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Evaluation of the Stochastic Modelling on Options

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Abstract

Modern option pricing techniques are often considered among the most mathematically complex of all applied areas of financial engineering. In particular these techniques derive their impetus from four milestones of option pricing models: Bachelier model, Samuelson model, Black-Scholes-Merton model and Levy model. In this paper we evaluate all related option pricing models based on these milestones, by comparing the corresponding stochastic differential equations and option pricing formulas. In addition we also include some simulations to make the comparisons more transparent.

Key Words: European option, Pricing model, Brownian motion, Levy processes, Variance Gamma process

I. INTRODUCTION

Financial Engineering is a cross-disciplinary field which relies on financial mathematics, numerical methods and computer simulations to make trading, hedging and investment decisions. With the rapid development of financial derivatives like options, futures and swaps, it has aroused more and more interest especially the topic on pricing models. Modern option pricing techniques, with roots in stochastic calculus, are often considered among the most mathematically complex of all applied areas of financial engineering. In particular these modern techniques derive their impetus from four milestone of option pricing models.

Dating back to 1900, Bachelier first proposed to use the Brownian motion to model the dynamics of stock price in his Ph.D dissertation, and he also derived a closed formula for pricing the standard European options. He was considered to be a pioneer in the study of financial engineering and financial mathematics, although the process he used can generate shares that allowed both negative security prices and option prices that exceeded the price of the underlying asset. In 1959, Osborne refined Bachelier model by using the stochastic exponential of the Brownian motion to model stock price [1]. Later, Samuelson [2] extended to study the option pricing using the geometric Brownian motion in a more systematic manner, and he came with the idea to use the discount rate in pricing. Due to his great contributions, Samuelson was awarded the Nobel Memorial Prize in Economic Sciences in 1970.

A breakthrough was made by Black, Scholes and Merton in 1973, that the option price is explicitly connected to a hedging strategy which depends on the volatility of the stock price as well as other observable quantities. The most significant idea of Black-Scholes-Merton (BS) model was to realize that the expected return of the option price should be the risk-free rate and that by holding a certain amount of stock, now referred to as the delta; the option position could be dynamically completely hedged. Later, Scholes and Merton received the Nobel Prize in 1997 for their key discovery.

Shortly after the invention of BS model, attempts have been made to relax the stringent assumptions and to improve the performance of the model. Merton proposed a jump diffusion model [3], under which the log return has both a diffusion component and a jump component. And this discovery made him conclude that the exponential Brownian motion should be replaced by symmetric α -stable Levy process to better fit market data. This model is often called Merton Jump diffusion (MJD) model. He also claimed that the log price process is a superposition of a Brownian motion and an independent compound Poisson process with lognormally distributed jumps.

Empirical studies reveal that the stock price distribution usually has some properties contradicting to the traditional Black-Scholes assumptions, such as "fat tail", "self-similarity", "long-range dependence" see [4]. Mandelbrot and Taylor [5, 6] proposed that the stock market should take on the character of fractional Brownian motion, then Peters [7] introduced Fractional Brownian motion to model the dynamics of stock price. After this, many scholars have made outstanding contributions on this topic. In 2000, Duncan et al. [8], Hu and Oksendal [9] developed the fractional Black-Scholes formula by a stochastic integration with the fractional Brownian motion and Wick-product at initial time. In

2008, Ciprian Necula [10] obtained an explicit fractional Black-Scholes formula for the price of an option by using Fourier transform. During the past decades, much attention has been drawn to the constant volatility model. It was shown in [11, 12] that stock returns were heteroscedastic with considerable evidences. The constant elasticity volatility process was first proposed by Cox and Ross [13, 14]. And they applied this diffusion process to option pricing. Besides the implied volatility smile, it is discovered that when the price of the underlying asset decreases, the smile shifts to lower prices; when the price increases, the smile shifts to higher prices. In order to have a coherent estimate of volatility risk, the SABR model was developed in [15]. More importantly, the SABR model captures the correct dynamics of the smile, and thus yields stable hedges. In [16], Heston also introduced a stochastic volatility model and derived a closed-form formula for a European call option. This model is called the 2-dimensional BS model. It is critical to correctly handle market smiles and skewness for hedging. In 1996, Chen developed a 3-dimensional BS model to describe the evolution of interest rate [17]. Comparing with Heston model, it can be viewed as a "three-factor model" since it describes interest rate as driven by three sources of market risk, stochastic mean and stochastic volatility.

Considering Levy models with jumps, many improvements have been made based on MJD model. Bates found that stochastic volatility alone could not explain the "volatility smile" well. He further extends the Heston model to combine stochastic volatility and jump diffusion process (SVJD) under systematic jumps and volatility risk [18]. In 2002, Kou [19] proposed a double exponential jump diffusion model, which can capture the leptokurtic and fat tail properties better than MJD model. Madan and Seneta proposed Variance Gamma (VG) model in 1990, in which the log return of stock follows a variance gamma process [20]. And the asymmetric VG model was studied in [21]. Then the Normal Inverse Gaussian (NIG) model was proposed by Barndorff-Nielsen [22] in 1998. Later Eberlein and Prause [23] extended NIG model to the generalized hyperbolic class. In 2002, Carr, Geman, Madan and Yor developed CGMY model based on VG model [24].

This paper mainly gives an overview and evaluations of the development of major option pricing models, as well as their improvements and extensions. Comparing to other survey papers, Sundaresan [25] mainly concerned with derivative pricing theory, term structure theory and asset pricing but without pricing models; In [26], Hobson focused on stochastic methods of derivative pricing and descriptions of some barrier options. While in this paper, we not only review the major theory of option pricing but also give a detailed evaluation along the development of European option pricing models, which can be of great help to researchers on option pricing.

The paper is organized as follows. In section 2, we begin with the review and evaluation of the four important milestone models on European options: the Bachelier Model, which is the prototype of option pricing theory; the Samuelson-Osborne Model, which can be seen as the mid-step to BS model; then the Black-Scholes-Merton (BS) model, followed by the MJD model. The extensions of BS model are listed in Section 3. Besides from the model driven by the Fractional Brownian motion, we also evaluated CEV, SABR, Heston and Chen model. The Kou model, SVJD model as well as the VG, CGMY and NIG model are outlined in section 4 as they have greatly improved the MJD model. We also include some simulations on comparison of BS model with Bachelier Model as well as Heston model.

II. MILESTONES OF OPTION PRICING MODELS

We first briefly review the concept of option: A call (or put) option is a contract that gives an investor the right but not the obligation to buy (or sell) a stock or other underlying assets at a specified price (the strike price) within a certain time (the maturity). In this paper we mainly consider the European options, which can only be exercised at its maturity, and for simplicity we assume the underlying assets are stocks. For a European call option with the strike price K and the maturity T, if the price of its underlying asset is S(T), then the payoff of this option is given by

$$C(S(T),T)=(S(T)-K)^{+} \qquad (1)$$

Consider the stock price process {S(t): $t\geq 0$}, equipped with the increasing sequence of σ-algebra $\{F_t\}$ that contains all the historical information at time t. Besides its real world expectation E[S(t)], it is also useful to consider the risk-neutral expectation $E^R[S(t)]$ under the risk-neutral probability $\mathbf{P}^R$, which is absolutely continuous with respect to **P**, the original probability. Then the price of a European call option at time t can be expressed as the expectation of discounted payoffs at maturity T under certain measure:

$$C(S(t),t)=\frac{1}{D(t)}E[D(T)(S(T)-K)^{+}\mid F_t] \qquad (2)$$

where $E(\cdot)$ denotes either the real world or risk-neutral expectation in different models and D(t) is a discount process. From now on, we assume t=0 for simplicity.

Since the price of put option can be derived from the put-call parity [27], we only consider call options in this paper. Next we list all general assumptions for the following models:

A1) The option is a European call option with strike price K and maturity time T;

A2) The stock price evolves as a continuous Markov process, homogeneous both in time and space;

A3) There are no transaction costs in buying or selling stocks, options, i.e. the market is frictionless;
A4) No dividends are paid on the underlying stock during the option life;
A5) The market does not admit arbitrage.

Note that the assumption **A3-A4)** can be dropped easily, by replacing the stock price by the effective stock price in all the models below.

2.1 Bachelier model

In [28] Bachelier first proposed a mathematic model to price stock price, assuming that the discount rate is zero and the dynamics of stock price satisfies the following SDE:

$$dS(t) = S(0)\sigma dB(t) \qquad (3)$$

where S(t) denotes the spot price of the underlying security at time t; B(t) is the standard Brownian motion and σ is the volatility of stock price. Considering a European call option defined as above, its option at initial time satisfies:

$$C(S(0),0) = E[(S(T)-K)^+],$$

where $E(\cdot)$ denotes the expectation with respect to the real world. Although the solution was not precisely derived in his early work, based on (2) and (3) one can obtain:

$$C(S(0),0) = (S(0)-K)F_N(d) + S(0)\sigma\sqrt{T}\psi_N(d),$$

where $d = \dfrac{S(0)-K}{S(T)\sigma\sqrt{T}}$, $F_N(\cdot)$ and $\psi_N(\cdot)$ denote the cumulative distribution function and the probability density function of standard normal variable, respectively.

Bachelier proved that the one dimensional distribution of this stock process satisfies the relation known as the Chapman-Kolmogorov equation and the Gaussian property with linearly increasing variance [29]. But the hypothesis of absolute Brownian motion (3) leads to negative stock price with positive probability. And the fact that it ignores any discounting contradicts the reality. Despite of these restrictions, the Bachelier model is widely celebrated as a landmark in the history of pricing theory. It has great influence on the whole development of stochastic calculus and financial engineering.

2.2 Samuelson/Osborne model

In 1959, Osborne refined Bachelier model by using the stochastic exponential of Brownian motion to model the stock price process [1]. He justified this approach in psychological terms based on the Weber-Fechner law, which states that humans perceive the intensity of stimuli on a log scale rather than a linear scale. Based on this argument, Osborne inferred that the logreturn $\log(S(t+\tau)/S(t))$ should follow a normal distribution with mean zero and variance $\sigma^2\tau$, for any small $\tau > 0$.

Later Samuelson applied this process to price options in a more systematic way [2]. Besides the general hypotheses, he assumed that the option derived from a stock has a constant expected return rate α. And it is greater than that of the underlying security. The evolution of stock price in Samuelson model satisfies the following SDE:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t) \qquad (4)$$

where μ denotes the expected return rate of S(t), σ is same as defined above. Again Samuelson did not give an explicit solution for the option price, but we can derive it from (2) and (4):

$$\mathrm{C(S(0),0)} = \mathrm{e}^{-\alpha T}E[(S(T)-K)^+] = S(0)\mathrm{e}^{(\mu-\alpha)T}F_N(d_1) - K\mathrm{e}^{-\alpha T}F_N(d_2)$$

where $d_1 = [\log(S(T)/K) + (\mu + \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}$, $d_2 = d_1 - \sigma\sqrt{T}$ and α, μ being the expected return rate of option value and stock price, respectively.

Although this pricing formula is very close to Black-Scholes formula, it did not receive much attention, mainly because the two expected return values are both unknown quantities from the market. As they depend heavily upon the unique risk characteristics of the underlying stock and the option, different investors might propose different values in terms of their own level of risk aversion. This uncertainty makes Samuelson model difficult for buyers and sellers with different risk aversions to agree upon a uniform price. On the other hand, it may give certain investors a more accurate forecasting based on personal keen insight into the market.

2.3 Black-Scholes-Merton (BS) model

In 1973, Black and Scholes derived the Black-Scholes formula, which posed a momentous significance in the history of option pricing theory. The outstanding feature of this model is introducing the risk neutral measure and claiming that the discounted value of derivative security under this measure is indeed a martingale [30].

The model assumes that the stock price follows the SDE (4) defined in the Samuelson model. To price a European call option, they further derived that its price process C(S(t), t) satisfies the following PDE:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0.$$

Thus one obtains the so-called Black-Scholes formula

$$C_{BS}(S(0), 0) = S(0)F_N(d_1) - Ke^{-rT}F_N(d_2) \quad (5)$$

Where $d_1 = [\log(S(0)/K) + (r + \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}$, $d_2 = d_1 - \sigma\sqrt{T}$ ·

Note that there is only one unknown parameter in the above formula: the volatility with all other parameters observable from the market. Besides this, from the derivation of this formula [30], BS model has a key advantage of posing an explicit hedging strategy for replication. These make Black-Scholes formula very easy to use and rather attractive. However, empirical studies find that the market price is generally higher than computed from the formula. One of the possibilities is that the existence of risk neutral measure needs efficient market hypotheses, which is definitely not so in reality. The assumption of frictionless market, such as no costs or fees for transactions, continuous time trading, may also lead to liquidity risk and gap risk. Apart from that, volatility surface from market data implies that the volatility is not constant. One of the most serious limitations is the underestimation of extreme moves. It is found that the return distribution has higher peak and heavier tails comparing with normal distribution, which disagree with the "ideal conditions". In general, this formula is popular with practioners for its simplicity and easeness to use.

To better understand the advantage of Black-Scholes formula, we now compare the Bachelier model and BS model in figure 1. Let the parameters be (S(0), K, σ , T, r)=(10, 11, 0.4, 2, 0). We simulate the paths of S(t) under the Bachelier model and BS model, respectively. The blue path denotes the path in the Bachelier model, whereas the red one refers to that in BS model. As shown in figure 1, the paths obtained from these two models fit very well at the beginning of the period. However, the difference is enlarged when the time interval gets longer. This implies that the Bachelier model can be seen as a linearization of BS model with short maturity and vanishing drift term.

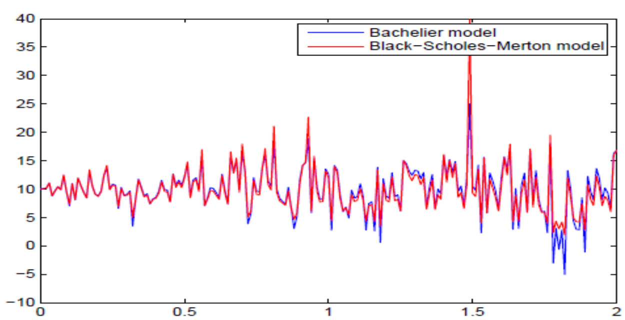

Figure 1: Simulated stock paths under Bachelier and BS model with $T = 2$

2.4 Levy model---MJD model

Levy Process is defined as a continuous-time stochastic process {X=X(t): t$\geq$0} starts at the origin and possesses the properties of independent increments, stationary increments and stochastic continuity. Besides Markovian and stationary distribution properties assumption on stock price, it is also important to model the situation when S(t) has the properties that large jumps for extreme market movements and small jumps and diffusion for instantaneous trading [31].

Note that critical assumptions in the BS model include trading takes place continuously in time and the dynamics of the stock has a continuous sample path with probability one. However, enough evidences show that the existence of jumps in response to certain announcements. In 1976, Merton [3] proposed the following jump diffusion model, where the stock price is the solution of the following SDE:

$$\frac{dS(t)}{S(t)} = (\mu - \lambda\gamma)dt + \sigma dB(t) + d\sum_{i=1}^{N(t)} Y_i.$$

where N(t) is a Poisson process, λ denotes the mean number of jumps per unit time; Y_i is a random variable characterizing the percentage change of the stock price when jump occurs, and $\gamma = EY_i$. The assumption that $Y_i > -1$ is to guarantee the fact that the stock price could not jump from a positive to a negative value or to zero. The stock price at time t can be obtained under the risk-neutral measure:

$$S(t) = S(0)\exp[(r - \overline{\lambda}\overline{\gamma} - \frac{1}{2}\sigma^2)t + \sigma\overline{B}(t)]\prod_{i=1}^{N(t)}(Y_i + 1). \quad (6)$$

Here Y_i+1 are independent and have identically lognormal distributions under the risk-neutral measure, i.e. $\log(Y_i+1)$ follow normal distributions $N(m,\delta^2)$. Thus we can obtain $\bar{\gamma}=E^R Y_i=\exp(m+\frac{\delta^2}{2})-1$ in (6). For a European call option, its price process C(S(t), t) satisfies the following PDE:

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+(r-\bar{\lambda}\bar{\gamma})S\frac{\partial C}{\partial S}+\frac{\partial C}{\partial t}-rC+\bar{\lambda}[E^R C(SY_i,t)-C(S,t)]=0.$$

By conditioning on the number of jumps N(T), we obtain

$$C(S(0),0)=\sum_{n\geq 0}\frac{e^{-\bar{\lambda}T}(\bar{\lambda}T)^n}{n!}C_{BS}(S_n(0),0),$$

where $C_{BS}(S_n(0),0)$ is defined as the Black-Scholes formula (5) with $\sigma_n^2=\sigma^2+n\delta^2/T$ and

$$S_n(0)=S(0)\exp[nm+\frac{n\delta^2}{2}-\bar{\lambda}e^{m+\frac{\delta^2}{2}}T+\bar{\lambda}T].$$

The MJD model has greatly improved BS model by introducing jump process to model the stock price. It can better fit market data especially when there are many outliers or certain announcements. Besides, this model leads to the leptokurtic feature and the implied volatility smile. The hypothesis of lognormal distribution makes the good property of the distribution of the sum of jump random variables and Brownian motion. Thus a closed-form solution can be derived. On the other hand, this model has its own drawback. For example, the hypothesis about jump process is too strict and the fat tail feature is not obvious under this model. We will review some improvements of this model in section 4.

III. Improvements and Extensions on Black-Scholes-Merton Model

As empirical studies find that the price of stock sometimes does not follow the solution of the Black-Scholes formula, many models have been constructed to make related improvements.

3.1 Fractional Brownian Motion model

For 0<H<1, let $\{B_H(t), t\in R\}$ be the fractal Brownian motion, which has zero means and covariance given by:

$$Cov_H(t,s)=E[B_H(t)B_H(s)]=\frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H})$$

Obviously, $B_H(t)$ coincides with the standard Brownian motion B(t) when $H=1/2$, and it has homogeneous increments and continuous paths. Under the fractional Brownian motion framework, the dynamics of stock price can be modeled by the following SDE:

$$\frac{dS(t)}{S(t)}=\mu dt+\sigma dB_H(t).$$

The corresponding fractional Black-Scholes formula was developed in [8, 9]. For a European call option, its price process C(S(t), t) satisfies the following PDE:

$$H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}+\frac{\partial C}{\partial t}-rC=0.$$

Solving the above PDE obtains the pricing formula: $C(S(0),0)=S(0)F_N(d_1)-Ke^{-rT}F_N(d_2)$, where

$$d_1=(\log\frac{S(0)}{K}+rT+\frac{\sigma^2}{2}T^{2H})/(\sigma T^H),\ d_2=d_1-\sigma T^H.$$

Comparing with BS model, this formula is similar to Black-Scholes formula except that there is one extra parameter---Hurst index. If $H=1/2, B_H(t)$ is a standard Brownian motion, then the increments of the process are independent. If $H>1/2$, the increments of the process are positively correlated and the process is persistent. While if $H<1/2$, the increments of the process are negatively correlated and in this case the process is anti-persistent. Of course by using Fractional Brownian motion to model the stock price, one needs to estimate the Hurst index, which indeed constrains the feasibility of this model.

3.2 Constant Elasticity Volatility (CEV) model

The constant volatility hypothesis in BS model often leads to results which are inconsistent with the market data. To improve the discrepancy, the constant elasticity of variance diffusion process was proposed in [13] to model the heteroscedasticity and the leverage effect in returns of common stocks. The CEV model can well capture the heteroscedasticity of the volatility of stock returns. Recently, this model has been further investigated in [19, 32], and the corresponding SDE is:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma S(t)^{\beta-1} dB(t) \qquad (7)$$

where $\beta \geq 0$, is a parameter that controls the relationship between volatility and price. The variance of log(S(t)) is $\upsilon(S,t) = \sigma^2 S(t)^{2\beta-2}$ and thus $\frac{d\upsilon(S,t)/dS}{\upsilon(S,t)/S} = 2\beta - 2$, which implies that the elasticity is $2\beta - 2$.

The equilibrium price of a call price for $\beta < 1$ was investigated in [13], while the case for $\beta > 1$ was done in [33]. In summary, the CEV solution consists of a pair of infinite summations of gamma density and survivor functions:

$$C(S(0),0) = S(0)P_1 - Ke^{-rT}P_2,$$

where $P_i = P_i(S(0),T;\beta,\sigma)$ denotes the probability of the call option expiring in-the-money. Its derivation rests on the risk-neutral pricing theory, see [13, 33] for further information.

Attempts were made in early 1980s, as a three-stage procedure to estimate σ, β was proposed in [33]. However, the results were not satisfactory. A breakthrough was made by using the non-central Chi-squared distribution, which facilitates the computations significantly with suitable statistical software, see [33] for further reference.

If $\beta = 1$, the elasticity is zero, so the stock price is lognormally distributed, which is just the case of BS model; if $\beta < 1$, it is observed from equity markets that the volatility of a stock increases when its price falls. Conversely, for $\beta > 1$, one often observes the so-called inverse leverage effect whereby the volatility of the price of a commodity tends to increase when its price increases. In empirical study, we mainly focus on the case of $\beta < 1$ as the relationship between the stock price and its return volatility usually is negative.

3.3 SABR model

The SABR model, i.e. stochastic alpha, beta, rho model, is purposed to capture the volatility smile in derivative markets. This model has been widely used by practitioners in financial industry. And it was first introduced in [15]. Since SABR model can be used to describe a single forward, a forward swap rate or a forward stock price, we just denote the underlying asset by S(t). The stochastic process S(t) satisfies the SDE (7), but σ also follows a stochastic process:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(t) S(t)^{\beta-1} dB_1(t), \;\; d\sigma(t) = \delta\sigma(t) dB_2(t)$$

where $dB_1(t)B_2(t) = \rho dt$, β, δ and $|\rho| < 1$ are constant. $\sigma(t)$ is a volatility-like parameter, whereas δ is thought to be the volatility of the volatility. This model was introduced to model the discounted stock price process under the risk-neutral measure and thus the drift term μdt vanishes under this transformation [15]. Considering a European call option, the implied volatility is obtained by using series expansion, see [15] for further reference.

To obtain the option price, we can replace the volatility in Black-Scholes formula by the value obtained from this model. Empirical studies found that although the formula is complicated, the approximate solution is quite accurate if parameters are relatively small. Note that if $\delta = 0$ the volatility is no longer stochastic and the SABR model coincides with the CEV model (7). Consequently, we can also replace the volatility in CEV model by the valude derived from SABR model and then obtain the option price. Theoretically, it is difficult to obtain the closed-form solution or efficient estimate, but it is feasible to simulate with computer programming.

The parameter $\beta \in [0,1]$ determines the relationship between the underlying asset pricing and at the money volatility. $\beta \approx 1$ indicates that the investor believes that if the market is to move up or down in an orderly fashion, the at the money volatility level would not be significantly affected; $\beta \approx 0$ indicates that if the market is to move up or down, then at the money volatility would move in the opposite direction.

3.4 Heston model

In [16], Heston introduced a stochastic volatility model and derived a closed-form solution for the price of a European call option. The stock price process {S(t)} satisfies:

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{\sigma(t)} dB_1(t),$$

where $\sigma(t)$ follows a CIR process,

$$d\sigma(t) = \kappa(\bar{\sigma} - \sigma(t))dt + \delta\sqrt{\sigma(t)} dB_2(t),$$

with $dB_1(t)B_2(t) = \rho dt$ and $|\rho|<1, \mu, \kappa, \bar{\sigma}, \delta$ are constant. No arbitrage assumption implies that the option value should depend on $S(t), \sigma(t)$ and t. Heston was able to obtain the following result:

$$C(S(t), \sigma(t), t) = S(t)P_1 - Ke^{-r(T-t)}P_2$$

where $P_j = P_j(S(t), \sigma(t), t)$ represent the probability of the call option expiring in-the-money, which can be obtained from the Fourier Transform.

Comparing with BS model, one attractive feature of this model is the updating structure of its volatility. And based on solving two partial differential equations about $P_j,\ j=1, 2,$ a closed-form solution for option value is derived under this stochastic volatility hypothesis. In Heston model, skewness is generated by the correlation parameter ρ, while kurtosis is generated by the volatility of volatility parameter δ, which can explain the discrepancy between the BS model and market data.

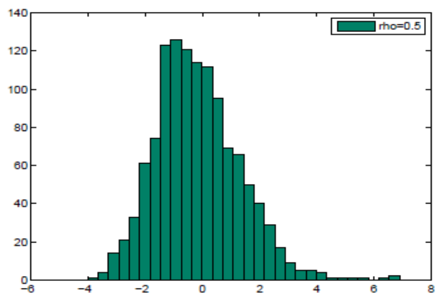

Figure 2: Distribution of log(S(T)/S(0)) in Heston model with $\rho = 0.5$

Since in BS model, log(S(T)/S(0)) follows a normal distribution, which is symmetric about its mean. Here we use MATLAB to simulate the paths of log(S(T)/S(0)) under Heston model. Let $(r, \kappa, \bar{\sigma}, \delta, \rho)$ =(0.04, 0.2, 0.3, 0.2, 0.5) and the initial value of volatility $\sigma(0) = 0.4$. We do the simulation for 1000 times and then plot the histogram of the distribution of log(S(T)/S(0)). As shown in figure 2, the distribution is positively skewed comparing with stand normal distribution. This implies that a positive correlation results in higher variance when the spot asset rises. And therefore, the Heston model explains the fat tail phenomenon observed in empirical studies.

3.5 Chen/Three-factor model

As demonstrated in [34], nearly all the variation in bond returns can be explained by three factors: the level of interest rate, the slope of the yield curve and the curvature of the yield curve. Based on this argument, in 1996, Chen developed a three-factor model describing the evolution of bond, which is now called Chen model. It describes interest rate driven by three sources of market risk---the interest rate, the stochastic mean and stochastic volatility. In [17, 35], the dynamics of the instantaneous interest rate are proposed to follow the SDEs:

$$dr(t) = \mu(\theta(t) - r(t))dt + \sqrt{\sigma(t)}dB_1(t), \tag{8}$$

$$d\theta(t) = \zeta(\bar{\theta} - \theta(t))dt + \eta\sqrt{\theta(t)}dB_2(t), \tag{9}$$

$$d\sigma(t) = \kappa(\bar{\sigma} - \sigma(t))dt + \delta\sqrt{\sigma(t)}dB_3(t). \tag{10}$$

Here $B_i(t), i = 1, 2, 3$ are assumed to be independent, and $\mu, \zeta, \bar{\theta}, \eta, \kappa, \bar{\sigma}, \delta$ are all constant. Considering a default free discount bond which promises to pay one unit at time T, the bond price satisfies a certain PDE (see [35] for further reference). Under corresponding boundary conditions, the solutions can be obtained using numerical methods.

In Chen model, also known as the three-factor model, both mean and volatility of the short interest rate are stochastic. This property fits the market data much better and provides additional insights and explanatory powers. Although it is difficult to obtain a closed-form formula, we can obtain the solution by simulation. But how to evaluate the risk prices of the three factors from market data poses a great challenge, which can be an interesting topic of future research in the field of financial engineering.

Note that the interest rate follows a stochastic process in Chen model, thus this model can not be applied to option pricing. Instead, we conjecture the following model by replacing (8) with

$$dS(t) = \mu(\theta(t) - S(t))dt + \sqrt{\sigma(t)}dB_1(t) \tag{11}$$

Considering a European call option on this stock, its price should be obtained by the option pricing formula (2) under risk-neutral measure with $D(t)=e^{-rt}$. It would be very interesting to obtain either a closed-form solution or numerical solutions for the option price in the most general case.

IV. Improvements and Extensions on Levy Models

In this section, we will introduce four models: the SVJD model, which can be seen as the combination of Heston model and MJD model; the Kou model, a jump diffusion model; Variance Gamma (VG) and CGMY model, finally is the Normal Inverse Gaussian (NIG) model.

4.1 SVJD model

In 1996, Bates [18] proposed a model that the evolution of the underlying asset should follow:

$$\frac{dS(t)}{S(t)}=(\mu-\lambda\gamma)dt+\sqrt{\sigma(t)}dB_1(t)+d\sum_{i=1}^{N(t)}Y_i,\ \ d\sigma(t)=\kappa(\overline{\sigma}-\sigma(t))dt+\delta\sqrt{\sigma(t)}dB_2(t).$$

Here it is assumed that $dB_1(t)B_2(t)=\rho dt, \gamma=EY_i,$ N(t) is a Poisson process with intensity of λt, and Y_i is the random percentage of price change conditional on jumps occurring. And $\log(Y_i+1)$ follows the normal distribution same as MJD model. The price of a European call option on this underlying asset was derived:

$$C(S(t),t)=S(t)P_1-Ke^{-r(T-t)}P_2$$

where P_1 and P_2 are defined same as Heston model and can be obtained by Fourier Transform.

If the volatility process degenerates into constant, this model coincides with MJD model. The stochastic volatility property makes this model capture the feature of smile volatility better than MJD model. In addition, if the jump parameters are set to zero, the SVJD model turns into the Heston model. Thus this model outperforms Heston model especially when there exist outliers in the market. In general, more parameters make SVJD model more flexible.

4.2 Kou model

In the MJD model, the hypothesis of normal jump diffusion implies symmetric distribution of $\log(Y_i+1)$, which is not consistent with studies in behavioral finance [19]. As a result, a new model was proposed by Kou to price options by considering the psychology element of investors within the frame of traditional Brownian motion and efficient market hypothesis. The stochastic process of the asset price S(t) satisfies the SDE:

$$\frac{dS(t)}{S(t)}=\mu dt+\sigma dB(t)+d\sum_{i=1}^{N(t)}Y_i\cdot$$

It is assumed that $\log(Y_i+1)$ are independent and identically distributed with asymmetric double exponential distribution. The probability density function is as below:

$$f(y)=p\cdot\eta_1e^{-\eta_1 y}1_{y\geq0}+q\cdot\eta_2e^{-\eta_2 y}1_{y\leq0},$$

where p, q>0, p+q=1 represent the probabilities of upward and downward jumps respectively.

Thus the price of a European call option can be obtained by the formula (2) under the risk-neutral measure with $D(t)=e^{-rt}$. To obtain an explicit form, one needs to understand the distribution of the sum of the double exponential random variables and normal random variables. Kou [19] introduced the Hh function, a special function of mathematical physicals and derived an analytical formula.

Comparing with the MJD model, although both models can explain the following two empirical phenomena: the asymmetric leptokurtic feature and implied volatility smile; Kou model definitely has some good features. It can also be used to price path-dependent options, and embedded into a rational expectations equilibrium framework. However, Kou model also has certain limitations: although the closed-form solution can be obtained, it needs computer programming for its complexity. Most importantly, the jump process makes the market inefficient and thus the hedging strategy is impracticable.

4.3 Variance Gamma/CGMY model

Under the assumptions of BS model, a series of continuous time models have been developed to allow both finite or infinite jumps during finite time interval [21, 24, 36]. The stock price process can be expressed as a subordinated Brownian motion with a time changing process. In VG model, the stock price process S(t) satisfies:

$$\log S(t)=\mu G_t^{\upsilon}+\sigma B(G_t^{\upsilon}),$$

where G_t^{υ} is a Gamma distributed process with mean t and variance υt with υ be a positive constant. The absence of arbitrage implies that for a European call option, its price can be obtained from formula (2) under risk-neutral measure with $D(t) = e^{-rt}$. More precisely, it was shown in [30] that:

$$C(S(0),0) = S(0)e^{(\mu+\sigma)^2 T/2}(1-\upsilon(\mu+\sigma)^2/2)^{T/\upsilon}F_N(d_1) - Ke^{-rt+\mu^2 T/2}(1-\upsilon\mu^2/2)^{T/\upsilon}F_N(d_2)$$

where $d_1 = \frac{\log(S(0)/K)}{\sigma\sqrt{T}} + [\frac{r+(1/\upsilon)\log[(1-\upsilon(\mu+\sigma)^2/2)/(1-\upsilon\mu^2/2)]}{\sigma} + \mu + \sigma]\sqrt{T}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

In 2002, Carr, Geman, Madan and Yor developed CGMY model based on VG model [22]. Let $X(t) = \log S(t)$. Then its characteristic function is:

$$\varphi_{VG}(u, G_t^{\upsilon}) = E[\exp(iuX(t))] = (\frac{1}{1 - i\mu\upsilon u + \sigma^2\upsilon u^2/2})^{\frac{t}{\upsilon}}$$

Considering the infinitely divisibility property of VG process, the process X(t) can be expressed as

$$X(t,\mu,\upsilon,\sigma) = X_p(t,\mu_p,\upsilon_p,\sigma) - X_q(t,\mu_q,\upsilon_q,\sigma),$$

where $\mu_p = (\sqrt{\frac{\mu^2\upsilon^2}{4} + \frac{\sigma^2\upsilon}{2}} + \frac{\mu\upsilon}{2})/\upsilon$, $\mu_q = (\sqrt{\frac{\mu^2\upsilon^2}{4} + \frac{\sigma^2\upsilon}{2}} - \frac{\mu\upsilon}{2})/\upsilon$, $\upsilon_p = \mu_p^2\upsilon$, $\upsilon_q = \mu_q^2\upsilon$. This implies that the Levy density for the VG process is [36]

$$k_{VG}(x) = \begin{cases} \frac{\mu_p^2}{\upsilon_p}\frac{\exp(-\frac{\mu_p}{\upsilon_p}|x|)}{|x|} & for\ \ x>0 \\ \frac{\mu_q^2}{\upsilon_q}\frac{\exp(-\frac{\mu_q}{\upsilon_q}|x|)}{|x|} & for\ \ x<0 \end{cases}.$$

In [24], the authors developed the CGMY model---a transformation of VG model. They mainly change the parameters in the above Levy density:

$$k_{CGMY}(x) = \begin{cases} C\frac{\exp(-G|x|)}{|x|^{1+Y}} & for\ \ x<0 \\ C\frac{\exp(-M|x|)}{|x|^{1+Y}} & for\ \ x>0 \end{cases},$$

where C>0, $G \geq 0$, $M \geq 0$ and Y<2. In CGMY model, the stock price can be given by:

$$S(t) = S(0)\exp[(\mu+\varpi)t + X_{CGMY}(t;C,G,M,Y)],$$

where ϖ is a convexity correction defined by $\exp(-\varpi t) = \varphi_{CGMY}(-i;t,C,G,M,Y)$. The analytical solution in the CGMY framework could not be obtained. However, by Fast Fourier Transformation, we can use numerical methods to simulate the paths of stock price and then evaluate the price of option. Thus the characteristic function plays a key role. Also notice that VG model is a special case of CGMY model when

$$Y = 0, C = \frac{1}{\upsilon}, G = \frac{1}{\upsilon\mu_q}, M = \frac{1}{\upsilon\mu_p}.$$

Comparing with previous models, both VG model and CGMY model can capture the long tailedness for daily returns, although the returns under these models over long periods approach normality. In the MJD model, there are only finite jumps in a finite time interval, while in the VG and CGMY model there can be infinite number of jumps. In MJD model, closed-form solution has been derived but in these models the option price can only be obtained by Fast Fourier Transformation or numerical methods.

4.4 Normal Inverse Gaussian (NIG) model

Empirical studies found that the log return of financial assets can often be fitted well by Normal Inverse Gaussian (NIG) distributions. And this process was first introduced to model the log return of stock prices by Barndorff and Nielsen in [22]. We can relate the NIG process to a time-changing Brownian motion by introducing an independent inverse Gaussian process. Let T_t^{υ} be the first time that a Brownian motion with drift υ reaches the positive level t. And the dynamics of stock price follows:

$$\log S(t) = \mu T_t^{\upsilon} + \sigma B(T_t^{\upsilon}),$$

where μ, σ and B(t) are defined same as above. Define X(t)=log(S(t)), then the characteristic function can be obtained. By Fast Fourier Transform, we can obtain the option price based on (2).

Comparing with the VG model, NIG model differs in the different distribution of the time changing process. It is somewhat difficult to say one model outperforms the other as the good fitting also depends on data. In general, Levy process opens a new way to improve the BS model, because with jump process these models can always capture the long tailedness and heavier peak. However, due to the stationarity of increments of Levy process, the stock price returns for a fixed time horizon should have the same law. It is therefore impossible to incorporate any kind of new market information into the model, which disagrees with the reality.

V. CONCLUSION

Financial Engineering is a multidisciplinary field involving the theory of finance, the methods of engineering, the tools of mathematics and the practice of programming. And pricing models acts like the link between these topics. Beginning from the work of Bachelier, the theory of pricing options and other financial derivatives had been greatly improved. The process of the stock price can be described by the arithmetic Brownian motion in the Bachelier model and Geometric Brownian motion in Samuelson/Orsborne model BS model etc. Recently the Levy process with Jump process such as MJD, SVJD, Kou model, VG, NIG and CGMY model are developed. Apart from that, the Fractional Brownian motion is proposed. On the other hand, the volatility is improved from a constant to a stochastic process in the Heston model, SABR model, Chen model and SVJD model.

In applications, how to choose an appropriate model is a big challenge. Since the key criteria is to compare the errors between the data generated between the model and obtained from the market. A good model should fit market and provide additional insights and explanatory powers. From this point of view, the three-factor model has more advantages. There are evidences in [34] that the three-factor model, which involves the underlying asset process together with the stochastic mean and stochastic volatility processes, can fit bond returns much better.

Although more parameters make a model more flexible, the solution may not have closed-form. Thus one needs simulation to get the solution, and sometimes it is a rather difficult task to estimate these parameters. This requires efficient computer programmings and accurate algorithm methods. Consequently, calibrating carries more and more weight in Financial Engineering, which also poses a challenge problem in numerical computations. Mainly there are three numerical methods to obtain the solution of a SDE model: Fast Fourier Transformation, Finite Differential method and the Monte Carlo. In the future study of Financial engineering, one important question should be related to validate these models with market data, and meanwhile get better control on predicting the financial market.

Note that all the models we review in this paper can be used to calculate the premier of general derivatives with underlying assets that are not restricted to stocks. For simplicity we only consider derivatives with underlying assets to be stocks. In addition, when costs of transactions or dividend of stocks are considered, the models can still be applied by using the effective stock prices. Finally all models listed in the paper are under the assumption of independent increment for the stock price, except the Fractional Brownian motion model. But empirical studies convince us that there exists dependency consecutive periods. So it is another challenging question to set up proper models by considering the dependency of the stock price process.

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