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Reliability Estimation of Stress-Strength Model with Non-Identical Component Strengths: The Exponentiated Pareto Case

Amal S. Hassan*, Heba M. Basheikh**
*(Department of Mathematical Statistics in Statistical Studies and Research (ISSR), Cairo University, Egypt)
** (Department of Statistics in Faculty of Science, King Abdul-Aziz University, Jeddah, Saudi Arabia)

ABSTRACT
This article deals with the Bayesian and non-Bayesian estimation of reliability of an $\boldsymbol{s}$-out-of-$\boldsymbol{k}$system with non-identical component strengths which are subjected to a common stress. Both stress and strength are assumed to have an exponentiated Pareto distribution with common and known shape parameter. Five non-Bayesian methods of estimation will be used which are maximum likelihood, moments, percentile, least squares and weighted least squares. The Bayesian estimation will be studied under squared error and LINEX loss functions using Lindley's approximation. Based on a Monte Carlo simulation, comparison studies are made between the different estimators of system reliability by obtaining their absolute biases and mean squared errors. Comparison study revealed that the maximum likelihood estimator works the best among the competitors.

Keywords- Bayes estimator, exponentiated Pareto distribution, least squares estimator, maximum likelihood stress–strength model,

1. Introduction
The estimation of reliability is a very common problem in statistical literature. The most widely approach applied for reliability estimation is the well-known stress-strength model. This model is used in many applications of physics and engineering such as strength failure and the system collapse.In the statistical approach to the stress-strength model, most of the considerations depend on the assumption that the component strengths are independently and identically distributed (iid). In many practical situations, the components of a system are of different structure so that the assumption of identical strength distributions may not be quite realistic. This is often the case with systems having standby components.

Consider a system made up of k non-identical components. Out of these k components, k_1 are of one category and their strengths$Y_1,\dots,Y_{k_1}$are iid random variablesdistributed as exponentiated Pareto (EP) suggested by Gupta et al. [1] with parameters (θ_1,λ). The remaining components$k_2=k-k_1$are of different category and their strengths $Y_{k_1+1},\dots,Y_k$are iid random variables distributed as EP with parameters (θ_2,λ). This system is subjected to a common stress X which is independent random variable distributed as EP with parameters (θ_3,λ). Let $f_1(y_1;\theta_1,\lambda)$ be a common probability density function (PDF) of strengths$Y_1,\dots,Y_{k_1}$, $f_2(y_2;\theta_2,\lambda)$ be a common PDF of strengths$Y_{k_1+1},\dots,Y_k$ and $g(x;\theta_3,\lambda)$ be PDFof stress$X$.The corresponding cumulative distribution functions are given, respectively, by

$$\left.\begin{aligned}F_1(y_1;\theta_1,\lambda)&=[1-(1+y_1)^{-\lambda}]^{\theta_1};\ y_1>0,\theta_1>0,\lambda>0,\\ F_2(y_2;\theta_2,\lambda)&=[1-(1+y_2)^{-\lambda}]^{\theta_2};\ y_2>0,\theta_2>0,\lambda>0,\\ G(x;\theta_3,\lambda)&=[1-(1+x)^{-\lambda}]^{\theta_3};\ x>0,\theta_3>0,\lambda>0.\end{aligned}\right\}\quad(1)$$

The system operates successfully if at least s out of k components withstand the stress. According to Johnson [2], the system reliability with non-identical component strengths $R_{(s,k)}$ is given by

$$R_{(s,k)}=\sum_{j_1,j_2}\binom{k_1}{j_1}\binom{k_2}{j_2}\int_{-\infty}^{\infty}[1-F_1(x)]^{j_1}[F_1(x)]^{k_1-j_1}[1-F_2(x)]^{j_2}\times[F_2(x)]^{k_2-j_2}dG(x),\quad(2)$$

where the summation is over all possible pair (j_1,j_2) with $0\le j_1\le k_1$and $0\le j_2\le k_2$ such that $s\le j_1+j_2\le k$.It is important to note thatthe system reliability can be extended to more than two groups of components.

The reliability of s-out-of-k system with non-identical components for EP can be computed by substituting equations (1) in equation (2) and simplifying

$$R_{(s,k)}=\theta_3\sum_{j_1=s_1}^{k_1}\binom{k_1}{j_1}\sum_{j_2=s_2}^{k_2}\binom{k_2}{j_2}\sum_{m=0}^{j_1}\binom{j_1}{m}(-1)^m\sum_{n=0}^{j_2}\binom{j_2}{n}(-1)^n\times[\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)+\theta_3]^{-1}.\quad(3)$$

Note that the above expression depends on θ_1,θ_2 and θ_3 only and does not depend on λ.

The problem of estimation of system reliability was originally viewed as an extension of the stress-strength model to a multi-component system. The estimation of reliability of s-out-of-k stress–strength system has been discussed by many authors. Few authors considered the strengths are independently but not all identically distributed and are subjected to a common stress such as Pandey et al. [3] and Paul and BorhanUddin[4].

The main aim of this article is estimating the reliability in multi-component stress-strength model of an s-out-of-$k$system. Assuming both stress and strength are independently distributed asEP with common and known shape parameter λ. This problem is studied when the strengths of the components are independently but not all identically distributed. Maximum likelihood estimator (MLE), moments estimator (ME), percentile estimator (PCE), least squares estimator (LSE) and weighted least squares estimator (WLSE) are obtained. Also, the Bayes estimators under squared error and LINEX loss functions are discussed using Lindley's approximation. Monte Carlo simulation is performed for comparing different methods of estimation.

The rest of the article is organized as follows. In Section 2, different methods of estimation of $R_{(s,k)}$ are discussed. In Section 3, numerical illustration is carried out to illustrate theoretical results. In Section 4, simulation results are displayed. Finally, conclusions are presented in Section 5.

2. Different Methods of Estimation of $R_{(s,k)}$

It is well known that the method of maximum likelihood estimation has invariance property. When the method of estimation of unknown parameter is changed from maximum likelihood to any other traditional method, this invariance principle does not hold good to estimate the parametric function. However, such an adoption of invariance property for other optimal estimators of the parameters to estimate a parametric function is attempted in different situations by different authors (see SrinivasaRao and Kantam [5]). In this direction, in the following subsections some methods of estimation for the reliability of an s-out-of-k system in stress–strength model will be proposed by considering the estimators of model parameters.

2.1 Maximum likelihood estimator of $R_{(s,k)}$

Let$Y_{11}, Y_{12}, \dots, Y_{1m_1}$ be a random sample of size m_1 drawn from EP(θ_1,λ), then $Y_{1(1)} < Y_{1(2)} <..< Y_{1(m_1)}$ denotes the corresponding order statistic sample. Let $Y_{21}, Y_{22}, \dots, Y_{2m_2}$ be a random sample of size m_2 drawn from EP(θ_2,λ), then $Y_{2(1)} < Y_{2(2)} <..< Y_{2(m_2)}$ denotes the corresponding order statistic sample.Let$X_1, X_2, .., X_n$ be a random sample of size n drawn from EP(θ_3,λ),then $X_{(1)} < X_{(2)} <..< X_{(n)}$ denotes the corresponding order statistic sample.Then the likelihood function is given by

$$L = L(\theta_1,\theta_2,\theta_3,\lambda; \underline{y}_1, \underline{y}_2, \underline{x})$$

$$= \theta_3^n \theta_1^{m_1} \theta_2^{m_2} \lambda^{n+m_1+m_2} \prod_{i=1}^{n} [1-(1+x_i)^{-\lambda}]^{\theta_3 - 1} \prod_{i=1}^{n} (1+x_i)^{-(\lambda+1)} \times \prod_{j=1}^{m_1} [1-(1+y_{1j})^{-\lambda}]^{\theta_1-1} \prod_{j=1}^{m_1} (1+y_{1j})^{-(\lambda+1)} \times \prod_{v=1}^{m_2} [1-(1+y_{2v})^{-\lambda}]^{\theta_2-1} \prod_{v=1}^{m_2} (1+y_{2v})^{-(\lambda+1)}. \quad (4)$$

The first derivatives of the log-likelihood function with respect to θ_1,θ_2 and $\theta_3$are given, respectively, by

$$\left.\begin{aligned}
\frac{\partial \ln L}{\partial \theta_1} &= \frac{m_1}{\theta_1} + \sum_{j=1}^{m_1} \ln[1-(1+y_{1j})^{-\lambda}] = 0,\\
\frac{\partial \ln L}{\partial \theta_2} &= \frac{m_2}{\theta_2} + \sum_{v=1}^{m_2} \ln[1-(1+y_{2v})^{-\lambda}] = 0,\\
\frac{\partial \ln L}{\partial \theta_3} &= \frac{n}{\theta_3} + \sum_{i=1}^{n} \ln[1-(1+x_i)^{-\lambda}] = 0.
\end{aligned}\right\} \quad (5)$$

Then the MLE's of θ_1,θ_2 and θ_3, denotedby$\hat{\theta}_{1(MLE)}$,$\hat{\theta}_{2(MLE)}$ and$\hat{\theta}_{3(MLE)}$, respectively, can be obtained as the solution of equations (5) as

$$\left.\begin{aligned}
\hat{\theta}_{1(MLE)} &= -\frac{m_1}{\sum_{j=1}^{m_1} \ln[1-(1+y_{1j})^{-\lambda}]},\\
\hat{\theta}_{2(MLE)} &= -\frac{m_2}{\sum_{v=1}^{m_2} \ln[1-(1+y_{2v})^{-\lambda}]},\\
\hat{\theta}_{3(MLE)} &= -\frac{n}{\sum_{i=1}^{n} \ln[1-(1+x_i)^{-\lambda}]}.
\end{aligned}\right\} (6)$$

The MLE of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)MLE}$, is obtained by substitute $\hat{\theta}_{1(MLE)}$,$\hat{\theta}_{2(MLE)}$ and$\hat{\theta}_{3(MLE)}$ in equation (3).

2.2 Moments estimator of$R_{(s,k)}$

Since the strengths $Y_1, \dots, Y_{k_1}$ follow EP (θ_1,λ), the strengths $Y_{k_1+1}, \dots, Y_k$follow EP $(\theta_2,\lambda)$and the stress X follows EP (θ_3,λ), then their population means are given by

$$\left.\begin{aligned}
\mu_{Y_1} &= E(Y_1) = \theta_1 B(\theta_1, 1-\tfrac{1}{\lambda}) - 1 \,; \lambda > 1,\\
\mu_{Y_2} &= E(Y_2) = \theta_2 B(\theta_2, 1-\tfrac{1}{\lambda}) - 1 \,; \lambda > 1,\\
\mu_X &= E(X) = \theta_3 B(\theta_3, 1-\tfrac{1}{\lambda}) - 1 \,; \lambda > 1,
\end{aligned}\right\} \quad (7)$$

where $B(.,.)$ stands for the beta function.

According to the method of moments, equating the samples means with the corresponding populations means. Then,

$$\left.\begin{aligned}
\bar{y}_1 &= \theta_1 B(\theta_1, 1-\tfrac{1}{\lambda}) - 1 \,; \lambda > 1,\\
\bar{y}_2 &= \theta_2 B(\theta_2, 1-\tfrac{1}{\lambda}) - 1 \,; \lambda > 1,\\
\bar{x} &= \theta_3 B(\theta_3, 1-\tfrac{1}{\lambda}) - 1 \,; \lambda > 1.
\end{aligned}\right\} (8)$$

The ME's ofθ_1,θ_2 and θ_3, denoted by$\hat{\theta}_{1(ME)}$,$\hat{\theta}_{2(ME)}$ and$\hat{\theta}_{3(ME)}$, respectively, can be obtained by solving the non-linear equations (8)numerically. The ME of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)ME}$,is obtained by substitute$\hat{\theta}_{1(ME)}$, $\hat{\theta}_{2(ME)}$ and $\hat{\theta}_{3(ME)}$in equation (3).

2.3 Percentile estimator of $R_{(s,k)}$

The percentile estimators can be obtained by equating the sample percentile points with the population percentile points.In case of a EP distribution it is possible to use the same concept to obtain the estimators based on the percentiles, because of the structure of its distribution function.According to Kao [6,7] several estimators of p_{1j}, p_{2v} and p_{3i}, where p_{1j}, p_{2v} and p_{3i} are the samples percentile, can be used as estimates for populations percentile$F_1(y_{1(j)};\theta_1,\lambda)$, $F_2(y_{2(v)};\theta_2,\lambda)$ and $G(x_{(i)};\theta_3,\lambda)$.

The following formulas will be considered in this work:

$p_{1j} = \frac{j}{m_1+1}$; $j = 1,2,..,m_1$, $p_{2v} = \frac{v}{m_2+1}$; $v = 1,2,..,m_2$and

$$p_{3i} = \frac{i}{n+1};\ i = 1,2,..,n,$$

which are the expected values of $F_1(Y_{1(j)})$,$F_2(Y_{2(v)})$ and $G(X_{(i)})$respectively.

Then the PCE's of θ_1,θ_2 and θ_3 can be obtained by minimizing the following equationswith respect to θ_1,$\theta_2$and θ_3, respectively,

$$\left.\begin{aligned}
&\sum_{j=1}^{m_1} [\ln(p_{1j}) - \theta_1 \ln[1-(1+y_{1(j)})^{-\lambda}]]^2,\\
&\sum_{v=1}^{m_2} [\ln(p_{2v}) - \theta_2 \ln[1-(1+y_{2(v)})^{-\lambda}]]^2,\\
&\sum_{i=1}^{n} [\ln(p_{3i}) - \theta_3 \ln[1-(1+x_{(i)})^{-\lambda}]]^2.
\end{aligned}\right\} \quad (9)$$

Then the PCE's of θ_1,θ_2 and θ_3, denoted by$\hat{\theta}_{1(PCE)}$,$\hat{\theta}_{2(PCE)}$ and$\hat{\theta}_{3(PCE)}$, respectively, take the forms

$$\left.\begin{aligned}\hat{\theta}_{1(PCE)} &= \frac{\sum_{j=1}^{m_1}\ln(p_{1j})\ln[1-(1+y_{1(j)})^{-\lambda}]}{\sum_{j=1}^{m_1}\left[\ln[1-(1+y_{1(j)})^{-\lambda}]\right]^2},\\ \hat{\theta}_{2(PCE)} &= \frac{\sum_{v=1}^{m_2}\ln(p_{2v})\ln[1-(1+y_{2(v)})^{-\lambda}]}{\sum_{v=1}^{m_2}[\ln[1-(1+y_{2(v)})^{-\lambda}]]^2},\\ \hat{\theta}_{3(PCE)} &= \frac{\sum_{i=1}^{n}\ln(p_{3i})\ln[1-(1+x_{(i)})^{-\lambda}]}{\sum_{i=1}^{n}[\ln[1-(1+x_{(i)})^{-\lambda}]]^2}.\end{aligned}\right\}(10)$$

The PCE of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)PCE}$, is obtained by substitute $\hat{\theta}_{1(PCE)}$,$\hat{\theta}_{2(PCE)}$and$\hat{\theta}_{3(PCE)}$ in equation (3).

2.4 Least squares and weighted least squares estimatorsof $R_{(s,k)}$

Least squares estimators are obtained by minimizing the sum of squared errors between the value and it's expected value. This estimation method is very popular for model fitting, especially in linear and non-linear regression. According to Johnson et al. [8]:

$$E\big(F_1(Y_{1(j)})\big) = \frac{j}{m_1+1}; j = 1,..,m_1, E\big(F_2(Y_{2(v)})\big) = \frac{v}{m_2+1}; v = 1,..,m_2, E\big(G(X_{(i)})\big) = \frac{i}{n+1}; i = 1,..,n,$$

and

$$V\big(F_1(Y_{1(j)})\big) = \frac{j(m_1-j+1)}{(m_1+1)^2(m_1+2)},\quad V\big(F_2(Y_{2(v)})\big) = \frac{v(m_2-v+1)}{(m_2+1)^2(m_2+2)},$$

$$V\big(G(X_{(i)})\big) = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

Using the expectations and the variances of $F_1(Y_{1(j)})$,$F_2(Y_{2(v)})$ and $G(X_{(i)})$, two variants of the least squares methods can be used. The LSE's of θ_1,θ_2 and θ_3, denoted by$\hat{\theta}_{1(LSE)}$,$\hat{\theta}_{2(LSE)}$and$\hat{\theta}_{3(LSE)}$, respectively, can be obtained by minimizing the following equationswith respect to θ_1,θ_2 and θ_3

$$\left.\begin{aligned}&\sum_{j=1}^{m_1}[F_1(Y_{1(j)}) - E\big(F_1(Y_{1(j)})\big)]^2,\\ &\sum_{v=1}^{m_2}[F_2(Y_{2(v)}) - E\big(F_2(Y_{2(v)})\big)]^2,\\ &\sum_{i=1}^{n}[G(X_{(i)}) - E\big(G(X_{(i)})\big)]^2.\end{aligned}\right\}$$
(11)

The LSE of$R_{(s,k)}$, denoted by $\hat{R}_{(s,k)LSE}$, is obtained by substitute $\hat{\theta}_{1(LSE)}$,$\hat{\theta}_{2(LSE)}$ and$\hat{\theta}_{3(LSE)}$ in equation (3).Also, the WLSE's of θ_1, $\theta_2$andθ_3,denoted by$\hat{\theta}_{1(WLSE)}$,$\hat{\theta}_{2(WLSE)}$ and$\hat{\theta}_{3(WLSE)}$, respectively, can be obtained by minimizing the following equationswith respect to θ_1, $\theta_2$and θ_3

$$\left.\begin{aligned}&\sum_{j=1}^{m_1} w_{1j}[F_1(Y_{1(j)}) - E\big(F_1(Y_{1(j)})\big)]^2,\\ &\sum_{v=1}^{m_2} w_{2v}[F_2(Y_{2(v)}) - E\big(F_2(Y_{2(v)})\big)]^2,\\ &\sum_{i=1}^{n} w_{3i}[G(X_{(i)}) - E\big(G(X_{(i)})\big)]^2.\end{aligned}\right\}$$
(12)

where,

$$w_{1j} = \frac{1}{V(F_1(Y_{1(j)}))} = \frac{(m_1+1)^2(m_1+2)}{j(m_1-j+1)},\quad w_{2v} = \frac{1}{V(F_2(Y_{2(v)}))} = \frac{(m_2+1)^2(m_2+2)}{v(m_2-v+1)} \text{and} w_{3i} = \frac{1}{V(G(X_{(i)}))} = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

The WLSE of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)WLSE}$, is obtained by substitute $\hat{\theta}_{1(WLSE)}$, $\hat{\theta}_{2(WLSE)}$ and$\hat{\theta}_{3(WLSE)}$ in equation (3).

In the following subsection the approximate Bayes estimators (BE's) of$R_{(s,k)}$ are obtained. The approximate Bayes estimators under squared error and LINEX loss functions, by using Lindley's approximation, denoted by BESL and BELL, respectively, are discussed.

2.5 Bayes estimators of $R_{(s,k)}$

Assume θ_1, $\theta_2$and θ_3 are independent random variables. Following Afify[9] the noninformative type of priors for parameters θ_1, $\theta_2$and θ_3 are assumed. Therefore, the joint prior density of $(\theta_1,\theta_2,\theta_3)$ is:

$$g(\theta_1,\theta_2,\theta_3) \propto \frac{1}{\theta_1}\frac{1}{\theta_2}\frac{1}{\theta_3}\quad ;0<\theta_i<\infty\ ,i = 1,2,3.$$

Combining the joint prior density of $(\theta_1,\theta_2,\theta_3)$and the likelihood function given in equation (4) to obtain the joint posterior density of $(\theta_1,\theta_2,\theta_3)$ as

$$(\theta_1,\theta_2,\theta_3|\underline{y}_1,\underline{y}_2,\underline{x}) = \frac{g(\theta_1,\theta_2,\theta_3)L(\theta_1,\theta_2,\theta_3,\lambda;\underline{y}_1,\underline{y}_2,\underline{x})}{\int_0^\infty\int_0^\infty\int_0^\infty g(\theta_1,\theta_2,\theta_3)L(\theta_1,\theta_2,\theta_3,\lambda;\underline{y}_1,\underline{y}_2,\underline{x})d\theta_1 d\theta_2 d\theta_3}, = \frac{b}{b_1}; 0<\theta_i<\infty\ ,i = 1,2,3,(13)$$

where,

$$b = \theta_3^{n-1}\theta_1^{m_1-1}\theta_2^{m_2-1}\prod_{i=1}^{n}[1-(1+x_i)^{-\lambda}]^{\theta_3-1}\times\prod_{j=1}^{m_1}[1-(1+y_{1j})^{-\lambda}]^{\theta_1-1}\prod_{v=1}^{m_2}[1-(1+y_{2v})^{-\lambda}]^{\theta_2-1},$$

$$b_1 = \int_0^\infty\int_0^\infty\int_0^\infty \theta_3^{n-1}\theta_1^{m_1-1}\theta_2^{m_2-1}\prod_{i=1}^{n}[1-(1+x_i)^{-\lambda}]^{\theta_3-1}\times\prod_{j=1}^{m_1}[1-(1+y_{1j})^{-\lambda}]^{\theta_1-1}\prod_{v=1}^{m_2}[1-(1+y_{2v})^{-\lambda}]^{\theta_2-1}\,d\theta_1 d\theta_2 d\theta_3.$$

Under squared error and LINEX loss functions, the BE's of$R_{(s,k)}$denoted by$\hat{R}_{(s,k)BES}$ and $\hat{R}_{(s,k)BEL}$, respectively, defined as:

$$\hat{R}_{(s,k)BES} = E(R_{(s,k)}|\underline{y}_1,\underline{y}_2,\underline{x}) = \int_0^\infty\int_0^\infty\int_0^\infty R_{(s,k)}\pi(\theta_1,\theta_2,\theta_3|\underline{y}_1,\underline{y}_2,\underline{x})d\theta_1 d\theta_2 d\theta_3,$$

$$\hat{R}_{(s,k)BEL} = -\frac{1}{a}\ln[E(e^{-aR_{(s,k)}}|\underline{y}_1,\underline{y}_2,\underline{x})] = -\frac{1}{a}\ln[\int_0^\infty\int_0^\infty\int_0^\infty e^{-aR_{(s,k)}}\pi(\theta_1,\theta_2,\theta_3|\underline{y}_1,\underline{y}_2,\underline{x})d\theta_1 d\theta_2 d\theta_3],$$

this integrals cannot be obtained in a simple closed form. Alternatively, using the approximation of Lindley [10] to compute the approximate BE of$R_{(s,k)}$.

Using Lindley's approximation,the approximate BE's of$R_{(s,k)}$under squared error and LINEX loss functions,denoted by$\hat{R}_{(s,k)BESL}$ and $\hat{R}_{(s,k)BELL}$, respectively, take the following forms

$$\left.\begin{aligned}\hat{R}_{(s,k)BESL} &= \tilde{R}_{(s,k)} + \tfrac{1}{2}[U_{11}\tau_{11} + U_{22}\tau_{22} + U_{33}\tau_{33} + \\ &\quad Q_{111}U_1\tau_{11}^2 + Q_{222}U_2\tau_{22}^2 + +Q_{333}U_3\tau_{33}^2],\\ \hat{R}_{(s,k)BELL} &= -\tfrac{1}{a}\ln[e^{-a\tilde{R}_{(s,k)}} + \tfrac{1}{2}[U_{11}\tau_{11} + U_{22}\tau_{22} + U_{33}\tau_{33} + \\ &\quad Q_{111}U_1\tau_{11}^2 + Q_{222}U_2\tau_{22}^2 + +Q_{333}U_3\tau_{33}^2],\end{aligned}\right\} \quad (14)$$

where all functions in equations (14) defined in appendix A and evaluated at the posterior mode $\tilde{\theta}_1 = -\frac{(m_1-1)}{\sum_{j=1}^{m_1}\ln[1-(1+y_{1j})^{-\lambda}]}$, $\tilde{\theta}_2 = -\frac{(m_2-1)}{\sum_{v=1}^{m_2}\ln[1-(1+y_{2v})^{-\lambda}]}$, and $\tilde{\theta}_3 = -\frac{(n-1)}{\sum_{i=1}^{n}\ln[1-(1+x_i)^{-\lambda}]}$.

3. Numerical Experiments and Discussions

In this Section, a numerical experiment will be presented to compare the performance of the different estimatorsof $R_{(s,k)}$ proposed in the previous subsections with respect to their absolute biases and mean squared errors (MSE's). Monte Carlo simulation is applied for different sample sizes, different parameter values and for different s-out-of-k systems. The absolute biases and MSE's are computed for the different estimators over 5000 replications for different cases. The simulation procedures are described through the following steps:

Step 1: A random samples $Y_{11}, Y_{12}, \ldots, Y_{1m_1}$, $Y_{21}, Y_{22}, \ldots, Y_{2m_2}$ and $X_1, X_2, \ldots, X_n$ of sizes (m_1, m_2, n) = (10,10,10), (10,10,30), (10,10,50), (30,30,10), (30,30,30), (30,30,50), (50,50,10), (50,50,30) and (50,50,50) are generated from EP distributions.

Step 2: The parameters values are selected as $(\theta_1, \theta_2, \theta_3)$ = (2,1.5,0.5), (0.5,1.5,2) for λ= 3 in all cases are considered here. The selected values for s-out-of-k systems are (s_1, s_2, k_1, k_2) = (1,1,2,2), (1,2,2,2), (2,1,2,2) and (2,2,2,2).

Step 3: The estimation of the parameters θ_1,$\theta_2$andθ_3 are considered. The MLE's and PCE's of θ_1,$\theta_2$andθ_3 can be obtained from equations (6) and (10), respectively. The ME's of θ_1,$\theta_2$and$\theta_3$can be obtained by solving the non-linear equations (8). Also, The LSE's and WLSE's of θ_1,$\theta_2$andθ_3 can be obtained by minimizing equations (11) and (12) with respect to θ_1,$\theta_2$andθ_3, respectively.

Step 4: The MLE, ME, PCE, LSE and WLSE of $R_{(s,k)}$ are computed by using the estimates of θ_1,$\theta_2$andθ_3 obtained in step 3.

Step 5: The approximate Bayes estimates of $R_{(s,k)}$under squared error and LINEX loss functions, at a=1, using Lindley's approximation can be computed from equations (14).

Step 6: Repeat the pervious steps from 1 to 5 r times representing r different samples, where r = 5000. Then, the absolute average bias and MSE of the estimates of $R_{(s,k)}$ are computed.

4. Simulation Results

All simulated studies presented here are obtained via MathCAD (14). The results are reported in Tables 1 and 2. From Tables 1 and 2 many conclusions can be made on the performance of all methods of estimationof $R_{(s,k)}$. These conclusions are summarized as follows:

1- The value of $R_{(s,k)}$increases as the value of $\theta_1$and $\theta_2$increase and as the value of $\theta_3$decreases (see Tables 1 and 2).

2- For all the methods it is clear that when $m_1 = m_2 = n$ and m_1, m_2, n increase then the MSE's decrease. For fixed m_1, m_2, as n increases then the MSE's decrease. For fixed n, as m_1, m_2 increase then the MSE's decrease. In addition, the biases decrease in almost all values expect for some few cases.

3- For fixed k_i, as s_i, i = 1,2 increases then the value of $R_{(s,k)}$ decreases.

4- Comparing the MSE's of all estimators, the MLE performs the best estimators for $R_{(s,k)}$ for most different values of $(\theta_1, \theta_2, \theta_3)$ and (s_1, s_2, k_1, k_2) considered here. The performance of the BESL and BELL are quite close to the MLE.

5- Comparing the biases of different estimators, it is noted that the MLE's have the minimum biases in almost all of the cases. In few cases, The PCE's have minimum biases.

6- The BESL works the best estimators for $R_{(s,k)}$ at (s_1, s_2, k_1, k_2) = (1,2,2,2) for most different values of (m_1, m_2, n) when $(\theta_1, \theta_2, \theta_3)$= (2,1.5,0.5) in terms of MSE's (see Table 1).

7- At (s_1, s_2, k_1, k_2) = (1,1,2,2) and (1,2,2,2) the BELL works the best estimators for $R_{(s,k)}$when $(\theta_1, \theta_2, \theta_3)$ = (0.5,1.5,2) for most different sample sizes with respect to MSE's (see Table 2).

8- In all cases, WLSE works better estimator for $R_{(s,k)}$than LSE.

9- The ME performs the worst estimators for $R_{(s,k)}$with respect to MSE's in all the cases.

10- In the context of computational complexities, MLE, PCE, BESL and BELL are easiest to compute. They do not involve any non-linear equation solving, whereas the ME, LSE and WLSE involve solving non-linear equations and they need to be calculated by some iterative processes.

5. Conclusions

This article deals with the estimation problem of reliability of s-out-of-k system with non-identical components. All the components are subjected to a common stress X. Assuming that, both stress and strength are independent and have EP distribution with common and known shape parameter λ. Comparing the performance of all estimators, it is observed that the MLE performs the best among the others relative to their absolute biases and MSE's. Furthermore, the reliability of s-out-of-k system increases for large value of θ_1 and θ_2 and small value of $\theta_3$in all cases.

This article may give a chance to other studies. The following are some suggestions that might be considered in future researches:

1- Estimate the reliability of an s-out-of-k system when all the strengths are not identically distributed assuming that each strength follow independent exponentiated Pareto distribution.

2- Estimate the reliability of an s-out-of-k system when the strengths are dependent and identically distributed random variables follow multivariate Pareto distribution.

Table 1: Results of simulation study of absolute bias and MSE of estimates of reliability with non-identical components for θ_1=2 ,θ_2=1.5 ,θ_3=0.5 , λ=3 , a=1 and 5000 replications

$(s_1, s_2, k_1, k$	True $R_{(s,k)}$	(m_1, m_2, n)	Method of estimations						
			MLE	MME	PCE	LSE	WLSE	BESL	BELL
(1,1,2,2)	0.813	(10,10,10)	0.01519	0.03490	0.01792	0.03286	0.03129	0.02982	0.03217
			0.00488	0.02246	0.00572	0.02063	0.01912	0.00573	0.00604
		(10,10,30)	0.00529	0.04179	0.02100	0.00739	0.00859	0.02156	0.02295
			0.00224	0.01376	0.00341	0.00310	0.00297	0.00287	0.00299
		(10,10,50)	0.00252	0.04001	0.02316	0.00409	0.00469	0.01906	0.02024
			0.00189	0.01124	0.00317	0.00246	0.00239	0.00242	0.00251
		(30,30,10)	0.01372	0.01205	0.00171	0.02635	0.02208	0.01681	0.01836
			0.00367	0.01668	0.00367	0.01550	0.01308	0.00371	0.00385
		(30,30,30)	0.00486	0.01655	0.00581	0.00651	0.00601	0.00957	0.01026
			0.00136	0.00815	0.00169	0.00197	0.00182	0.00145	0.00148
		(30,30,50)	0.00346	0.01649	0.00892	0.00566	0.00531	0.00850	0.00901
			0.00099	0.00591	0.00137	0.00134	0.00122	0.00108	0.00110
		(50,50,10)	0.01438	0.00678	0.00199	0.03045	0.02501	0.01522	0.01667
			0.00362	0.01533	0.00336	0.01918	0.01590	0.00355	0.00367
		(50,50,30)	0.00451	0.00868	0.00104	0.00458	0.00418	0.00699	0.00757
			0.00119	0.00728	0.00137	0.00172	0.00158	0.00123	0.00125
		(50,50,50)	0.00284	0.00996	0.00377	0.00362	0.00336	0.00565	0.00605
			0.00081	0.00483	0.00103	0.00108	0.00098	0.00085	0.00086
(1,2,2,2)	0.632	(10,10,10)	0.01383	0.01882	0.01669	0.02493	0.02394	0.02642	0.03293
			0.01245	0.04181	0.01432	0.02471	0.02320	0.01203	0.01280
		(10,10,30)	0.00126	0.03982	0.02461	0.00132	0.00418	0.01927	0.02359
			0.00744	0.02978	0.00976	0.01043	0.00980	0.00754	0.00791
		(10,10,50)	0.00267	0.04019	0.02907	0.00325	0.00235	0.01643	0.02030
			0.00667	0.02607	0.00908	0.00971	0.00933	0.00675	0.00704
		(30,30,10)	0.01633	0.00662	0.00373	0.02376	0.01952	0.01550	0.01978
			0.00916	0.03398	0.00982	0.01801	0.01606	0.00855	0.00890
		(30,30,30)	0.00491	0.01211	0.00582	0.00630	0.00585	0.00920	0.01131
			0.00412	0.01916	0.00509	0.00551	0.00510	0.00409	0.00418
		(30,30,50)	0.00291	0.01454	0.01081	0.00572	0.00549	0.00827	0.00993
			0.00322	0.01493	0.00417	0.00429	0.00392	0.00323	0.00330
		(50,50,10)	0.01776	0.01272	0.00882	0.02693	0.02173	0.01421	0.01809
			0.00873	0.03189	0.00895	0.01951	0.01713	0.00806	0.00836
		(50,50,30)	0.00510	0.00108	0.00089	0.00400	0.00370	0.00664	0.00834
			0.00348	0.01732	0.00413	0.00463	0.00426	0.00342	0.00348
		(50,50,50)	0.00290	0.00634	0.00408	0.00331	0.00318	0.00548	0.00675
			0.00253	0.01259	0.00319	0.00329	0.00301	0.00251	0.00255
(2,1,2,2)	0.683	(10,10,10)	0.01823	0.03108	0.02144	0.03046	0.02989	0.03427	0.03956
			0.01096	0.03759	0.01267	0.02361	0.02214	0.01132	0.01209
		(10,10,30)	0.00574	0.04844	0.02828	0.00826	0.00983	0.02616	0.02954
			0.00566	0.02601	0.00798	0.00769	0.00736	0.00631	0.00665
		(10,10,50)	0.00182	0.04804	0.03120	0.00342	0.00511	0.02310	0.02606
			0.00490	0.02106	0.00746	0.00635	0.00610	0.00548	0.00574
		(30,30,10)	0.01666	0.00231	0.00155	0.02595	0.02156	0.01818	0.02171
			0.00771	0.02970	0.00807	0.01756	0.01540	0.00737	0.00771
		(30,30,30)	0.00553	0.01583	0.00642	0.00747	0.00686	0.01100	0.01266
			0.00326	0.01610	0.00402	0.00443	0.00407	0.00332	0.00340
		(30,30,50)	0.00402	0.01828	0.01208	0.00683	0.00641	0.01030	0.01158
			0.00251	0.01245	0.00340	0.00330	0.00302	0.00260	0.00265
		(50,50,10)	0.01810	0.00367	0.00671	0.02965	0.02410	0.01663	0.01987
			0.00763	0.02812	0.00750	0.01989	0.01718	0.00717	0.00746
		(50,50,30)	0.00526	0.00601	0.00043	0.00512	0.00457	0.00774	0.00911
			0.00275	0.01430	0.00323	0.00373	0.00343	0.00274	0.00279
		(50,50,50)	0.00309	0.01020	0.00404	0.00426	0.00390	0.00638	0.00737
			0.00197	0.01013	0.00248	0.00260	0.00236	0.00199	0.00202

Continued Table 1

$(s_1, s_2, k_1, k$	True $R_{(s,k)}$	(m_1, m_2, n)	Method of estimations						
			MLE	MME	PCE	LSE	WLSE	BESL	BELL
(2,2,2,2)	0.572	(10,10,10)	0.01903	0.02529	0.02251	0.02957	0.02887	0.03579	0.04257
			0.01346	0.04439	0.01554	0.02324	0.02195	0.01332	0.01413
		(10,10,30)	0.00599	0.05301	0.03269	0.00887	0.01121	0.02991	0.03404
			0.00703	0.03067	0.00971	0.00927	0.00885	0.00768	0.00804
		(10,10,50)	0.00160	0.05420	0.03715	0.00409	0.00522	0.02698	0.03055
			0.00608	0.02573	0.00893	0.00786	0.00759	0.00668	0.00696
		(30,30,10)	0.01814	0.00816	0.00397	0.02433	0.02001	0.01764	0.02252
			0.01029	0.03775	0.01110	0.01816	0.01640	0.00956	0.00994
		(30,30,30)	0.00645	0.01570	0.00757	0.00843	0.00778	0.01239	0.01459
			0.00430	0.02031	0.00533	0.00572	0.00528	0.00431	0.00441
		(30,30,50)	0.00462	0.01976	0.01394	0.00805	0.00762	0.01190	0.01355
			0.00322	0.01553	0.00427	0.00426	0.00389	0.00329	0.00336
		(50,50,10)	0.01935	0.01598	0.01034	0.02668	0.02141	0.01525	0.01981
			0.01015	0.03634	0.01049	0.01951	0.01743	0.00932	0.00965
		(50,50,30)	0.00603	0.00242	0.00070	0.00521	0.00473	0.00831	0.01018
			0.00378	0.01887	0.00448	0.00501	0.00461	0.00372	0.00379
		(50,50,50)	0.00373	0.00937	0.00504	0.00468	0.00438	0.00734	0.00865
			0.00263	0.01333	0.00333	0.00343	0.00313	0.00264	0.00267

Note: The first entry is the simulated about absolute biases.
The second entry is the simulated about MSE's.

Table 2: Results of simulation study of absolute bias and MSE of estimates of reliability with non-identical components for θ_1=0.5 ,θ_2=1.5 ,θ_3=2 , λ=3 , α=1 and 5000 replications

$(s_1, s_2, k_1, k$	True $R_{(s,k)}$	(m_1, m_2, n)	Method of estimations						
			MLE	MME	PCE	LSE	WLSE	BESL	BELL
(1,1,2,2)	0.267	(10,10,10)	0.00290	0.00372	0.00334	0.00677	0.00623	0.00709	0.00234
			0.00873	0.02630	0.00982	0.01387	0.01300	0.00797	0.00767
		(10,10,30)	0.00881	0.02085	0.01056	0.01305	0.01026	0.00218	0.00045
			0.00524	0.01827	0.00568	0.00897	0.00831	0.00471	0.00460
		(10,10,50)	0.00954	0.02436	0.01661	0.01247	0.00958	0.00068	0.00155
			0.00455	0.01661	0.00496	0.00806	0.00743	0.00406	0.00399
		(30,30,10)	0.00302	0.02019	0.01709	0.00396	0.00197	0.00913	0.00542
			0.00649	0.02101	0.00830	0.00917	0.00863	0.00634	0.00614
		(30,30,30)	0.00203	0.00139	0.00195	0.00280	0.00258	0.00347	0.00192
			0.00302	0.01178	0.00368	0.00400	0.00369	0.00294	0.00290
		(30,30,50)	0.00266	0.00306	0.00398	0.00166	0.00146	0.00188	0.00074
			0.00217	0.00955	0.00264	0.00289	0.00262	0.00210	0.00208
		(50,50,10)	0.00523	0.02555	0.02202	0.00567	0.00331	0.00854	0.00506
			0.00632	0.02070	0.00833	0.00900	0.00859	0.00623	0.00605
		(50,50,30)	0.00011	0.00511	0.00741	0.00153	0.00155	0.00321	0.00189
			0.00254	0.01007	0.00333	0.00332	0.00305	0.00250	0.00247
		(50,50,50)	0.00136	0.00026	0.00144	0.00156	0.00146	0.00223	0.00131
			0.00178	0.00780	0.00221	0.00235	0.00215	0.00175	0.00173
(1,2,2,2)	0.146	(10,10,10)	0.00830	0.01357	0.00942	0.01399	0.01272	0.01708	0.01366
			0.00602	0.01743	0.00684	0.00916	0.00858	0.00574	0.00536
		(10,10,30)	0.01051	0.00598	0.00499	0.01473	0.01182	0.00771	0.00600
			0.00361	0.01059	0.00367	0.00610	0.00556	0.00318	0.00306
		(10,10,50)	0.01057	0.01024	0.01055	0.01351	0.01055	0.00541	0.00400
			0.00311	0.00930	0.00307	0.00538	0.00490	0.00268	0.00260
		(30,30,10)	0.00180	0.03024	0.01938	0.00365	0.00478	0.01669	0.01387
			0.00437	0.01521	0.00622	0.00565	0.00539	0.00465	0.00439
		(30,30,30)	0.00336	0.00824	0.00367	0.00473	0.00433	0.00646	0.00540
			0.00204	0.00734	0.00251	0.00271	0.00248	0.00202	0.00197

Continued Table 2

$(s_1, s_2, k_1, k$	True $R_{(s,k)}$	(m_1, m_2, n)	Method of estimations						
			MLE	MME	PCE	LSE	WLSE	BESL	BELL
(1,2,2,2)	0.146	(30,30,50)	0.00324	0.00294	0.00202	0.00297	0.00261	0.00393	0.00317
			0.00146	0.00572	0.00174	0.00193	0.00174	0.00141	0.00139
		(50,50,10)	0.00027	0.03486	0.02404	0.00232	0.00391	0.01635	0.01367
			0.00429	0.01580	0.00645	0.00552	0.00540	0.00465	0.00441
		(50,50,30)	0.00191	0.01083	0.00850	0.00355	0.00340	0.00622	0.00530
			0.00172	0.00655	0.00237	0.00227	0.00208	0.00174	0.00171
		(50,50,50)	0.00244	0.00490	0.00282	0.00289	0.00270	0.00433	0.00370
			0.00124	0.00480	0.00156	0.00161	0.00148	0.00123	0.00121
(2,1,2,2)	0.061	(10,10,10)	0.00993	0.02825	0.01123	0.02079	0.01927	0.02077	0.01946
			0.00240	0.01257	0.00278	0.00858	0.00781	0.00293	0.00275
		(10,10,30)	0.00980	0.01416	0.00220	0.02016	0.01782	0.01273	0.01209
			0.00159	0.00780	0.00142	0.00777	0.00703	0.00162	0.00156
		(10,10,50)	0.00961	0.01129	0.00086	0.01926	0.01689	0.01099	0.01046
			0.00140	0.00679	0.00115	0.00753	0.00678	0.00137	0.00132
		(30,30,10)	0.00436	0.02892	0.01431	0.00598	0.00638	0.01579	0.01486
			0.00136	0.00834	0.00217	0.00173	0.00165	0.00181	0.00172
		(30,30,30)	0.00387	0.01357	0.00453	0.00511	0.00471	0.00744	0.00710
			0.00066	0.00415	0.00083	0.00090	0.00082	0.00073	0.00071
		(30,30,50)	0.00351	0.00998	0.00132	0.00382	0.00346	0.00556	0.00531
			0.00049	0.00336	0.00056	0.00066	0.00059	0.00051	0.00051
		(50,50,10)	0.00291	0.02936	0.01580	0.00478	0.00539	0.01435	0.01350
			0.00121	0.00739	0.00211	0.00156	0.00153	0.00163	0.00155
		(50,50,30)	0.00230	0.01218	0.00598	0.00363	0.00338	0.00594	0.00566
			0.00050	0.00300	0.00074	0.00069	0.00063	0.00056	0.00055
		(50,50,50)	0.00223	0.00854	0.00267	0.00292	0.00267	0.00436	0.00417
			0.00036	0.00234	0.00046	0.00049	0.00044	0.00038	0.00038
(2,2,2,2)	0.041	(10,10,10)	0.00838	0.02205	0.00948	0.01428	0.01338	0.01749	0.01661
			0.00150	0.00752	0.00176	0.00346	0.00331	0.00188	0.00178
		(10,10,30)	0.00747	0.00848	0.00109	0.01245	0.01092	0.00907	0.00871
			0.00089	0.00386	0.00078	0.00273	0.00261	0.00088	0.00085
		(10,10,50)	0.00712	0.00568	0.00163	0.01146	0.00993	0.00729	0.00701
			0.00075	0.00315	0.00059	0.00254	0.00242	0.00070	0.00068
		(30,30,10)	0.00435	0.02557	0.01297	0.00607	0.00635	0.01500	0.01431
			0.00093	0.00564	0.00159	0.00116	0.00112	0.00133	0.00127
		(30,30,30)	0.00320	0.01054	0.00373	0.00428	0.00394	0.00621	0.00600
			0.00041	0.00226	0.00052	0.00056	0.00051	0.00046	0.00045
		(30,30,50)	0.00270	0.00706	0.00080	0.00295	0.00266	0.00426	0.00412
			0.00029	0.00172	0.00033	0.00039	0.00035	0.00030	0.00030
		(50,50,10)	0.00342	0.02712	0.01471	0.00531	0.00582	0.01429	0.01363
			0.00086	0.00549	0.00162	0.00108	0.00108	0.00126	0.00120
		(50,50,30)	0.00213	0.01030	0.00533	0.00327	0.00305	0.00539	0.00521
			0.00033	0.00180	0.00050	0.00045	0.00041	0.00038	0.00037
		(50,50,50)	0.00193	0.00667	0.00233	0.00248	0.00227	0.00374	0.00362
			0.00023	0.00128	0.00029	0.00031	0.00028	0.00024	0.00024

Note: The first entry is the simulated about absolute biases.
The second entry is the simulated about MSE's.

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APPENDIEX A

$\tau_{11} = \frac{\theta_1^2}{m_1-1}$.(15)

$\tau_{22} = \frac{\theta_2^2}{m_2-1}$.(16)

$\tau_{33} = \frac{\theta_3^2}{n-1}$.(17)

$Q_{111} = \frac{2(m_1-1)}{\theta_1^3}$.(18)

$Q_{222} = \frac{2(m_2-1)}{\theta_2^3}$.(19)

$Q_{333} = \frac{2(n-1)}{\theta_3^3}$.(20)

$$\left.\begin{aligned}
U_1 &= \frac{\partial R_{(s,k)}}{\partial \theta_1} = \theta_3 \sum_{j_1}^{k_1}\binom{k_1}{j_1}\sum_{j_2}^{k_2}\binom{k_2}{j_2}\sum_{m=0}^{j_1}\binom{j_1}{m}(-1)^m \sum_{n=0}^{j_2}\binom{j_2}{n} \times (-1)^n \frac{-(m+k_1-j_1)}{[\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)+\theta_3]^2},\\
U_2 &= \frac{\partial R_{(s,k)}}{\partial \theta_2} = \theta_3 \sum_{j_1}^{k_1}\binom{k_1}{j_1}\sum_{j_2}^{k_2}\binom{k_2}{j_2}\sum_{m=0}^{j_1}\binom{j_1}{m}(-1)^m \sum_{n=0}^{j_2}\binom{j_2}{n} \times (-1)^n \frac{-(n+k_2-j_2)}{[\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)+\theta_3]^2},\\
U_3 &= \frac{\partial R_{(s,k)}}{\partial \theta_3} = \sum_{j_1}^{k_1}\binom{k_1}{j_1}\sum_{j_2}^{k_2}\binom{k_2}{j_2}\sum_{m=0}^{j_1}\binom{j_1}{m}(-1)^m \sum_{n=0}^{j_2}\binom{j_2}{n}(-1)^n \times \frac{\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)}{[\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)+\theta_3]^2},\\
U_{11} &= \frac{\partial^2 R_{(s,k)}}{\partial \theta_1^2} = \theta_3 \sum_{j_1}^{k_1}\binom{k_1}{j_1}\sum_{j_2}^{k_2}\binom{k_2}{j_2}\sum_{m=0}^{j_1}\binom{j_1}{m}(-1)^m \sum_{n=0}^{j_2}\binom{j_2}{n} \times (-1)^n \frac{2(m+k_1-j_1)^2}{[\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)+\theta_3]^3},\\
U_{22} &= \frac{\partial^2 R_{(s,k)}}{\partial \theta_2^2} = \theta_3 \sum_{j_1}^{k_1}\binom{k_1}{j_1}\sum_{j_2}^{k_2}\binom{k_2}{j_2}\sum_{m=0}^{j_1}\binom{j_1}{m}(-1)^m \sum_{n=0}^{j_2}\binom{j_2}{n} \times (-1)^n \frac{2(n+k_2-j_2)^2}{[\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)+\theta_3]^3},\\
U_{33} &= \frac{\partial^2 R_{(s,k)}}{\partial \theta_3^2} = \sum_{j_1}^{k_1}\binom{k_1}{j_1}\sum_{j_2}^{k_2}\binom{k_2}{j_2}\sum_{m=0}^{j_1}\binom{j_1}{m}(-1)^m \sum_{n=0}^{j_2}\binom{j_2}{n}(-1)^n \times \frac{-2[\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)]}{[\theta_1(m+k_1-j_1)+\theta_2(n+k_2-j_2)+\theta_3]^3},
\end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned}
W_1 &= -a\left(\frac{\partial R_{(s,k)}}{\partial \theta_1}\right)e^{-aR_{(s,k)}},\\
W_2 &= -a\left(\frac{\partial R_{(s,k)}}{\partial \theta_2}\right)e^{-aR_{(s,k)}},\\
W_3 &= -a\left(\frac{\partial R_{(s,k)}}{\partial \theta_3}\right)e^{-aR_{(s,k)}},\\
W_{11} &= a\,e^{-aR_{(s,k)}}\left[a\left(\frac{\partial R_{(s,k)}}{\partial \theta_1}\right)^2 - \frac{\partial^2 R_{(s,k)}}{\partial \theta_1^2}\right],\\
W_{22} &= a\,e^{-aR_{(s,k)}}\left[a\left(\frac{\partial R_{(s,k)}}{\partial \theta_2}\right)^2 - \frac{\partial^2 R_{(s,k)}}{\partial \theta_2^2}\right],\\
W_{33} &= a\,e^{-aR_{(s,k)}}\left[a\left(\frac{\partial R_{(s,k)}}{\partial \theta_3}\right)^2 - \frac{\partial^2 R_{(s,k)}}{\partial \theta_3^2}\right].
\end{aligned}\right\} \quad (22)$$

where, $\frac{\partial R_{(s,k)}}{\partial \theta_1}$, $\frac{\partial R_{(s,k)}}{\partial \theta_2}$, $\frac{\partial R_{(s,k)}}{\partial \theta_3}$, $\frac{\partial^2 R_{(s,k)}}{\partial \theta_1^2}$, $\frac{\partial^2 R_{(s,k)}}{\partial \theta_2^2}$ and $\frac{\partial^2 R_{(s,k)}}{\partial \theta_3^2}$ are given, respectively, in equations (21).