ON RPS-CONNECTED SPACES

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ABSTRACT

The authors introduced rps-closed sets and rps-open sets in topological spaces and established their relationships with some generalized sets in topological spaces. Connected spaces constitute the most important classes of topological spaces. In this paper we introduce the concept "rps-connected" in topological spaces.

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1. INTRODUCTION

In topology and related branches of mathematics a connected space is a topological space that cannot be represented as the union of two disjoint non empty open subsets. Connectedness is one of the principal topological properties that are used to distinguish topological spaces. In this paper we introduce rps-connected spaces. A topological space X is said to be rps-connected if X cannot be written as the disjoint union of two non empty rps-open sets in X.

2. PRELIMINARIES

Throughout this paper (X,τ) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. For a subset A of a topological space X, *cl*A and *int*A denote the closure of A and the interior of A respectively. X \ A denotes the complement of A in X. Throughout the paper □ indicates the end of the proof. We recall the following definitions and results.

Definition 2.1
A subset A of a space (X,τ) is called
(i) regular-open [1] if A = *int cl*A and regular-closed if A = *cl int*A,
(ii) pre-open [2] if A $\subseteq$ *int cl*A and pre-closed if *cl int*A $\subseteq$ A,
(iii) semi-pre-open [3] if A $\subseteq$ *cl int cl*A and semi-pre-closed if *int cl int*A $\subseteq$ A,
(iv) π -open [4] if A is a finite union of regular-open sets.

The semi-pre-closure of a subset A of X is the intersection of all semi-pre-closed sets containing A and is denoted by *spcl*A and the pre-closure of a subset A of X is the intersection of all pre-closed sets containing A and is denoted by *pcl*A.
Andrijevic [3] established the relationships among the above operators.

Lemma 2.2[3]
For any subset A of a topological space X, the following relations hold:
(i) *pcl*A = A $\cup$ *cl int*A,
(ii) *spcl*A = A $\cup$ *int cl int*A.

Definition 2.3
A subset A of a space X is called
(i) regular generalized closed or rg-closed if *cl*A $\subseteq$ U whenever A $\subseteq$ U and U is regular-open,[5]
(ii) generalized pre-regular closed or gpr-closed if *pcl*A $\subseteq$ U whenever A $\subseteq$ U and U is regular-open. [6]

The complement of an rg-closed set is rg-open and the complement of a gpr-closed set is gpr-open.

Definition 2.4
A subset A of a space X is called
(i) regular pre-semiclosed or rps-closed if s*pcl*A $\subseteq$ U whenever A $\subseteq$ U and U is rg-open, [7]
(ii) π -generalized pre-closed or π gp-closed if *pcl*A $\subseteq$ U whenever A $\subseteq$ U and U is

π -open, [8]
(iii) pre-generalized pre-regular closed or pgpr-closed if $pcl A \subseteq U$ whenever
$A \subseteq U$ and U is rg-open. [9]
The complement of an rps-closed set is rps-open, the complement of a π gp-closed set is π gp-open and the complement of a pgpr-closed set is pgpr-open.

Definition 2.5 [10]
A function f: $(X,\tau) \rightarrow (Y,\sigma)$ is called rps-continuous if $f^{-1}(V)$ is rps-closed in (X,τ) for every closed set V in (Y,σ).

Definition 2.6 [10]
A function f: $(X,\tau) \rightarrow (Y,\sigma)$ is called rps-irresolute if $f^{-1}(V)$ is rps-closed in (X,τ) for every rps-closed set V in (Y,σ).

Definition 2.7
A topological space (X,τ) is said to be
(i) pgpr-connected if X cannot be written as the union of two non empty disjoint pgpr-open sets in X, [11]
(ii) gpr-connected if X cannot be written as the union of two non empty disjoint gpr-open sets in X, [12]
(iii) π gp-connected if X cannot be written as the union of two non empty disjoint π gp-open sets in X. [13]

Definition 2.8 [14]
A function f: $(X,\tau) \rightarrow (Y,\sigma)$ is called contra rps-continuous if $f^{-1}(V)$ is rps-closed in (X,τ) for each open set V in (Y,σ).

Lemma 2.9 [10]
Let f: $(X,\tau) \rightarrow (Y,\sigma)$ be a function. Then the following are equivalent.
(i) f is rps-continuous.
(ii) The inverse image of each closed set in Y is rps-closed in X.
(iii) The inverse image of each open set in Y is rps-open in X.

Theorem 2.10 [10]
A function f: $X \rightarrow Y$ is rps-irresolute if and only if the inverse image of every rps-open set in Y is rps-open in X.

Lemma 2.11 [11]
For a topological space X, the following are equivalent.
(i) X is pgpr-connected
(ii) The only subsets of X which are both pgpr-open and pgpr-closed are the empty set and X.

Lemma 2.12
(i) Every pgpr-closed set is rps-closed. [7]
(ii) Every pgpr-open set is rps-open. [15]

Remark 2.13 [7]
If A is rps-closed in(X, τ), then A is closed in (X, τ_{rps}) provided τ_{rps} is a topology.

Definition 2.14 [16]
A space (X,τ) is called regular pre-semi-$T_{3/4}$ (briefly rps-$T_{3/4}$) if every rps-closed set is pre-closed.

Diagram 2.15 [7]
pre-closed $\rightarrow$ *pgpr-closed* $\rightarrow$ *rps-closed*

3. RPS-CONNECTED SPACES

Definition 3.1
A topological space X is said to be rps-connected if X cannot be written as the disjoint union of two non empty rps-open sets in X.

Definition 3.2
A subset S of a topological space X is said to be rps-connected relative to X if S cannot be written as the disjoint union of two non empty rps-open sets in X.

Theorem 3.3
For a topological space X, the following are equivalent.
(i) X is rps-connected

(ii) The only subsets of X which are both rps-open and rps-closed are the empty set and X.
(iii) Each rps-continuous function of X in to a discrete space Y with at least two points is a constant map.

Proof

Suppose X is rps-connected. Let S be a proper subset which is both rps-open and rps-closed in X. Then its complement $X \setminus S$ is also rps-open and rps-closed. Then $X = S \cup (X \setminus S)$, a disjoint union of two non empty rps-open sets which contradicts (i). Therefore $S = \emptyset$ or X. This proves (i) $\Rightarrow$(ii).

Suppose (ii) holds. Let $X = A \cup B$ where A and B are disjoint non empty rps-open subsets of X. Since $A = X \setminus B$ and $B = X \setminus A$, A and B are both rps-open and rps-closed. By assumption, $A = \emptyset$ or X which is a contradiction. Therefore X is rps-connected. This proves (ii)$\Rightarrow$(i).

Now to prove (ii) $\Rightarrow$(iii). Suppose (ii) holds. Let f: $X \rightarrow Y$ be an rps-continuous function where Y is a discrete space with atleast two points. Then $f^{-1}(\{y\})$ is rps-closed and rps-open for each $y \in Y$. Since (ii) holds, $f^{-1}(\{y\}) = \emptyset$ or X. If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$, f will not be a function. That implies $f^{-1}(\{y\}) = X$ for some $y \in Y$. Therefore for the fixed y, $f(x) = y$ for all $x \in X$. This proves that f is a constant map. This proves (ii) $\Rightarrow$(iii).

Now suppose (iii) holds. Let S be both rps-open and rps-closed in X. Suppose $S \neq \emptyset$.
Let f: $X \rightarrow Y$ be an rps-continuous function defined by $f(S) = \{y\}$ and $f(X \setminus S) = \{w\}$ for some distinct points y and w in Y. By (iii) f is a constant function. Therefore S = X. Hence (ii) holds. This proves (iii) $\Rightarrow$(ii).

□

Theorem 3.4

Let f: $X\rightarrow Y$ be a function.
(i) If X is rps-connected and if f is rps-continuous, surjective, then Y is connected.
(ii) If X is rps-connected and if f is rps-irresolute, surjective, then Y is rps-connected.

Proof

Let X be rps-connected and f be rps-continuous surjective. Suppose Y is disconnected. Then $Y = A \cup B$, where A and B are disjoint non empty open subsets of Y. Since f is rps-continuous surjective, by using Theorem 2.9, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$, $f^{-1}(B)$ are disjoint non empty rps-open subsets of X. This contradicts the fact that X is rps-connected. Therefore Y is connected. This proves (i).

Let X be rps-connected and f be rps-irresolute surjective. Suppose Y is not rps-connected. Then $Y = A \cup B$ where A and B are disjoint non empty rps-open subsets of Y. Since f is rps-irresolute surjective, by using Theorem 2.10, $X = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$, $f^{-1}(B)$ are disjoint non empty rps-open subsets of X. This implies X is not rps-connected, a contradiction. Therefore Y is rps-connected. This proves (ii).

□

Theorem 3.5

Every rps-connected space is connected.

Proof

Let X be an rps-connected space. Suppose X is not connected. Then there exists a proper non empty subset B of X which is both open and closed in X. Since every closed set is rps-closed, B is a proper non empty subset of X which is both rps-open and rps-closed in X. Then by using Theorem 3.3, X is not rps-connected. This proves the theorem.
The converse of Theorem 3.5 is not true as shown in the following example.

Example 3.6

Let $X = \{a,b,c\}$ with $\tau = \{\emptyset,\{a,b\},X\}$. Then we see that the topological space (X,τ) is connected. However, since $\{a\}$, $\{b\}$, $\{a,c\}$ and $\{b,c\}$ are both rps-open and rps-closed, X is not rps-connected.

Theorem 3.7

Every rps-connected space is pgpr-connected.

Proof

Let X be an rps-connected space. Suppose X is not pgpr-connected. Then by using Lemma 2.11,there exists a proper non empty subset B of X which is both pgpr-open and pgpr-closed in X. Using Theorem 2.12(i) and 2.12(ii), B is a proper non empty subset of X which is both rps-open and rps-closed in X. Then by using Theorem 3.3, X is not rps-connected. This proves the theorem.
The converse of Theorem 3.7 is not true as shown in the following example.

Example 3.8

Let $X = \{a,b,c\}$ with $\tau = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. Then the topological space (X,τ) is not rps-connected. The only subsets of X which are both pgpr-open and pgpr-closed are the empty set and X. Therefore X is pgpr-connected.

The concepts of rps-connectedness and π gp-connectedness are independent of each other as shown in the following examples.

Example 3.9

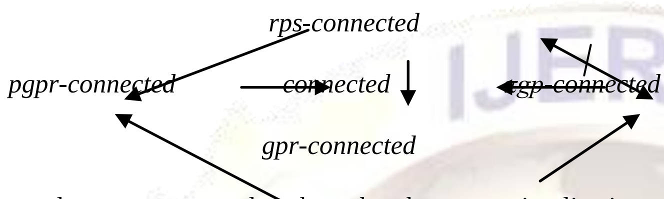
Let X = {a,b,c}, τ = {Ø,{a},{b},{a,b},X}. Then we see that (X,τ) is π gp-connected but not rps-connected, because {a} is both rps-open and rps-closed in (X,τ).

Example 3.10

Let X = {a,b,c}, τ = {Ø,{b},X}. Then (X,τ) is rps-connected but not π gp-connected, because {a,b} is both π gp-closed and π gp-open in (X,τ).

Thus we have the following implication diagram

Diagram 3.11



Examples are constructed to show that the reverse implications are not true as shown in Example 3.6 and Example 3.8.

Theorem 3.12

Suppose X is a topological space with $\tau_{rps} = \tau$. Then X is connected if and only if X is rps-connected.

Proof

Suppose X is not rps-connected. Then there exists a proper non empty subset B of X which is both rps-open and rps-closed in X. Since $\tau_{rps} = \tau$, using Remark 2.13, every rps-closed set is closed. Therefore B is both open and closed in X that implies X is not connected. This proves that connectedness implies rps-connectedness. The converse follows from Theorem 3.5.

Theorem 3.13

Suppose X is an rps-$T_{3/4}$ space. Then X is rps-connected if and only if X is pgpr-connected.

Proof

Suppose X is rps-connected. Then by using Theorem 3.7, X is pgpr-connected. Conversely we assume that X is pgpr-connected. Suppose X is not rps-connected. Then there exists a proper non empty subset B of X which is both rps-open and rps-closed in X. Since X is rps-$T_{3/4}$ by using Definition 2.14, B is both pre-open and pre-closed in X. Again using Diagram 2.15, B is both pgpr-open and pgpr-closed in X which shows that X is not pgpr-connected, a contradiction. Therefore X is rps-connected.

□

Theorem 3.14

A contra rps-continuous image of an rps-connected space is connected.

Proof

Let f :(X,τ) → (Y,σ) be a contra rps-continuous function from an rps-connected space X on to a space Y. Assume that Y is disconnected. Then Y = A ∪ B where A and B are non empty clopen sets in Y with A ∩ B = Ø.

Since f is contra rps-continuous, we have that $f^{-1}(A)$ and $f^{-1}(B)$ are non empty rps-open sets in X with $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(Ø) = Ø$. This means that X is not rps-connected, which is a contradiction. This proves the theorem.

Theorem 3.15

Let $\{A_\alpha: \alpha \in \Delta\}$ be a locally finite family of clopen sets in X such that they have a common point. If each A_α is an rps-connected subspace of X then their union is an rps-connected subspace of X.

Proof

Let p be a point A_α for every $\alpha \in \Delta$. Let $Y = \bigcup_{\alpha \in \Delta} A_\alpha$. Then Y is clopen. Suppose Y = C ∪ D where C and D are two disjoint non empty rps-open subsets of Y. The point p is in one of the sets C or D. If $p \in C$ then $A_\alpha \subseteq C$ for every α, so that $\bigcup_{\alpha \in \Delta} A_\alpha \subseteq C$. This shows that D = Ø. Therefore Y = C is rps-connected.

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