepsilon)\log^{[n]}\exp^{[n-1]}r$$

$$i.e.,\ \log^{[m]}M(\exp^{[n-1]}r,g)\le(\lambda_g(m,n)+\varepsilon)\log r$$

$$i.e.,\ \log^{[m]}M(\exp^{[n-1]}r,g)\le\log r^{(\lambda_g(m,n)+\varepsilon)}$$

$$i.e.,\ \exp^{[m-1]}\log^{[m]}M(\exp^{[n-1]}r,g)\le\exp^{[m-1]}\log r^{(\lambda_g(m,n)+\varepsilon)}$$

$$i.e.,\ \exp^{[m-1]}\log^{[m]}M(\exp^{[n-1]}r,g)\le\exp^{[m-2]}r^{(\lambda_g(m,n)+\varepsilon)}. \tag{10}$$

Now from (3) and (10) we have for a sequence of values of r tending to infinity that

$$\log T(\exp^{[n-1]}r, f\circ g)\le(\rho_f+\varepsilon)\exp^{[m-1]}\exp^{[m-2]}r^{(\lambda_g(m,n)+\varepsilon)}+O(1)$$

$$i.e.,\ \log^{[2]}T(\exp^{[n-1]}r, f\circ g)\le\exp^{[m-3]}r^{(\lambda_g(m,n)+\varepsilon)}+O(1)$$

$$i.e.,\ \log^{[m-1]}T(\exp^{[n-1]}r, f\circ g)\le\log^{[m-3]}\exp^{[m-3]}r^{(\lambda_g(m,n)+\varepsilon)}+O(1)$$

$$i.e.,\ \log^{[m-1]}T(\exp^{[n-1]}r, f\circ g)\le r^{(\lambda_g(m,n)+\varepsilon)}+O(1). \tag{11}$$

Combining (6) and (11) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]}T(\exp^{[n-1]}r, f\circ g)}{\log T(r,P_0[h])}\le\frac{r^{(\lambda_g(m,n)+\varepsilon)}+O(1)}{r^{(\lambda_h-\varepsilon)}}. \tag{12}$$

Now in view of (1) it follows from (12) that

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}T(\exp^{[n-1]}r, f\circ g)}{T(r,P_0[h])}=0. \tag{13}$$

Thus the theorem follows from (9) and (13).

Remark 2 The conclusion of Theorem 2 can also deduced if we replace $\sum_{a\ne\infty}\Theta(a;f)=2$ and $\sum_{a\ne\infty}\Theta(a;h)=2$ by $\Theta(a;f)=\sum_{a\ne\infty}\delta_p(a;f)=1$ and $\Theta(a;h)=\sum_{a\ne\infty}\delta_p(a;h)=1$ or $\delta(a;f)=\sum_{a\ne\infty}\delta(a;f)=1$ and $\delta(a;h)=\sum_{a\ne\infty}\delta(a;h)=1$ respectively.

Theorem 3 Let f and h be any two meromorphic functions of finite order or non zero lower order such that $\delta(a;f)=\sum_{a\ne\infty}\delta(a;f)=1$ and $\delta(a;h)=\sum_{a\ne\infty}\delta(a;h)=1$. Also let g be an entire function with $\lambda_{f\circ g}=\infty$. Then if f and h share six small functions $a_i\in S(f)\cap S(h)$ *for* $i=1,2,\ldots 6$ ignoring multiplicities then outside a set of r of finite linear measure, and for every $A(>0)$

$$\lim_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(r^A,P_0[f])}=\infty.$$

Proof. Let us suppose that

$$\lim_{r\to\infty}\frac{\log T(r,f\circ g)}{\log T(r^A,P_0[h])}=\infty$$

do not hold. Then we can find a constant $\beta>0$ such that for a sequence of values of r tending to infinity

$$\log T(r,f\circ g)\le\beta\log T(r^A,P_0[h]). \tag{14}$$

Again from the definition of $P_0[h]$ it follows that for all sufficiently large values of r

$$\log T(r^A, P_0[h]) \le (\rho_h + \varepsilon) A \log r. \tag{15}$$

Thus from (14) and (15) we have for a sequence of values of r tending to infinity that

$$\log T(r, f \circ g) \le \beta A(\rho_h + \varepsilon)\log r$$

$$i.e., \frac{\log T(r, f \circ g)}{\log r} \le \frac{\beta A(\rho_h + \varepsilon)\log r}{\log r}$$

$$i.e., \liminf_{r\to\infty} \frac{\log T(r, f \circ g)}{\log r} = \lambda_{f\circ g} \le \infty.$$

This is a contradiction. Therefore

$$\lim_{r\to\infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P_0[h])} = \infty. \tag{16}$$

Since f and h share six small functions $a_i \in S(f) \cap S(h)$ for $i = 1,2,...6$ ignoring multiplicities then in view of Lemma 12 we may write outside a set of r of finite linear measure,

$$\lim_{r\to\infty} \frac{\log T(r, P_0[h])}{\log T(r, P_0[f])} = 1. \tag{17}$$

Thus the theorem follows from (16) and (17).

Remark 3 Theorem 3 is also valid with "limit superior" instead of "limit" if $\lambda_{f\circ g} = \infty$ is replaced by $\rho_{f\circ g} = \infty$.

Corollary 1 Under the assumptions of Remark 3,

$$\limsup_{r\to\infty} \frac{T(r, f \circ g)}{T(r^A, P_0[f])} = \infty.$$

Proof. From Remark 3 we obtain for all sufficiently large values of r and for $K > 1$,

$$\log T(r, f \circ g) > K \log T(r^A, P_0[f])$$

$$i.e., \log T(r, f \circ g) > \{\log T(r^A, P_0[f])\}^K,$$

from which the corollary follows.

Remark 4 Theorem 3, Remark 3 and Corollary 1 are all valid if we take $\sum_{a\neq\infty} \Theta(a; f) = \sum_{a\neq\infty} \Theta(a; h) = 2$ or $\Theta(a; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ and $\Theta(a; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ instead of $\delta(a; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and $\delta(a; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ and the other conditions remain the same.

Theorem 4 Let f and h be any two meromorphic functions of finite order or of non zero lower order such that $\delta(a; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and $\delta(a; h) = \sum_{a\neq\infty} \delta(a; h) = 1$. Also let g be entire such that $\rho_g(m, n) < \infty$ where m, n are positive integers with $m > n$. Also if f and h share six small functions $a_i \in S(f) \cap S(h)$ for $i = 1,2,...6$ ignoring multiplicities then outside a set of r of finite linear measure,

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(\exp^{[n-1]} r, f \circ g)}{\log T(r, P_0[f])} \le \frac{\rho_g(m, n)}{\lambda_h}.$$

Proof. In view of Lemma 8 we have for all sufficiently large values of r

$$\log T(r, P_0[h]) \ge (\lambda_{P_0[h]} - \varepsilon)\log r$$

$$i.e., \log T(r, P_0[h]) \ge (\lambda_h - \varepsilon)\log r. \tag{18}$$

Now from (5) and (18) we obtain for all sufficiently large values of r that

$$\frac{\log^{[m]} T(\exp^{[n-1]} r, f \circ g)}{\log T(r, P_0[h])} \le \frac{(\rho_g(m, n) + \varepsilon)\log r + O(1)}{(\lambda_h - \varepsilon)\log r}$$

$$i.e., \frac{\log^{[m]} T(\exp^{[n-1]} r, f \circ g)}{\log T(r, P_0[h])} \le \frac{\rho_g(m, n) + \varepsilon}{\lambda_h - \varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[m]}T(\exp^{[n-1]}r, f\circ g)}{\log T(r,P_0[h])}\leq\frac{\rho_g(m,n)}{\lambda_h}. \tag{19}$$

Since f and h share six small functions $a_i\in S(f)\cap S(h)$ for $i=1,2,\ldots 6$ ignoring multiplicities then in view of Lemma 12 we may write outside a set of r of finite linear measure,

$$\lim_{r\to\infty}\frac{\log T(r,P_0[h])}{\log T(r,P_0[f])}=1. \tag{20}$$

Thus the theorem follows from (19) and (20).

Remark 5 If we replace $\delta(a;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$ and $\delta(a;h)=\sum\limits_{a\neq\infty}\delta(a;h)=1$ by $\sum\limits_{a\neq\infty}\Theta(a;f)=2$ and $\sum\limits_{a\neq\infty}\Theta(a;h)=2$ or $\Theta(a;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ and $\Theta(a;h)=\sum\limits_{a\neq\infty}\delta_p(a;h)=1$ in Theorem 4 and the other conditions remain the same then also the conclusion of Theorem 4 holds.

Theorem 5 Let f be a meromorphic function such that $\lambda_f(p,q)$ is finite where p,q are any two positive integers with $p>q$ and g and h be any two transcendental non constant entire functions of finite order or of non zero lower order such that $\sum\limits_{a\neq\infty}\Theta(a;g)=2$ and $\sum\limits_{a\neq\infty}\Theta(a;h)=2$ .Also suppose that there exist entire functions $d_i\,(i=1,2,\ldots n; n\leq\infty)$ satisfying

(A) $T(r,d_i)=o\{T(r,g)\}$ as $r\to\infty$ and

(B) $\sum\limits_{i}^{n}\delta(d_i,g)=1.$

Now if g and h share six small functions $a_i\in S(f)\cap S(h)$ for $i=1,2,\ldots 6$ ignoring multiplicities outside a set of r of finite linear measure. Also for any two distinct small functions b and c of h with $b\neq\infty$ and $c\neq\infty$ if h and h' share b and c IM, then

$$(i)\qquad \liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log T(r,P_0[h'])}\leq\lambda_f(p,q).\pi.\Gamma_{P_0[h]}.$$

$$(ii)\qquad \limsup_{r\to\infty}\frac{\log^{[p]}T(r,f\circ g)}{\log^{[q]}T(r,P_0[h'])}\leq 1$$

and

$$(iii)\qquad \liminf_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[q-1]}T(r,P_0[h'])}\leq\lambda_f(p,q)\quad if\ q>2.$$

Proof. Since $\varepsilon(>0)$ is arbitrary and $T(r,g)\leq\log^{+}M(r,g),$ we have from Lemma 1 for all sufficiently large values of r,

$$\log^{[p-1]}Tr, f\circ g)\leq\log^{[p-1]}T(M(r,g),f)+O(1). \tag{21}$$

So from (21) we get for a sequence of values of r tending to infinity that

$$\log^{[p-1]}T(r,f\circ g)\leq(\lambda_f(p,q)+\varepsilon)\log^{[q]}M(r,g)+O(1)$$
$$i.e.,\log^{[p-1]}T(r,f\circ g)\leq(\lambda_f(p,q)+\varepsilon)\log M(r,g)+O(1)$$
$$i.e.,\frac{\log^{[p-1]}T(r,f\circ g)}{T(r,P_0[g])}\leq\frac{(\lambda_f(p,q)+\varepsilon)\log M(r,g)+O(1)}{T(r,P_0[g])}. \tag{22}$$

Again from (21) we get for all sufficiently large values of r that

$$\frac{\log^{[p]} T(r, f\circ g)}{\log^{[q]} T(r, P_0[g])} \le \frac{\log^{[q+1]} M(r,g) + O(1)}{\log^{[q]} T(r, P_0[g])}. \tag{23}$$

Also from (21) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[q-1]} T(r, P_0[g])} \le \frac{(\lambda_f(p,q)+\varepsilon)\log^{[q]} M(r,g) + O(1)}{\log^{[q-1]} T(r, P_0[g])}. \tag{24}$$

Since $\varepsilon(>0)$ is arbitrary by Lemma 2 and Lemma 4 it follows from (22) that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{T(r, P_0[g])} \le (\lambda_f(p,q)+\varepsilon) \limsup_{r\to\infty} \frac{\log M(r,g)}{T(r, P_0[g])}$$

$$i.e., \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{T(r, P_0[g])} \le (\lambda_f(p,q)+\varepsilon) \limsup_{r\to\infty} \frac{\log M(r,g)}{T(r,g)} . \lim_{r\to\infty} \frac{T(r,g)}{T(r, P_0[g])}$$

$$i.e., \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{T(r, P_0[g])} \le \lambda_f(p,q).\pi.\frac{1}{\Gamma_{P_0[g]}}. \tag{25}$$

Since g *and* h share six small functions $a_i \in S(f)\cap S(h)$ *for* $i = 1,2,...6$ ignoring multiplicities then in view of Lemma 10 we may write from (25) outside a set of r of finite linear measure that

$$i.e., \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{T(r, P_0[h])} \le \lambda_f(p,q).\pi.\Gamma_{P_0[h]}. \tag{26}$$

Thus in view of Lemma 13, the first part of Theorem 5 follows from (26).
Again by Lemma 3 and Lemma 4 it follows from (23) that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T(r, f\circ g)}{\log^{[q]} T(r, P_0[g])} \le \limsup_{r\to\infty} \frac{\log^{[q+1]} M(r,g) + O(1)}{\log^{[q]} T(r, P_0[g])}$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[p]} T(r, f\circ g)}{\log^{[q]} T(r, P_0[g])} \le \limsup_{r\to\infty} \frac{\log^{[q+1]} M(r,g) + O(1)}{\log^{[q]} T(r,g)} . \limsup_{r\to\infty} \frac{\log^{[q]} T(r,g)}{\log^{[q]} T(r, P_0[g])}$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[p]} T(r, f\circ g)}{\log^{[q]} T(r, P_0[g])} \le 1. \tag{27}$$

Since g *and* h share six small functions $a_i \in S(f)\cap S(h)$ *for* $i = 1,2,...6$ ignoring multiplicities then in view of Lemma 10 we may write from (27) outside a set of r of finite linear measure that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T(r, f\circ g)}{\log^{[q]} T(r, P_0[h])} \le 1. \tag{28}$$

Thus by Lemma 13 the second part of the theorem follows from (28).
Similarly in view of Lemma 3 and Lemma 4 and as $\varepsilon(>0)$ is arbitrary we obtain from (24) that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[q-1]} T(r, P_0[g])} \le (\lambda_f(p,q)+\varepsilon) \limsup_{r\to\infty} \frac{\log^{[q]} M(r,g) + O(1)}{\log^{[q-1]} T(r, P_0[g])}$$

$$i.e., \ \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[q-1]} T(r, P_0[g])} \le (\lambda_f(p,q)+\varepsilon) \limsup_{r\to\infty} \frac{\log^{[q]} M(r,g) + O(1)}{\log^{[q-1]} T(r,g)} . \lim_{r\to\infty} \frac{\log^{[q-1]} T(r,g)}{\log^{[q-1]} T(r, P_0[g])}$$

$$i.e., \ \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[q-1]} T(r, P_0[g])} \le \lambda_f(p,q). \tag{29}$$

As g *and* h share six small functions $a_i \in S(f) \cap S(h)$ *for* $i = 1,2,\ldots 6$ ignoring multiplicities then in view of Lemma 10 we may write from (27) outside a set of r of finite linear measure that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f \circ g)}{\log^{[q-1]} T(r, P_0[h])} \le \lambda_f(p,q). \tag{29}$$

In view of Lemma 13 the third part of Theorem 5 follows from (30).

In the line of Theorem 5 one may easily prove the following corollary.

Corollary 2 Let f be a meromorphic function such that $\lambda_f(p,q)$ is finite where p,q are any two positive integers with $p > q$ and g and h be any two transcendental non constant entire functions of finite order or of non zero lower order such that $\Theta(a;g) = \sum_{a\ne\infty} \delta_p(a;g) = 1$ and $\Theta(a;h) = \sum_{a\ne\infty} \delta_p(a;h) = 1$ or $\delta(a;g) = \sum_{a\ne\infty} \delta(a;g) = 1$ and $\delta(a;h) = \sum_{a\ne\infty} \delta(a;h) = 1$. Also suppose that there exist entire functions $d_i (i = 1,2,\ldots n; n \le \infty)$ satisfying

(A) $T(r,d_i) = o\{T(r,g)\}$ *as* $r \to \infty$ and

(B) $\sum_{i}^{n} \delta(d_i, g) = 1.$

Now if g *and* h share six small functions $a_i \in S(f) \cap S(h)$ *for* $i = 1,2,\ldots 6$ ignoring multiplicities outside a set of r of finite linear measure. Also for any two distinct small functions b and c of h with $b \ne \infty$ and $c \ne \infty$ if h and h' share b and c IM, then

$$(i) \qquad \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f \circ g)}{\log T(r, P_0[h'])} \le \pi.\lambda_f(p,q).\gamma_{P_0[h]}$$

$$(ii) \qquad \limsup_{r\to\infty} \frac{\log^{[p]} T(r, f \circ g)}{\log^{[q]} T(r, P_0[h'])} \le 1$$

and

$$(iii) \qquad \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r, f \circ g)}{\log^{[q-1]} T(r, P_0[h'])} \le \lambda_f(p,q) \quad \text{if } q > 2.$$

Theorem 6 Let f be a meromorphic function such that $\rho_f(p,q)$ is finite where p,q are any two positive integers with $p > q$ and g and h be any two transcendental non constant entire functions of finite order or of non zero lower order such that $\Theta(a;g) = \sum_{a\ne\infty} \delta_p(a;g) = 1$ and $\Theta(a;h) = \sum_{a\ne\infty} \delta_p(a;h) = 1$ or $\delta(a;g) = \sum_{a\ne\infty} \delta(a;g) = 1$ and $\delta(a;h) = \sum_{a\ne\infty} \delta(a;h) = 1$. Also suppose that there exist entire functions $d_i (i = 1,2,\ldots n; n \le \infty)$ satisfying

(A) $T(r,d_i) = o\{T(r,g)\}$ *as* $r \to \infty$ and

(B) $\sum_{i}^{n} \delta(d_i, g) = 1.$

Now if g *and* h share six small functions $a_i \in S(f) \cap S(h)$ *for* $i = 1,2,\ldots 6$ ignoring multiplicities outside a set of r of finite linear measure. Also for any two distinct small functions b and c of h with $b \ne \infty$ and $c \ne \infty$ if h and h' share b and c IM, then

$$(i) \qquad \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f \circ g)}{\log^{[q-1]} T(r, P_0[h'])} \le \rho_f(p,q) \quad \text{if } q > 2$$

$$(ii) \qquad \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f \circ g)}{\log T(r, P_0[h'])} \le \pi.\rho_f(p,q).\gamma_{P_0[h]}$$

and

$$(iii) \qquad \limsup_{r\to\infty} \frac{\log^{[p]} T(r, f\circ g)}{\log^{[q]} T(r, P_0[h'])} \le 1.$$

Proof. Since $\varepsilon(>0)$ is arbitrary and $T(r,g) \le \log^+ M(r,g)$, we have from Lemma 1 for all sufficiently large values of r,

$$\log^{[p-1]} T(r, f\circ g) \le (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r,g) + O(1). \tag{31}$$

Again from (31) it follows for all sufficiently large values of r that

$$\log^{[p-1]} T(r, f\circ g) \le (\rho_f(p,q)+\varepsilon)\log M(r,g) + O(1). \tag{32}$$

Since $\varepsilon(>0)$ is arbitrary , in view of Lemma 3 and Lemma 5 we obtain from (31) that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[q-1]} T(r, P_0[g])} \le (\rho_f(p,q)+\varepsilon) \limsup_{r\to\infty} \frac{\log^{[q]} M(r,g)}{\log^{[q-1]} T(r, P_0[g])}$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[q-1]} T(r, P_0[g])} \le \rho_f(p,q) \limsup_{r\to\infty} \frac{\log^{[q]} M(r,g)}{\log^{[q-1]} T(r,g)} . \limsup_{r\to\infty} \frac{\log^{[q-1]} T(r,g)}{\log^{[q-1]} T(r, P_0[g])}$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[q-1]} T(r, P_0[g])} \le \rho_f(p,q). \tag{33}$$

Since g *and* h share six small functions $a_i \in S(f) \cap S(h)$ *for* $i = 1,2,\ldots 6$ ignoring multiplicities then in view of Lemma 11 we may write from (33) outside a set of r of finite linear measure that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{\log^{[q-1]} T(r, P_0[h])} \le \rho_f(p,q). \tag{34}$$

Thus in view of Lemma 13, the first part of the theorem is established.
Again by Lemma 3 and as $\varepsilon(>0)$ is arbitrary it follows from (32) that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{T(r, P_0[g])} \le (\rho_f(p,q)+\varepsilon) \limsup_{r\to\infty} \frac{\log M(r,g) + O(1)}{\log^{[q-1]} T(r, P_0[g])}$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{T(r, P_0[g])} \le \rho_f(p,q) \limsup_{r\to\infty} \frac{\log M(r,g)}{T(r,g)} . \limsup_{r\to\infty} \frac{T(r,g)}{T(r, P_0[g])}$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{T(r, P_0[g])} \le \rho_f(p,q).\pi.\frac{1}{\gamma_{P_0[g]}}. \tag{35}$$

Since g *and* h share six small functions $a_i \in S(f) \cap S(h)$ *for* $i = 1,2,\ldots 6$ ignoring multiplicities then in view of Lemma 11 we may write from (33) outside a set of r of finite linear measure that

$$i.e., \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, f\circ g)}{T(r, P_0[h])} \le \rho_f(p,q).\pi.\gamma_{P_0[h]}. \tag{36}$$

In view of Lemma 13 the second part of the theorem follows from (36) .

Now in the line of Theorem 5 one may easily prove the third part of Theorem 6.

Corollary 3 Let f be a meromorphic function such that $\rho_f(p,q)$ is finite where p,q are any two positive integers with $p > q$ and g and h be any two transcendental non constant entire functions of finite order or of non zero lower order such that $\sum_{a\neq\infty} \Theta(a;g) = 2$ and $\sum_{a\neq\infty} \Theta(a;h) = 2$. Also suppose that there exist entire functions $d_i (i = 1,2,\ldots n; n \le \infty)$ satisfying

(A) $T(r,d_i)=o\{T(r,g)\}\ as\ r\to\infty$ and

(B) $\sum_{i}^{n}\delta(d_i,g)=1.$

Now if $g\ and\ h$ share six small functions $a_i\in S(f)\cap S(h)\quad for\ i=1,2,...6$ ignoring multiplicities outside a set of r of finite linear measure. Also for any two distinct small functions b and c of h with $b\neq\infty$ and $c\neq\infty$ if h and $h^{/}$ share b and c IM, then

$$(i)\qquad \limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log^{[q-1]}T(r,P_0[h^{/}])}\leq\rho_f(p,q)\quad if\ q>2$$

$$(ii)\qquad \limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f\circ g)}{\log T(r,P_0[h^{/}])}\leq\pi.\rho_f(p,q).\Gamma_{P_0[h]}$$

and

$$(iii)\qquad \limsup_{r\to\infty}\frac{\log^{[p]}T(r,f\circ g)}{\log^{[q]}T(r,P_0[h^{/}])}\leq 1.$$

Proof. In view of Lemma 10, Corollary 3 follows from Theorem 6.

4. References

[1] Bergweiler, W.: On the Nevanlinna Characteristic of a composite function, Complex Variables, Vol. 10(1988), p.p. 225-236.

[2] Bhattacharjee, N. and Lahiri, I.: Growth and value distribution of differential polynomials, Bull. Math. Soc. Sc. Math. Roumanie Tome, Vol 39(87), No.1-4(1996), pp.85-104.

[3] Doeringer, W.: Exceptional values of differential polynomials, Pacific J Math., Vol. 98, No.1 (1982), pp.55-62.

[4] Hayman, W. K.: Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[5] Juneja, O. P., Kapoor, G. P. and Bajpai, S. K.: On the (p,q)-order and lower (p,q)-order of an entire function, J. Reine Angew. Math., Vol. 282(1976), pp.53-67.

[6] Lin, Q. and Dai, C. : On a conjecture of Shah concerning small functions, Kexue Tong bao, Vol. 31, No 4(English Ed.)(1986) , pp. 220-224.

[7] Lahiri, I.: Deficiencies of differential polynomials, Indian J. Pure Appl. Math.,Vol.30, No.5(1999), pp.435-447.

[8] Sato, D.: On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., Vol. 69 (1963), pp. 411-414.

[9] Sons, L.R.: Deficiencies of monomials, Math. Z, Vol.111 (1969), pp.53-68.

[10] Qiu, G.: Uniqueness of entire functions that share some small functions, Kodai Math. J., Vol. 23 (2000), pp. 1-11.

[11] Singh, A. P., Charak, K. S. and Sharma, J. L. : Meromorphic functions that share small functions, J. Ramanujan Math. Soc., Vol. 11, No.1 (1996), pp.73-84.

[12] Valiron, G.: Lectures on the General Theory of Integral Functions, Chelsea Publishing Company, 1949.

[13] Yi, H. X.: On a result of Singh, Bull. Austral. Math. Soc., Vol. 41 (1990), pp.417-420.